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MATHEMATICS

The field of rational constants of the Volterra derivation

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Abstract. We describe the field of rational constants of the four-variable Volterra derivation. Thus, we determine all rational first integrals of its corresponding system of differential equations. Such derivations play a role in population biology, laser physics, and plasma physics. Moreover, they play an important part in the derivation theory itself, since they are factorizable derivations. The problem is also linked to the invariant theory.

Key words: commutative algebra, Volterra derivation, Lotka–Volterra derivation, factorizable derivation, rational constant, rational first integral.

1. INTRODUCTION

The main result of the paper is Theorem 2, which gives the description of the field of rational constants of the four-variable Volterra derivation. The motivations of our study are the following:

- applications of Volterra and Lotka–Volterra systems in population biology, laser physics, and plasma physics (see, for instance, [1,3]);
- Lagutinsky's procedure of the association of the factorizable derivation (examples of such derivations are Lotka–Volterra derivations) with any given derivation;
- link to the invariant theory (for every connected algebraic group $G \subseteq \mathrm{GL}_n(k)$ there exists a derivation d such that $k[X]^G = k[X]^d$, see [7]).

Let us fix some notations:
k – a field of characteristic zero,
$\mathbb{N}$ – the set of nonnegative integers,
n – an integer ≥ 3,
$k[X] := k[x_1, \ldots, x_n]$,
$k(X) := k(x_1, \ldots, x_n)$.

Recall that if R is a commutative k-algebra, then a k-linear map $d : R \to R$ is called a *derivation* of R if for all $a, b \in R$

$$d(ab) = ad(b) + d(a)b.$$

We call $R^d = \ker d$ the *ring of constants* of the derivation d. Then $k \subseteq R^d$ and a *nontrivial* constant of d is an element of the set $R^d \setminus k$. If $f_1, \ldots, f_n \in k[X]$, and there exists exactly one derivation $d : k[X] \to k[X]$ such that $d(x_1) = f_1, \ldots, d(x_n) = f_n$. A derivation $d : k[X] \to k[X]$ is said to be *factorizable* if $d(x_i) = x_i f_i$, where the polynomials f_i are of degree 1 for $i = 1, \ldots, n$. We may associate the factorizable derivation with any given derivation of $k[X]$, and that construction helps to establish new facts on constants, especially rational constants, of the initial derivation (see, for instance, [6,8]).

There is no general effective procedure for determining $k[X]^d$ of a derivation $d : k[X] \to k[X]$, nor even deciding whether it is finitely generated (it may not be finitely generated for $n \geq 4$, see [5]). Even for a given derivation the problem may be difficult, see for instance counterexamples to Hilbert's fourteenth problem (all of them are of the form $k[X]^d$; however, it took more than half a century to find at least one of them, for more details we refer the reader to [5,7]) or Jouanolou derivations (where the rings of constants are trivial, see [6,7]).

2. LOTKA–VOLTERRA DERIVATIONS

Let $C_1, \ldots, C_n \in k$. From now on, $d : k[X] \to k[X]$ is a derivation of the form

$$d(x_i) = x_i(x_{i-1} - C_i x_{i+1})$$

for $i = 1, \ldots, n$ (we adhere to the convention that $x_{n+1} = x_1$ and $x_0 = x_n$). We call d a *Lotka–Volterra derivation* with parameters $C_1, \ldots, C_n$.

We will call a polynomial $g \in k[X]$ *strict* if it is homogeneous and not divisible by the variables $x_1,\ldots,x_n$. For $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$, we denote by X^α the monomial $x_1^{\alpha_1}\ldots x_n^{\alpha_n} \in k[X]$. Every nonzero homogeneous polynomial $f \in k[X]$ has the unique representation $f = X^\alpha g$, where X^α is a monomial and g is strict.

A nonzero polynomial f is said to be a *Darboux polynomial* of a derivation $\delta : k[X] \to k[X]$ if $\delta(f) = \Lambda f$ for some $\Lambda \in k[X]$. We will call Λ a *cofactor* of f. Thus, constants of a derivation δ are precisely its Darboux polynomials with cofactor 0. Denote by $k[X]_{(m)}$ the group of homogeneous polynomials of $k[X]$ of degree m. A derivation $\delta : k[X] \to k[X]$ is called *homogeneous of degree s* if $\delta(k[X]_{(m)}) \subseteq k[X]_{(m+s)}$ for every m. Since d is a homogeneous derivation of degree 1, the cofactor of each homogeneous polynomial is a linear form.

Lemma 1. (**[12] 3.2**). *Let $n = 4$. Let $g \in k[X]_{(m)}$ be a Darboux polynomial of d with the cofactor $\lambda_1x_1 + \ldots + \lambda_4x_4$. Let $i \in \{1,2,3,4\}$. If g is not divisible by x_i, then $\lambda_{i+1} \in \mathbb{N}$. More precisely, if $g(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_4) = x_{i+2}^{\beta_{i+2}}G$ and $x_{i+2} \nmid G$, then $\lambda_{i+1} = \beta_{i+2}$ and $\lambda_{i+3} = -C_{i+2}\lambda_{i+1}$.*

Corollary 1. (**[12] 3.3**). *Let $n = 4$. If $g \in k[X]$ is a strict Darboux polynomial of d, then its cofactor is a linear form with coefficients in $\mathbb{N}$.*

If $C_i = 1$ for all i, then we call d a *Volterra derivation*. Such derivations were investigated for example in [2], [4], and [10].

Lemma 2. *All strict Darboux polynomials of the 4-variable Volterra derivation are its constants.*

Proof. Let $\Lambda = \lambda_1x_1 + \ldots + \lambda_4x_4$ be the cofactor of a strict Darboux polynomial of the 4-variable Volterra derivation. By Lemma 1 we have

$$\lambda_{i+3} = -\lambda_{i+1} \tag{2.1}$$

for all i in the cyclic sense. However, Corollary 1 gives $\lambda_i \in \mathbb{N}$ for $i = 1,\ldots,4$. Then the left-hand side of (2.1) is nonnegative, whereas its right-hand side is nonpositive. Consequently, $\lambda_1 = \ldots = \lambda_4 = 0$ and thus $\Lambda = 0$. □

3. THE FIELD OF RATIONAL CONSTANTS

We show how to use the results of the previous section to determine the field of rational constants. For any derivation $\delta : k[X] \to k[X]$ there exists exactly one derivation $\bar{\delta} : k(X) \to k(X)$ such that $\bar{\delta}_{|k[X]} = \delta$. By a *rational constant* of the derivation $\delta : k[X] \to k[X]$ we mean the constant of its corresponding derivation $\bar{\delta} : k(X) \to k(X)$. The rational constants of δ form a field. For simplicity, we write δ instead of $\bar{\delta}$.

Throughout the rest of this paper we assume $n = 4$ and $C_1 = C_2 = C_3 = C_4 = 1$, that is, d is the four-variable Volterra derivation. We know the ring of polynomial constants of d (and we use this in the proof of Theorem 2).

Theorem 1. (**[9] 3.1**). *If d is the four-variable Volterra derivation, then*

$$k[X]^d = k[x_1 + x_2 + x_3 + x_4, x_1x_3, x_2x_4].$$

A generalization of Theorem 1 can be found in [11]. We also need the following facts.

Proposition 1. (**[7] 2.2.2**). *Let $\delta : k[X] \to k[X]$ be a derivation and let f and g be nonzero relatively prime polynomials from $k[X]$. Then $\delta(\frac{f}{g}) = 0$ if and only if f and g are Darboux polynomials of δ with the same cofactor.*

Proposition 2. (**[7] 2.2.3**). *Let δ be a homogeneous derivation of $k[X]$ and let $f \in k[X]$ be a Darboux polynomial of δ with the cofactor $\Lambda \in k[X]$. Then Λ is homogeneous and each homogeneous component of f is also a Darboux polynomial of δ with the same cofactor Λ.*

Proposition 3. (**[7] 2.2.1**). *Let δ be a derivation of $k[X]$. If $f \in k[X]$ is a Darboux polynomial of δ, then all factors of f are Darboux polynomials of δ.*

Now we can describe the field of rational constants of d.

Theorem 2. *If d is the four-variable Volterra derivation, then*

$$k(X)^d = k(x_1 + x_2 + x_3 + x_4, x_1x_3, x_2x_4).$$

Proof. A short calculation gives $d(x_1 + x_2 + x_3 + x_4) = 0$, $d(x_1x_3) = 0$, and $d(x_2x_4) = 0$. Thus we have $k(x_1 + x_2 + x_3 + x_4, x_1x_3, x_2x_4) \subseteq k(X)^d$. We need to prove the inclusion $k(X)^d \subseteq k(x_1 + x_2 + x_3 + x_4, x_1x_3, x_2x_4)$.

Let then $\psi = \frac{f}{g} \in k(X)^d$, where $f, g \in k[X] \setminus \{0\}$ and $\gcd(f,g) = 1$. Since $d(\psi) = 0$, by Proposition 1 both f and g are Darboux polynomials of d with common cofactor Λ. Let $\Lambda = ax_1 + bx_2 + cx_3 + ex_4$, where $a,b,c,e \in k$. By Proposition 2 all homogeneous components of both f and g are also Darboux polynomials of d with the same cofactor Λ. Consider one of these components $h = x_1^rx_2^sx_3^tx_4^u\bar{h}$, where $\bar{h}$ is strict. It is easy to prove, using induction on $r + s + t + u$, that

$$d(x_1^rx_2^sx_3^tx_4^u) = x_1^rx_2^sx_3^tx_4^u((s-u)x_1 + (t-r)x_2 + (u-s)x_3 + (r-t)x_4).$$

By Proposition 3, $\bar{h}$ is a Darboux polynomial of d. Hence, by Lemma 2, $\bar{h}$ is a constant of d. Therefore we have

$$d(h) = h((s-u)x_1 + (t-r)x_2 + (u-s)x_3 + (r-t)x_4),$$

that is, h is a Darboux polynomial of d with cofactor $(s-u)x_1+(t-r)x_2+(u-s)x_3+(r-t)x_4$ which, on the other hand, must be Λ. Consequently we have equations:

$$a = s-u = -c, \quad b = t-r = -e.$$

Therefore $a,b,c,e \in \mathbb{Z}$. Assume first that $a > 0$. Then $s = u+a > 0$, which means that $x_2 \mid h$. Since h was chosen as an arbitrary homogeneous component of f or g, we have $x_2 \mid f$, $x_2 \mid g$, a contradiction with $\gcd(f,g) = 1$. Similarly, assuming $a < 0$, we get $u = s-a > 0$ yielding a contradiction $x_4 \mid f$, $x_4 \mid g$. This proves $a = c = 0$, and one can use similar arguments for proving $b = e = 0$. We have proven that $\Lambda = 0$, that is, both f and g are constants of d. Thus, by Theorem 1, $f,g \in k[x_1+x_2+x_3+x_4, x_1x_3, x_2x_4]$ but then obviously $\psi \in k(x_1+x_2+x_3+x_4, x_1x_3, x_2x_4)$. □

Note that in view of Theorems 1 and 2, if d is the four-variable Volterra derivation, then $k(X)^d$ is the field of fractions of $k[X]^d$ (which is not true in general).

4. CONCLUSIONS

If δ is a derivation of $k(X)$ such that $\delta(x_i) = f_i$ for $i = 1,\dots,n$, then the set $k(X)^\delta \setminus k$ coincides with the set of all rational first integrals of a system of ordinary differential equations

$$\frac{dx_i(t)}{dt} = f_i(x_1(t),\dots,x_n(t)),$$

where $i = 1,\dots,n$ (for more details we refer the reader to [7] 1.6). Therefore, we described both all rational constants of the four-variable Volterra derivation and all rational first integrals of its corresponding system of differential equations. We believe that Lemma 1 would be useful for solving the problem also for arbitrary four-variable Lotka–Volterra derivations.

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REFERENCES

1. Almeida, M. A., Magalhães, M. E., and Moreira, I. C. Lie symmetries and invariants of the Lotka–Volterra system. *J. Math. Phys.*, 1995, **36**, 1854–1867.
2. Bogoyavlenskii, O. I. Algebraic constructions of integrable dynamical systems-extension of the Volterra system. *Russ. Math. Surv.*, 1991, **46**(3), 1–64.
3. Cairó, L. and Llibre, J. Darboux integrability for 3D Lotka–Volterra systems. *J. Phys. A*, 2000, **33**, 2395–2406.
4. Hegedűs, P. The constants of the Volterra derivation. *Cent. Eur. J. Math.*, 2012, **10**, 969–973.
5. Kuroda, S. Fields defined by locally nilpotent derivations and monomials. *J. Algebra*, 2005, **293**, 395–406.
6. Maciejewski, A. J., Moulin Ollagnier, J., Nowicki, A., and Strelcyn, J.-M. Around Jouanolou non-integrability theorem. *Indag. Math. (N.S.)*, 2000, **11**, 239–254.
7. Nowicki, A. *Polynomial Derivations and Their Rings of Constants*. N. Copernicus University Press, Toruń, 1994.
8. Nowicki, A. and Zieliński, J. Rational constants of monomial derivations. *J. Algebra*, 2006, **302**, 387–418.
9. Ossowski, P. and Zieliński, J. Polynomial algebra of constants of the four variable Lotka–Volterra system. *Colloq. Math.*, 2010, **120**, 299–309.
10. Zieliński, J. The five-variable Volterra system. *Cent. Eur. J. Math.*, 2011, **9**, 888–896.
11. Zieliński, J. Rings of constants of four-variable Lotka–Volterra systems. *Cent. Eur. J. Math.*, 2013, **11**, 1923–1931.
12. Zieliński, J. and Ossowski, P. Rings of constants of generic 4D Lotka–Volterra systems. *Czechoslovak Math. J.*, 2013, **63**, 529–538.

Volterra tuletise ratsionaalsete konstantide korpus

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On antud neljamuutuja Volterra tuletise ratsionaalsete konstantide korpuse kirjeldus. Sellest järeldub vastava diferentsiaalvõrrandite süsteemi ratsionaalsete esimeste integraalide kirjeldus. Sellistel tuletistel on rakendusi populatsioonibioloogias, laser- ja plasmafüüsikas.