Proceedings of the Estonian Academy of Sciences,
2014, **63**, 1, 8–10
doi: 10.3176/proc.2014.1.03
Available online at www.eap.ee/proceedings

MATHEMATICS

Sums of slices in direct sums of Banach spaces

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Received 13 May 2013, accepted 10 June 2013, available online 14 March 2014

Abstract. Motivated by recent studies on the strong diameter 2 property, we exhibit a simple reason why, in many direct sums of Banach spaces, the arithmetic mean of two slices may have diameter less than 2.

Key words: geometry of Banach spaces, direct sums, uniformly convex norm, slices, strong diameter 2 property.

Let X be a Banach space (over the real field $\mathbb{R}$). Let B_X and S_X denote its closed unit ball and its unit sphere, respectively. A *slice* of B_X is a set $S(x^*, \alpha) := \{x \in B_X : x^*(x) > 1 - \alpha\}$, where $x^* \in S_{X^*}$ and $\alpha > 0$.

The maximal diameter of slices can be 2. The same clearly holds for *convex combinations* of slices: diam $(\sum_{i=1}^n \lambda_i S_i) \le 2$ when S_i are slices and $\lambda_i \ge 0$ satisfy $\sum_{i=1}^n \lambda_i = 1$. In studies on "big" slice phenomena, some recent attention has been paid to Banach spaces X where the diameter of every convex combination of slices equals 2 (see, e.g., [1,2,4] for results and references). Such spaces are said to have the *strong diameter 2 property* (for this and related terminology, see [1]). For instance, it is known that the direct sums $c_0 \oplus_1 c_0$ and $c_0 \oplus_\infty c_0$ have the strong diameter 2 property.

Answering (in the negative) a question posed in [1, Question (c)], R. Haller, J. Langemets, and M. Põldvere showed that the ℓ_p-sum $c_0 \oplus_p c_0$ fails the strong diameter 2 property whenever $1 < p < \infty$. More generally, they proved the theorem (see [4, Theorem 1]): *if $X \neq \{0\}$ and $Y \neq \{0\}$ are Banach spaces and $1 < p < \infty$, then $X \oplus_p Y$ fails the strong diameter 2 property.*

The same result was independently obtained in [2, Theorem 3.2] in a more general context of absolute norms. In both cases, among others, two slices S_1 and S_2 were exhibited, whose *arithmetic mean* $1/2(S_1 + S_2)$ had diameter strictly less than 2.

In this short note, we shall present a simple proof of the above result. Since we also would like to expose the *reason for the failure of the strong diameter 2 property* in $X \oplus_p Y$, we shall consider the following general case of $X \oplus Y$.

Definition. *Let X and Y be Banach spaces. We say that $X \oplus Y$ is* equipped with a uniformly convex $\mathbb{R}^2$-norm *if $\|(x,y)\| = |(\|x\|, \|y\|)|$ for all $x \in X$ and $y \in Y$, where $|(\cdot,\cdot)|$ is a uniformly convex norm on $\mathbb{R}^2$ such that $|(1,0)| = |(0,1)| = 1$ and $|(a,b)| \le |(c,d)|$ whenever $0 \le a \le c$ and $0 \le b \le d$.*

Recall (see, e.g., [5, pp. 59–60]) that a norm $\|\cdot\|$ of a Banach space X is *uniformly convex* if its *modulus of convexity* $\delta_X : (0,2] \to [0,1]$, defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in X, \|x\|, \|y\| \le 1, \|x-y\| \ge \varepsilon\right\},$$

is strictly positive (i.e., $\delta_X(\varepsilon) > 0$ when $\varepsilon > 0$).

The modulus of convexity measures (as nicely expressed in [3, p. 145]) the minimum depth of midpoints of line segments $[x,y]$ in B_X below the unit sphere S_X. If X is uniformly convex, then S_X contains no (non-trivial) line segments, i.e., X is *strictly convex*. (The converse holds when X is of finite dimension.)

Let S be a non-empty subset of a Banach space $X = (X, \|\cdot\|)$. In the following theorem, we shall use the notation

$$\|S\| := \sup\{\|x\| : x \in S\}.$$

Since $S \subset \|S\| B_X$, we clearly have

$$\operatorname{diam} \frac{1}{2} S \le \|S\|.$$

This is a rough estimate: if S is a slice, then $\|S\| = 1$, but diam S can be arbitrarily small (e.g., when $X = \ell_p^2, 1 < p < \infty$).

Theorem. *Let $X \neq \{0\}$ and $Y \neq \{0\}$ be Banach spaces. If $X \oplus Y$ is equipped with a uniformly convex $\mathbb{R}^2$-norm, then there are slices S_1 and S_2 of $B_{X\oplus Y}$ such that*

$$\|S_1 + S_2\| < 2.$$

Consequently,

$$\operatorname{diam} \frac{1}{2}(S_1 + S_2) < 2,$$

and $X \oplus Y$ fails the strong diameter 2 property.

Proof. Let $0 < \alpha < 1$. We choose α such that $(1-\alpha, 1-\alpha) \notin B_{\mathbb{R}^2}$. This can be done, because if $(1-\alpha, 1-\alpha) \in B_{\mathbb{R}^2}$ for all such α, then $(1,1) \in B_{\mathbb{R}^2}$, meaning that $(1,1) \in S_{\mathbb{R}^2}$. Hence, $S_{\mathbb{R}^2}$ would contain the line segment $[(1,0),(1,1)]$, which is a contradiction.

Denote $\sigma_1 = \{(a,b) \in B_{\mathbb{R}^2} : a \ge 1-\alpha\}$ and $\sigma_2 = \{(a,b) \in B_{\mathbb{R}^2} : b \ge 1-\alpha\}$. Since σ_1 and σ_2 are disjoint non-empty compact subsets, one has $\varepsilon := \operatorname{dist}(\sigma_1, \sigma_2) > 0$ and $\delta := \delta_{\mathbb{R}^2}(\varepsilon) > 0$. If $u_i \in \sigma_i$, $i = 1,2$, then $|u_1 - u_2| \ge \varepsilon$ and therefore $1 - |u_1 + u_2|/2 \ge \delta$. Hence $|u_1 + u_2| \le 2(1-\delta)$, meaning that

$$|\sigma_1 + \sigma_2| \le 2(1-\delta).$$

Take now any $x^* \in S_{X^*}$ and $y^* \in S_{Y^*}$. Consider the slices $S_1 = S((x^*,0),\alpha)$ and $S_2 = S((0,y^*),\alpha)$ of $B_{X\oplus Y}$. We shall show that

$$\|S_1 + S_2\| \le |\sigma_1 + \sigma_2|,$$

which would complete the proof of the theorem.

First, observe that if $(x,y) \in S_i$, then $(\|x\|, \|y\|) \in \sigma_i,\ i = 1,2$. Indeed, $|(\|x\|, \|y\|)| = \|(x,y)\| \le 1$ and, e.g.,

$$\|x\| \ge x^*(x) = (x^*,0)(x,y) > 1-\alpha.$$

Hence, for all $(x_i, y_i) \in S_i, i = 1,2$, denoting $u_i = (\|x_i\|, \|y_i\|) \in \sigma_i$, we have

$$\begin{aligned}\|(x_1,y_1) + (x_2,y_2)\| &= \|(x_1+x_2, y_1+y_2)\| \\ &= |(\|x_1+x_2\|, \|y_1+y_2\|)| \\ &\le |(\|x_1\| + \|x_2\|, \|y_1\| + \|y_2\|)| \\ &= |u_1+u_2|,\end{aligned}$$

as desired. □

ACKNOWLEDGEMENTS

The author is grateful to the referees for the comments and to María Acosta, Julio Becerra Guerrero, and Ginés López Pérez for sending her the preprint of [2]. This research was supported by Estonian Targeted Financing Project SF0180039s08 and by Estonian Science Foundation Grant 8976.

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Viilude summad Banachi ruumide otsesummades

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On antud lihtne olemuslik põhjendus nähtusele, miks paljudes Banachi ruumide otsesummades võib viilude aritmeetilise keskmise diameeter olla väiksem kui 2. Tulemus on ajendatud hiljutistest uuringutest tugeva diameeter-2 omaduse vallas.