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MATHEMATICS

A Voronovskaya-type theorem for the second derivative of the Bernstein–Chlodovsky polynomials

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Abstract. This paper is devoted to a Voronovskaya-type theorem for the second derivative of the Bernstein–Chlodovsky polynomials. This type of theorem was considered for the Bernstein–Chlodovsky polynomials by Jerzy Albrycht and Jerzy Radecki in 1960 and by Paul L. Butzer and the author in 2009, in case of the polynomials themselves and their first derivative, respectively.

Key words: Bernstein–Chlodovsky polynomials, Voronovskaya-type theorems.

1. INTRODUCTION

This paper is concerned with the classical Bernstein–Chlodovsky operators

$$(C_n f)(x) := \sum_{k=0}^{n} f\left(\frac{b_n}{n}k\right) p_{k,n}\left(\frac{x}{b_n}\right), \tag{1}$$

where f is a function defined on $[0,\infty)$ and bounded on every finite interval $[0,b] \subset [0,\infty)$ with a certain rate, with $p_{k,n}$ denoting as usual

$$p_{k,n}(x) = \left(\begin{array}{c} n \\ k \end{array}\right) x^k (1-x)^{n-k}, \ \ 0 \le x \le 1, \tag{2}$$

and $(b_n)_{n=1}^{\infty}$ being a positive increasing sequence of real numbers with the properties

$$\lim_{n\to\infty} b_n = \infty, \qquad \lim_{n\to\infty} \frac{b_n}{n} = 0. \tag{3}$$

These polynomials were introduced by Chlodovsky [4] in 1937 in generalization of the Bernstein polynomials $(B_n f)(x)$, the case $b_n = 1$, $n \in \mathbb{N}_0$, which approximate the function f on the interval $[0,1]$ (or, suitably modified on any fixed finite interval $[-b,b]$).

In fact, if $M(b;f) := \sup_{0\le x\le b} |f(x)|$, then Chlodovsky (see also Lorentz [5], p. 36) showed that if

$$\lim_{n\to\infty} \exp\left(-\alpha \frac{n}{b_n}\right) M(b_n;f) = 0 \tag{4}$$

for every $\alpha > 0$, then $(C_n f)(x)$ converges to $f(x)$ at each point of continuity of f.

As a corollary he states that if a function f belonging to $C[0,\infty)$ is of order $f(x) = \mathscr{O}(\exp x^p)$ for some $p > 0$, and if the sequence $\{b_n\}$ satisfies the condition

$$b_n \le n^{\frac{1}{p+1+\eta}},$$

where $\eta > 0$, no matter how small, then $(C_n f)(x)$ converges to $f(x)$ at each point $x \in \mathbb{R}^+$.

Albrycht and Radecki [1] proved that under the assumption

$$\lim_{n\to\infty} \frac{n}{b_n} \exp\left(-\alpha \frac{n}{b_n}\right) M(b_n; f) = 0 \tag{5}$$

for every $\alpha > 0$,

$$\lim_{n\to\infty} \frac{n}{b_n} [(C_n f)(x) - f(x)] = \frac{x}{2} f''(x) \tag{6}$$

at each point $x \ge 0$ for which $f''(x)$ exists.

Then Butzer and Karsli [3] established the counterpart of (6) for the first derivative, namely

$$\lim_{n\to\infty} \frac{n}{b_n} [(C_n f)'(x) - f'(x)] = \frac{f''(x) + x f'''(x)}{2} \tag{7}$$

holds at each fixed point $x \ge 0$ for which $f'''(x)$ exists, provided the growth condition (5) is satisfied for every $\alpha > 0$.

The present paper deals with the counterpart of (7) for the second derivative of the Chlodovsky polynomials. The theorem states that

$$\lim_{n\to\infty} \frac{n}{b_n} [(C_n f)''(x) - f''(x)] = \frac{2 f'''(x) + x f^{(iv)}(x)}{2} \tag{8}$$

at each fixed point $x \ge 0$ for which $f^{(iv)}(x)$ exists, provided that again the growth condition (5) is satisfied for every $\alpha > 0$. This will be Theorem 1 of Section 3. Section 2 includes some preliminaries.

The Bernstein–Chlodovsky polynomials, based on functions defined on $[0,\infty)$, which are bounded on every $[0,b_n] \subset [0,\infty)$ with a certain rate, such as (4), (5) or (14), are indeed true polynomials of degree n (in x/b_n), also having support $[0,b_n]$, with $\{b_n\}$ satisfying (3).

Thus they are a very natural polynomial process in approximating unbounded functions on the unbounded infinite interval $[0,\infty)$.

2. AUXILIARY RESULTS FOR BERNSTEIN–CHLODOVSKY POLYNOMIALS

In this section we present some results needed to prove our main theorem.

Lemma 1. *For the central moments of order $m \in N_0$, for any fixed $x \in [0,\infty)$,*

$$T^*_{n,m}(x) := \sum_{k=0}^{n} \left(\frac{k b_n}{n} - x\right)^m p_{k,n}\left(\frac{x}{b_n}\right),$$

one has

$$\begin{aligned} T^*_{n,0}(x) &= 1, \;\; T^*_{n,1}(x) = 0, \;\; T^*_{n,2}(x) = \frac{x(b_n - x)}{n}, \;\; T^*_{n,3}(x) = \frac{x(b_n - x)(b_n - 2x)}{n^2}, \\ T^*_{n,4}(x) &= \frac{x(b_n - x)\left[(b_n - x)(b_n - 2x) + x(4x - 3b_n) + 3nx(b_n - x)\right]}{n^3}, \end{aligned}$$

and for any fixed $x \in [0,\infty)$,

$$\left|T^*_{n,m}(x)\right| \leq A_m(x)\frac{x(b_n-x)}{b_n}\left(\frac{b_n}{n}\right)^{[(m+1)/2]} \quad (n \in \mathbb{N},\ n > b_n,\ 0 \leq x \leq b_n), \tag{9}$$

where $A_m(x)$ *denotes a polynomial in* x, *of degree* $[m/2]-1$, *with non-negative coefficients independent of* n, *and* $[a]$ *denotes the integral part of* a.

For the proof see Butzer and Karsli [3].

The first part of the next lemma is due to Chlodovsky [4].

Lemma 2. *For* $t \in [0,1]$ *the inequality*

$$0 \leq z \leq \frac{3}{2}\sqrt{nt(1-t)}$$

implies

$$\sum_{|k-nt|\geq 2z\sqrt{nt(1-t)}} p_{k,n}(t) \leq 2\exp\left(-z^2\right). \tag{10}$$

In particular, for $0 < \delta \leq x < b_n$ *and sufficiently large* n,

$$\sum_1^{*} := \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} p_{k,n}\left(\frac{x}{b_n}\right) \leq 2\exp\left(-\frac{\delta^2}{4x}\frac{n}{b_n}\right). \tag{11}$$

The proof of (11) is given in [1].

According to (1), (2), and (10), there follow by differentiation the two fundamental representations for $(C_nf)''(x)$, which are also needed:

$$(C_nf)''(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}b_n\right)\frac{d^2}{dx^2}p_{k,n}\left(\frac{x}{b_n}\right), \quad (0 < x < b_n), \tag{12}$$

where

$$\frac{d^2}{dx^2}p_{k,n}\left(\frac{x}{b_n}\right) = \left[\left(\frac{kb_n-nx}{x(b_n-x)}\right)^2 - \frac{n}{x(b_n-x)} - \frac{(b_n-2x)(kb_n-nx)}{x^2(b_n-x)^2}\right]p_{k,n}\left(\frac{x}{b_n}\right),$$

and

$$(C_nf)''(x) = \frac{n(n-1)}{(b_n-x)^2}\sum_{k=0}^{n-2}\left[f\left(\frac{k+2}{n}b_n\right) - 2f\left(\frac{k+1}{n}b_n\right) + f\left(\frac{k}{n}b_n\right)\right]p_{k,n-2}\left(\frac{x}{b_n}\right). \tag{13}$$

3. VORONOVSKAYA-TYPE THEOREM FOR $(C_nf)''(x)$

Theorem 1. *Let a function* f, *defined on* $[0,\infty)$, *satisfy the growth condition*

$$\lim_{n\to\infty}\left(\frac{n}{b_n}\right)^{3/2}\exp\left(-\alpha\frac{n}{b_n}\right)M(b_n;f) = 0 \ \text{ or } \ M(b_n;f) = o\left(\left(\frac{b_n}{n}\right)^{3/2}\exp\left(\alpha\frac{n}{b_n}\right)\right) \tag{14}$$

for every $\alpha > 0$, $\{b_n\}$ *being a positive sequence satisfying* (3). *Then there holds*

$$\lim_{n\to\infty}\frac{n}{b_n}[(C_nf)''(x)-f''(x)]=\frac{2f'''(x)+xf^{(iv)}(x)}{2}, \tag{15}$$

at each point $x\geq 0$ at which $f^{(iv)}(x)$ exists.

Proof. Note that $(C_nf)''(0)\neq f''(0)$, all $n\in\mathbb{N}$. Firstly, the asymptotic formula (15) is valid for $x=0$. Since

$$(C_nf)''(0)=\frac{n(n-1)}{b_n^2}\left[f(2b_n/n)-2f(b_n/n)+f(0)\right]$$

in view of (13), it suffices to show provided $f'''(x)$ and $f^{(iv)}(x)$ exist that

$$\lim_{n\to\infty}\frac{n}{b_n}\left\{\frac{n(n-1)}{b_n^2}\left[f(2b_n/n)-2f(b_n/n)+f(0)\right]-f''(0)\right\}=f'''(0). \tag{16}$$

Indeed, Taylor's formula (see below) readily yields if $f^{(iv)}(0)$ exists

$$f\left(\frac{2b_n}{n}\right)=f(0)+\left(\frac{2b_n}{n}\right)f'(0)+\left(\frac{2b_n}{n}\right)^2\frac{f''(0)}{2}+\left(\frac{2b_n}{n}\right)^3\frac{f'''(0)}{6}+\left(\frac{2b_n}{n}\right)^4\left[\frac{f^{(iv)}(0)}{24}+h\left(\frac{2b_n}{n}\right)\right],$$

and

$$\begin{aligned} f\left(\frac{b_n}{n}\right) &= f(0)+\left(\frac{b_n}{n}\right)f'(0)+\left(\frac{b_n}{n}\right)^2\frac{f''(0)}{2}+\left(\frac{b_n}{n}\right)^3\frac{f'''(0)}{6} \\ &\quad +\left(\frac{b_n}{n}\right)^4\left[\frac{f^{(iv)}(0)}{24}+h\left(\frac{b_n}{n}\right)\right], \end{aligned}$$

where $h(b_n/n)\to 0$ as $n\to\infty$. So we have

$$f\left(\frac{2b_n}{n}\right)-2f\left(\frac{b_n}{n}\right)+f(0)=\left(\frac{b_n}{n}\right)^2f''(0)+\left(\frac{b_n}{n}\right)^3f'''(0)+\frac{14}{24}\left(\frac{b_n}{n}\right)^4f^{(iv)}(0)+k\left(\frac{b_n}{n}\right), \tag{17}$$

where

$$k\left(\frac{b_n}{n}\right)=\left(\frac{2b_n}{n}\right)^4h\left(\frac{2b_n}{n}\right)-2\left(\frac{b_n}{n}\right)^4h\left(\frac{b_n}{n}\right)$$

and $k(b_n/n)\to 0$ as $n\to\infty$. If we use (17) in (16) for $x=0$, then we have

$$\frac{n}{b_n}[(C_nf)''(0)-f''(0)]=\frac{-f''(0)}{b_n}+\frac{n-1}{n}f'''(0)+\frac{14}{24}\frac{b_n(n-1)}{n^2}f^{(iv)}(0)+\frac{n^2(n-1)}{b_n^3}k\left(\frac{b_n}{n}\right).$$

Thus the assertion (15) now follows for $x=0$ as $n\to\infty$.

So let $b_n > x > 0$. By Taylor's formula we have

$$f\left(\frac{k}{n}b_n\right) = f(x)+\left(\frac{k}{n}b_n-x\right)f'(x)+\left(\frac{k}{n}b_n-x\right)^2\frac{f''(x)}{2}+\left(\frac{k}{n}b_n-x\right)^3\frac{f'''(x)}{6}$$
$$+\left(\frac{k}{n}b_n-x\right)^4\left[\frac{f^{(iv)}(x)}{24}+h\left(\frac{k}{n}b_n-x\right)\right], \tag{18}$$

where $h(y)$ converges to zero with y. Substituting (18) into the representation (12), we can write:

$$(C_nf)''(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}b_n\right)\left[\left(\frac{kb_n-nx}{x(b_n-x)}\right)^2-\frac{n}{x(b_n-x)}-\frac{(b_n-2x)(kb_n-nx)}{x^2(b_n-x)^2}\right]p_{k,n}\left(\frac{x}{b_n}\right)$$
$$= \frac{n^2}{x^2(b_n-x)^2}\sum_{k=0}^{n} f\left(\frac{k}{n}b_n\right)\left(\frac{k}{n}b_n-x\right)^2 p_{k,n}\left(\frac{x}{b_n}\right)$$
$$-\frac{(b_n-2x)n}{x^2(b_n-x)^2}\sum_{k=0}^{n} f\left(\frac{k}{n}b_n\right)\left(\frac{k}{n}b_n-x\right)p_{k,n}\left(\frac{x}{b_n}\right)$$
$$-\frac{n}{x(b_n-x)}\sum_{k=0}^{n} f\left(\frac{k}{n}b_n\right)p_{k,n}\left(\frac{x}{b_n}\right). \tag{19}$$

According to Lemma 1, we have for (19)

$$(C_nf)''(x) = -\frac{n}{x(b_n-x)}\left[f(x)+\frac{f''(x)}{2}T^*_{n,2}(x)+\frac{f'''(x)}{6}T^*_{n,3}(x)+\frac{f^{(iv)}(x)}{24}T^*_{n,4}(x)\right]+R_{n,3}(x)$$
$$-\frac{(b_n-2x)n}{x^2(b_n-x)^2}\left[f'(x)T^*_{n,2}(x)+\frac{f''(x)}{2}T^*_{n,3}(x)+\frac{f'''(x)}{6}T^*_{n,4}(x)+\frac{f^{(iv)}(x)}{24}T^*_{n,5}(x)\right]+R_{n,2}(x)$$
$$+\frac{n^2}{x^2(b_n-x)^2}\left[f(x)T^*_{n,2}(x)+f'(x)T^*_{n,3}(x)+\frac{f''(x)}{2}T^*_{n,4}(x)+\frac{f'''(x)}{6}T^*_{n,5}(x)+\frac{f^{(iv)}(x)}{24}T^*_{n,6}(x)\right]$$
$$+R_{n,1}(x), \tag{20}$$

where

$$R_{n,1}(x) := \frac{n^2}{x^2(b_n-x)^2}\sum_{k=0}^{n} h\left(\frac{k}{n}b_n-x\right)\left(\frac{k}{n}b_n-x\right)^6 p_{k,n}\left(\frac{x}{b_n}\right),$$

$$R_{n,2}(x) := -\frac{(b_n-2x)n}{x^2(b_n-x)^2}\sum_{k=0}^{n} h\left(\frac{k}{n}b_n-x\right)\left(\frac{k}{n}b_n-x\right)^5 p_{k,n}\left(\frac{x}{b_n}\right),$$

$$R_{n,3}(x) := -\frac{n}{x(b_n-x)}\sum_{k=0}^{n} h\left(\frac{k}{n}b_n-x\right)\left(\frac{k}{n}b_n-x\right)^4 p_{k,n}\left(\frac{x}{b_n}\right).$$

For simplicity we define

$$R_n(x) := R_{n,1}(x)+R_{n,2}(x)+R_{n,3}(x).$$

Again, by Lemma 1, we can rewrite (20) in the form

$$\begin{aligned}
(C_n f)''(x) &= f(x)\left[\frac{n^2}{x^2(b_n-x)^2}T^*_{n,2}(x)-\frac{n}{x(b_n-x)}\right]\\
&+f'(x)\left[\frac{n^2}{x^2(b_n-x)^2}T^*_{n,3}(x)-\frac{(b_n-2x)n}{x^2(b_n-x)^2}T^*_{n,2}(x)\right]\\
&+\frac{f''(x)}{2}\left[\frac{n^2}{x^2(b_n-x)^2}T^*_{n,4}(x)-\frac{(b_n-2x)n}{x^2(b_n-x)^2}T^*_{n,3}(x)-\frac{n}{x(b_n-x)}T^*_{n,2}(x)\right]\\
&+\frac{f'''(x)}{6}\left[\frac{n^2}{x^2(b_n-x)^2}T^*_{n,5}(x)-\frac{(b_n-2x)n}{x^2(b_n-x)^2}T^*_{n,4}(x)-\frac{n}{x(b_n-x)}T^*_{n,3}(x)\right]\\
&+\frac{f^{(iv)}(x)}{24}\left[\frac{n^2}{x^2(b_n-x)^2}T^*_{n,6}(x)-\frac{(b_n-2x)n}{x^2(b_n-x)^2}T^*_{n,5}(x)-\frac{n}{x(b_n-x)}T^*_{n,4}(x)\right]\\
&+R_n(x).
\end{aligned}$$

The first two expressions on the right hand side are zero on account of

$$\begin{aligned}
T^*_{n,2}(x) &= \frac{x(b_n-x)}{n},\\
T^*_{n,3}(x) &= \frac{x(b_n-x)(b_n-2x)}{n^2},
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{n}{b_n}\left[(C_n f)''(x)-f''(x)\right] &= -\frac{n}{b_n}\frac{f''(x)}{n}+\frac{n}{b_n}\frac{f'''(x)}{6}\left[\frac{6(n-1)(b_n-2x)}{n^2}\right]\\
&+\frac{n}{b_n}\frac{f^{(iv)}(x)}{24}\left[\frac{2(n-1)(7b_n^2+6b_n(n-6)x-6(n-6)x^2)}{n^3}\right]+\frac{n}{b_n}R_n(x).
\end{aligned}$$

Now, as $n\to\infty$, the first three terms on the right-hand side tend to zero, to $f'''(x)$ and $xf^{(iv)}(x)/2$, respectively.

In order to complete the proof, we have to prove

$$\lim_{n\to\infty}\frac{n}{b_n}R_n(x)=0.$$

For any $\varepsilon>0$ there exists a $\delta>0$ such that $|h(y)|<\varepsilon$ for $|y|\le\delta$, and we choose δ so small that $\delta\le x$. So we split the sum $R_{n,1}(x)$ into two parts as follows:

$$\begin{aligned}
R_{n,1}(x) = &\frac{n^2}{x^2(b_n-x)^2}\sum_{\left|\frac{kb_n}{n}-x\right|<\delta} h\left(\frac{k}{n}b_n-x\right)\left(\frac{k}{n}b_n-x\right)^6 p_{k,n}\left(\frac{x}{b_n}\right)\\
&+\frac{n^2}{x^2(b_n-x)^2}\sum_{\left|\frac{kb_n}{n}-x\right|\ge\delta} h\left(\frac{k}{n}b_n-x\right)\left(\frac{k}{n}b_n-x\right)^6 p_{k,n}\left(\frac{x}{b_n}\right)\\
=:&\ R_{n,1,1}(x)+R_{n,1,2}(x),
\end{aligned}$$

say. According to Lemma 1, once more, one has for $R_{n,1,1}(x)$ the inequality

$$\begin{aligned} R_{n,1,1}(x) &\leq \varepsilon \frac{n^2}{x^2(b_n-x)^2} T^*_{n,6}(x) \\ &\leq \varepsilon \frac{n^2}{x^2(b_n-x)^2} \frac{x(b_n-x)}{b_n} A_6(x) \left(\frac{b_n}{n}\right)^3 \\ &= \varepsilon \frac{b_n}{n} \frac{b_n}{x(b_n-x)} A_6(x), \end{aligned}$$

$\varepsilon > 0$ being arbitrary, which implies that

$$\lim_{n\to\infty} \frac{n}{b_n} R_{n,1,1}(x) = 0. \tag{21}$$

In order to estimate the term $R_{n,1,2}(x)$, we rewrite the representation (18) in the form

$$\begin{aligned} \left(\frac{k}{n}b_n - x\right)^4 h\left(\frac{k}{n}b_n - x\right) &= f\left(\frac{k}{n}b_n\right) - f(x) - \left(\frac{k}{n}b_n - x\right) f'(x) - \left(\frac{k}{n}b_n - x\right)^2 \frac{f''(x)}{2} \\ &\quad - \left(\frac{k}{n}b_n - x\right)^3 \frac{f'''(x)}{6} - \left(\frac{k}{n}b_n - x\right)^4 \frac{f^{(\iota v)}(x)}{24}, \end{aligned}$$

and hence we obtain

$$\begin{aligned} |R_{n,1,2}(x)| &= \left| \frac{n^2}{x^2(b_n-x)^2} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} h\left(\frac{k}{n}b_n - x\right)\left(\frac{k}{n}b_n - x\right)^4 \left(\frac{k}{n}b_n - x\right)^2 p_{k,n}\left(\frac{x}{b_n}\right) \right| \\ &\leq \frac{n^2}{x^2(b_n-x)^2} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left| f\left(\frac{k}{n}b_n\right)\right| \left|\frac{k}{n}b_n - x\right|^2 p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad + |f(x)| \frac{n^2}{x^2(b_n-x)^2} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{k}{n}b_n - x\right|^2 p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad + |f'(x)| \frac{n^2}{x^2(b_n-x)^2} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{k}{n}b_n - x\right|^3 p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad + |f''(x)| \frac{n^2}{2x^2(b_n-x)^2} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left(\frac{k}{n}b_n - x\right)^4 p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad + \left|f'''(x)\right| \frac{n^2}{6x^2(b_n-x)^2} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{k}{n}b_n - x\right|^5 p_{k,n}\left(\frac{x}{b_n}\right) \\ &\quad + \left|f^{(\iota v)}(x)\right| \frac{n^2}{24x^2(b_n-x)^2} \sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta} \left|\frac{k}{n}b_n - x\right|^6 p_{k,n}\left(\frac{x}{b_n}\right) \\ &=: \sum\nolimits_1^*(n) + \sum\nolimits_2^*(n) + \sum\nolimits_3^*(n) + \sum\nolimits_4^*(n) + \sum\nolimits_5^*(n) + \sum\nolimits_6^*(n), \end{aligned}$$

say.

In view of the Cauchy–Schwarz inequality,

$$\begin{aligned}
\sum_1^*(n) &:= \frac{n^2}{x^2(b_n-x)^2}\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left\{\left|f\left(\frac{kb_n}{n}\right)\right|\sqrt{p_{k,n}\left(\frac{x}{b_n}\right)}\left|\frac{kb_n}{n}-x\right|^2\sqrt{p_{k,n}\left(\frac{x}{b_n}\right)}\right\}\\
&\leq \frac{n^2}{x^2(b_n-x)^2}\left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|f\left(\frac{kb_n}{n}\right)\right|^2 p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2}\left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|\frac{kb_n}{n}-x\right|^4 p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2}\\
&=: \frac{n^2}{x^2(b_n-x)^2}\left(\sum_{1,1}^*\right)\left(\sum_{1,2}^*\right),
\end{aligned}$$

say. Since $\sqrt{\sup_{0\leq x\leq a}|f(x)|^2}=M(a;f)$,

$$\begin{aligned}
\sum_{1,1}^*(n) &\leq M(b_n;f)\left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2}\\
&\leq \sqrt{2}M(b_n;f)\exp\left(-\frac{\delta^2}{8x}\frac{n}{b_n}\right).
\end{aligned}$$

As to the second product term, noting Lemma 2 and inequality (9) (with $m=6$), we have

$$\begin{aligned}
\sum_{1,2}^*(n) &\leq \left\{\frac{1}{\delta^2}\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|\frac{kb_n}{n}-x\right|^6 p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2}\\
&\leq \left\{\frac{1}{\delta^2}T_{n,6}^*(x)\right\}^{1/2}\\
&\leq \frac{1}{\delta}\sqrt{A_6(x)}\sqrt{\frac{x(b_n-x)}{b_n}}\left(\frac{b_n}{n}\right)^{3/2}.
\end{aligned}$$

Thus, altogether we have

$$\lim_{n\to\infty}\frac{n}{b_n}\sum_1^*(n)\leq\lim_{n\to\infty}\left(\frac{n}{x(b_n-x)}\right)^{3/2}\frac{\sqrt{2A_6(x)}}{\delta}M(b_n;f)\exp\left(-\frac{\delta^2}{8x}\frac{n}{b_n}\right)=0.$$

So it remains to show that

$$\lim_{n\to\infty}\frac{n}{b_n}\sum_i^*(n)=0 \tag{22}$$

is valid for $i=2,3,4,5,6$.

Now we consider the second term, i.e.,

$$\begin{aligned}
\sum_2^*(n) &:= |f(x)|\frac{n^2}{x^2(b_n-x)^2}\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left\{\sqrt{p_{k,n}\left(\frac{x}{b_n}\right)}\left|\frac{kb_n}{n}-x\right|^2\sqrt{p_{k,n}\left(\frac{x}{b_n}\right)}\right\}\\
&\leq |f(x)|\frac{n^2}{x^2(b_n-x)^2}\left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2}\left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|\frac{kb_n}{n}-x\right|^4 p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2}\\
&=: |f(x)|\frac{n^2}{x^2(b_n-x)^2}\left(\sum_{2,1}^*\right)\left(\sum_{2,2}^*\right).
\end{aligned}$$

From (11) one has

$$\sum_{2,1}^{*}(n) \leq \sqrt{2}\exp\left(-\frac{\delta^2}{8x}\frac{n}{b_n}\right).$$

As to the second product term, noting Lemma 1 (with $m=6$), we have

$$\begin{aligned}
\sum_{2,2}^{*}(n) &\leq \left\{\frac{1}{\delta^2}\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|\frac{kb_n}{n}-x\right|^6 p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2} \\
&\leq \left\{\frac{1}{\delta^2}T_{n,6}^{*}(x)\right\}^{1/2} \\
&\leq \frac{1}{\delta}\sqrt{A_6(x)}\sqrt{\frac{x(b_n-x)}{b_n}}\left(\frac{b_n}{n}\right)^{3/2}
\end{aligned}$$

for all $n\in\mathbb{N}$. So from (9) we obtain

$$\lim_{n\to\infty}\frac{n}{b_n}\sum_{2}^{*}(n) \leq \lim_{n\to\infty}|f(x)|\left(\frac{n}{x(b_n-x)}\right)^{3/2}\frac{\sqrt{2A_6(x)}}{\delta}\exp\left(-\frac{\delta^2}{8x}\frac{n}{b_n}\right)=0$$

establishing (22) for $i=2$.

Since the case for $i=3,4,5$ and 6 is similar, this finally establishes

$$\lim_{n\to\infty}\frac{n}{b_n}R_{n,1}(x)=0.$$

By using the similar method, we have

$$\lim_{n\to\infty}\frac{n}{b_n}R_{n,2}(x)=0$$

and

$$\lim_{n\to\infty}\frac{n}{b_n}R_{n,3}(x)=0.$$

Thus the proof is complete. □

Observe that if f was bounded on the whole $[0,\infty)$, then $|h(y)|\leq M$ for all y, so that

$$R_{n,1,2}(x) \leq M\frac{n^2}{x^2(b_n-x)^2}\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|\frac{k}{n}b_n-x\right|^6 p_{k,n}\left(\frac{x}{b_n}\right)\to 0 \quad (n\to\infty);$$

$$\begin{aligned}
R_{n,1,2}(x) &\leq M\frac{n^2}{x^2(b_n-x)^2}\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left\{\sqrt{p_{k,n}\left(\frac{x}{b_n}\right)}\left|\frac{kb_n}{n}-x\right|^6\sqrt{p_{k,n}\left(\frac{x}{b_n}\right)}\right\} \\
&\leq M\frac{n^2}{x^2(b_n-x)^2}\left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2}\left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|\frac{kb_n}{n}-x\right|^{12}p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2}.
\end{aligned}$$

Again from Lemma 1 (with $m=12$), we have

$$\left\{\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\left|\frac{kb_n}{n}-x\right|^{12}p_{k,n}\left(\frac{x}{b_n}\right)\right\}^{1/2} \leq \left\{T_{n,12}^{*}(x)\right\}^{1/2} \leq \sqrt{A_{12}(x)}\sqrt{\frac{x(b_n-x)}{b_n}}\left(\frac{b_n}{n}\right)^3$$

for all $n \in \mathbb{N}$. Hence, considering inequality (9),

$$\lim_{n\to\infty} \frac{n}{b_n} R_{n,1,2}(x) \leq \lim_{n\to\infty} M \left(\frac{b_n}{x(b_n - x)} \right)^{3/2} \sqrt{2A_{12}(x)} \exp\left(-\frac{\delta^2}{8x} \frac{n}{b_n} \right) = 0.$$

Then the proof would already be complete shortly after relation (21). This beginning part had as its model the proof of Theorem 3.5 [2].

Corollary 1. *Let f belong to $C[0,\infty)$ and let it be of order $f(x) = O(\exp(x^p))$ on R^+ with some constant $p > 0$. If $b_n = o(n)$ satisfies the condition*

$$b_n = n^{1/(p+1+r)}$$

for any $r > 0$, no matter how small, then the asymptotic formula (15) *holds at each point $x \in R^+$ at which $f^{(iv)}(x)$ exists.*

In particular, the assertion holds for $b_n = n^{1/(p+2)}$.

4. CONCLUSION

In the classical book of Lorentz [5] there is a theorem on the variation detracting (or diminishing) property for the Bernstein operator. As far as we know, this is the first study on this topic. However, the importance of the variation detracting property appears after the paper by Bardaro et al. [2]. They point out that in order to obtain a convergence result in the variation seminorm, it is necessary and important to state the variation detracting property. After this fundamental study the convergence in variation seminorm becomes a new field in the theory of approximation. In addition, it is well known that the variation detracting property is related to the Voronovskaya-type theorems for the derivatives of the operators. The theorem presented in this work is the counterpart of the very recent result due to Butzer and Karsli [3], established for the first derivative of the same operators. In conclusion, the results presented in this work can be used to obtain such kind of convergence results for other approximation operators in variation seminorm.

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Voronovskaja tüüpi teoreem Bernšteini-Chlodovsky polünoomide teise tuletise kohta

Harun Karsli

On tõestatud Voronovskaja tüüpi koonduvusteoreem Bernšteini-Chlodovsky polünoomide teise tuletise ja funktsiooni teise tuletise vahest. Varem on sellised teoreemid tõestatud esimese tuletise ja funktsiooni enda kohta. On näidatud, et esimese tuletise juhuga võrreldes peab teise tuletise juhul Bernšteini-Chlodovsky polünoome defineeriv jada koonduma kiiremini.