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MATHEMATICS

Morita invariants for partially ordered semigroups with local units

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Abstract. We study Morita invariants for strongly Morita equivalent partially ordered semigroups with several types of local units. These include the greatest commutative images, satisfying a given inequality and the fact that strong Morita equivalence preserves various sublattices of the lattice of ideals.

Key words: ordered semigroup, strong Morita equivalence, Morita invariant.

1. INTRODUCTION

Since the 1970s a number of results have appeared on generalizing the Morita theory of rings with identity to monoids [4,10], associative rings [1,3,8] or semigroups [7,15,19]. Recently, there has been a resurgence of interest in the Morita theory of semigroups [11–14,18]. The current work is part of an attempt (initiated in [20]) to generalize some of those latest results from semigroups to partially ordered semigroups, and belongs to a line of research trying to establish generalizations of various results of semigroup theory to partially ordered semigroups (e.g. [6,16,17], an overview in [5]).

A Morita invariant is a property of (partially ordered) semigroups such that whenever a (po)semigroup S has it, any other (po)semigroup that is strongly Morita equivalent to S has it as well. A natural part of the investigation of Morita equivalence is the study of invariants. In the case of semigroups, the pioneering work of Talwar [19] already considered a number of such properties, e.g. being completely 0-simple or bisimple. More recently, Lawson [14], Laan [11], and Laan and Márki [12] have examined more invariants by using both Morita contexts and various new characterizations of Morita equivalence. For example, Lawson [14] implicitly shows that the property of being regular and locally inverse is a Morita invariant. Laan [11] does the same for complete simplicity and Laan and Márki [12] prove it for (a)periodic semigroups. We show that order-related properties are not Morita invariants in general, but some (which include satisfying a given inequality) may be in special cases. There are also classes of posemigroups for which the greatest commutative images are always isomorphic, generalizing the result (e.g. from [10]) that commutative Morita equivalent monoids are isomorphic. Furthermore, under weaker assumptions, the lattices of ideals, downwards and upwards closed ideals, and convex ideals of strongly Morita equivalent posemigroups are correspondingly isomorphic, which corresponds to the ring-theoretic result that the ideal lattices of Morita equivalent rings are isomorphic.

We use the category of partial orders and monotone maps, which will be denoted by Pos. We will also employ categories and functors enriched over Pos, i.e. categories with morphism posets and monotone composition, and monotone functors, respectively. For more details on Pos-categories, Pos-functors, and

Pos-equivalences, the reader is referred to [9], but we remark that a full and faithful Pos-functor must provide a poset isomorphism instead of a bijection between the corresponding morphism posets.

A partially ordered semigroup S (a *posemigroup* for short) is a (nonempty) semigroup that is endowed with a partial order so that its operation is monotone. For a fixed posemigroup S, (one-sided) *S-posets* are partially ordered S-acts where the S-action is monotone in both arguments. A left S-poset X is said to be *unitary* if $SX = X$. The notion for right S-posets is dual. A poset is called an *(S,T)-biposet* if it is a left S- and a right T-poset and its S- and T-actions commute with each other. An (S,T)-biposet is called *unitary* if it is unitary as both a left S- and a right T-poset. *Posemigroup homomorphisms* are monotone semigroup homomorphisms. A number of basic facts about S-posets over pomonoids can be found in [5].

The set of idempotents of a semigroup S will be denoted by $E(S)$. A posemigroup S is said to have *local units* if for any $s \in S$ there exist $e \in E(S)$ and $f \in E(S)$ such that

$$es = s = sf.$$

A posemigroup S is said to have *weak local units* if for any $s \in S$ there exist $e \in S$ and $f \in S$ such that $es = s = sf$. A posemigroup S is said to have *common* (*weak*) *local units* (cf. [12]) if for any $s, s' \in S$ there exist $e \in E(S)$ and $f \in E(S)$ ($e \in S$ and $f \in S$) such that $es = s = sf$ and $es' = s' = s'f$. It is said to have *two-sided* (*common*, *weak*) *local units* (cf. [12]) if for any $s \in S$ ($s, s' \in S$) there exist $e \in E(S)$ ($e \in S$) such that $es = s = se$ (and $es' = s' = s'e$).

Lemma 1.1. *If a posemigroup S has common* (*two-sided*, *weak*) *local units*, *then any finite subset of S also has common* (*two-sided*, *weak*) *local units*.

Proof. We only need to verify this for a three-element subset $\{s_1, s_2, s_3\} \subseteq S$. Take $e \in S$ such that $es_1 = s_1$ and $es_2 = s_2$. Now take $f \in S$ so that $fe = e$ and $fs_3 = s_3$. Then $fs_1 = fes_1 = es_1 = s_1$ and similarly $fs_2 = s_2$, as required. □

A posemigroup is called *factorizable* if $S^2 = S$. Having local units implies having weak local units, which in turn implies factorizability. Also, a posemigroup with (weak) common local units has (weak) local units. For any $e, f \in E(S)$, we say that the subposemigroup $eSf \subseteq S$ (with the order inherited from S) is a *generalized local subpomonoid* of S.

If S is a posemigroup and ρ is a reflexive relation on S, one can define a preorder $\leq_\rho$ on S by setting $s \leq_\rho t$ if there exist $n \in \mathbb{N}$, $s_i, t_i \in S$, $1 \leq i \leq n$ such that

$$s \leq s_1 \,\rho\, t_1 \leq s_2 \,\rho\, t_2 \leq \ldots \leq s_n \,\rho\, t_n \leq t.$$

A *posemigroup congruence* on S is a semigroup congruence θ on S such that the *closed chains condition* holds:

$$\text{if } s \leq_\theta t \leq_\theta s, \text{ then } s\theta t.$$

The posemigroup congruence generated by a relation $H \subseteq S \times S$ (denoted by $\theta(H)$) is the smallest posemigroup congruence on S that contains H. For a more detailed description, we refer the reader to [5], but we remark that if H is a semigroup congruence, then $\theta(H) = \leq_H \cap \geq_H$.

The *tensor product* $A \otimes_S B$ of a right S-poset A and a left S-poset B is the quotient poset $(A \times B)/\sim$ by the least poset congruence $\sim$ for which $(as, b) \sim (a, sb)$ for all $a \in A$, $b \in B$, $s \in S$.

If A is a (T,S)-biposet, then $A \otimes_S B$ is a left T-poset, where the action is defined by $t(a \otimes b) = (ta) \otimes b$. Similarly, if B is an (S,T)-biposet, then $A \otimes_S B$ is a right T-poset.

2. STRONG MORITA EQUIVALENCE

If S and T are posemigroups, we say that a 6-tuple

$$(S, T, P, Q, \langle -,- \rangle, [-,-])$$

is a *Morita context* if the following conditions hold:
(M1) P is an (S,T)-biposet and Q is a (T,S)-biposet;
(M2) $\langle -,- \rangle : P \otimes_T Q \to S$ is an (S,S)-biposet morphism and
$[-,-] : Q \otimes_S P \to T$ is a (T,T)-biposet morphism;
(M3) the following two conditions hold for all $p,p' \in P$ and $q,q' \in Q$:
(i) $\langle p,q \rangle p' = p[q,p']$,
(ii) $q\langle p,q' \rangle = [q,p]q'$.

A Morita context is called *unitary* if the biposets P and Q are unitary. We say that two posemigroups S and T are *strongly Morita equivalent* (a notion introduced for unordered semigroups by Talwar [19]) if there exists a unitary Morita context $(S,T,P,Q,\langle -,- \rangle,[-,-])$ such that the mappings $\langle -,- \rangle$ and $[-,-]$ are surjective.

For a small Pos-category $\mathscr{C}$, we will denote by $\mathscr{C}_0$ its set of objects and by $\mathscr{C}(B,A)$ its poset of morphisms from object A to object B.

If S is a posemigroup, then the *Cauchy completion* of S (cf. [14]) is the Pos-category $C(S)$ that has $C(S)_0 = E(S)$, morphism posets

$$C(S)(f,e) = \{(e,s,f) \,|\, s \in S, esf = s\},$$

with the order $(e,s,f) \leq (e,s',f)$ iff $s \leq s'$ in S and the composition rule $(e,s,f) \circ (f,s',g) = (e,ss',g)$.

From Theorem 2.1 of [20], we recall the following description of strong Morita equivalence.

Theorem 2.1. *Let S and T be posemigroups with local units. Then the following conditions are equivalent*:
(1) *S and T are strongly Morita equivalent*;
(2) *the categories $C(S)$ and $C(T)$ are* Pos-*equivalent.*

Also, condition (2) provides an important fact on the local structure of strongly Morita equivalent posemigroups.

Corollary 2.1. *If S and T are strongly Morita equivalent posemigroups with local units, then each generalized local subpomonoid S is isomorphic to a generalized local subpomonoid of T.*

3. INVARIANTS OF STRONGLY MORITA EQUIVALENT SEMIGROUPS

In this section we investigate which properties of posemigroups remain invariant under strong Morita equivalence.

First, we use Theorem 2.1 to derive the following result.

Proposition 3.1. *Let S be a posemigroup. Then S is strongly Morita equivalent to a one-element posemigroup if and only if S is a rectangular poband.*

Proof. Let $S = U \times V$ be a rectangular poband, E a one-element posemigroup, and take $e = (u_1,v_1), f = (u_2,v_2) \in E(S) = S$. Then $C(S)(e,f) = \{(f,(u_2,v_1),e)\}$ and $C(S)$ is a groupoid. It is therefore easy to see that any mapping $C(E) \to C(S)$ is a Pos-equivalence.

Conversely, if a posemigroup S is strongly Morita equivalent to a one-element posemigroup E, then $C(E)$ and $C(S)$ are Pos-equivalent by Theorem 2.1, implying that $C(E)$ and $C(S)$ are equivalent categories. Using the unitarity of the Morita context, it is not difficult to show directly that S is a poband and therefore has local units. Then by Theorem 1.1 of [14], S and E are Morita equivalent as semigroups, so the semigroup S must be a rectangular band by Theorem 16 of [11]. □

Due to the above, we can immediately conclude that

Corollary 3.1. *The invariants of strong Morita equivalence do not include any purely order-related property that does not model the entire* Pos. *Moreover, the congruence lattices of strongly Morita equivalent posemigroups are not necessarily isomorphic.*

Proof. The one-element posemigroup either satisfies an order-related property or not. By our assumption, there exists a poset that does not satisfy (or does satisfy, as the case may be) the same property. We turn that poset into a left zero posemigroup, which is a rectangular band and therefore strongly Morita equivalent to the one-element posemigroup by Proposition 3.1.

Similarly, the congruence lattice of the one-element posemigroup is singleton, while nontrivial rectangular bands have at least two congruences. □

Still, if we assume sufficiently "good" local units, we can prove that some properties are Morita invariants for such classes of posemigroups. The following use of common local units to establish the existence of Morita invariants is a very useful technique, and will be used to prove Proposition 3.3 as well.

Proposition 3.2. *Let S and T be strongly Morita equivalent posemigroups with common local units. If the order on S is either total, discrete, directed or a semiorder, then the order on T is also total, discrete, directed or a semiorder.*

Proof. We will prove only the claim about chains, since the others can be proved in the same way. Let S be a chain and take $t,t' \in T$. As T has common local units, there exist $i,j \in E(T)$ such that $t = it = tj$ and $t' = it' = t'j$, so $t,t' \in iTj$. By Corollary 2.1, $iTj \cong eSf$ for some $e,f \in E(S)$. But since S is a chain, eSf and therefore iTj are also chains. So either $t \leq t'$ or $t' \leq t$, as required. □

Theorem 3.1 (cf. Th. 5 of [12]). *Let S and T be strongly Morita equivalent posemigroups with common two-sided weak local units. If S satisfies an inequality, then T also satisfies the same inequality.*

Proof. The proof is a minor modification of Theorem 5 of [12]. Let there be two terms $w(x_1,\ldots,x_n) = x_{i_1}\ldots x_{i_k}$ and $w'(x_1,\ldots,x_n) = x_{j_1}\ldots x_{j_l}$. Suppose that S satisfies the inequality $w \leq w'$. Take any $t_1,\ldots,t_n \in T$ and fix $e = [q,p] \in T$ such that $t_i = et_i = t_ie$ for all $i = 1,\ldots,n$.

Then for all $m_1,\ldots,m_a \in \{1,\ldots,n\}$ we have

$$\begin{aligned} t_{m_1}\ldots t_{m_a} &= et_{m_1}e\ldots et_{m_{a-1}}et_{m_a}e = [q,p][t_{m_1}q,p]\ldots[t_{m_{a-1}}q,p][t_{m_a}q,p] \\ &= [q,\langle p,t_{m_1}q\rangle\ldots\langle p,t_{m_{a-1}}q\rangle\langle p,t_{m_a}q\rangle p]. \end{aligned}$$

So

$$\begin{aligned} w(t_1,\ldots,t_n) &= [q,\langle p,t_{i_1}q\rangle\ldots\langle p,t_{i_{k-1}}q\rangle\langle p,t_{i_k}q\rangle p] \\ &= [q,w(\langle p,t_1q\rangle,\ldots,\langle p,t_nq\rangle)p] \\ &\leq [q,w'(\langle p,t_1q\rangle,\ldots,\langle p,t_nq\rangle)p] \\ &= [q,\langle p,t_{j_1}q\rangle\ldots\langle p,t_{j_{l-1}}q\rangle\langle p,t_{j_l}q\rangle p] = w'(t_1,\ldots,t_n). \end{aligned}$$ □

There is also an alternative proof under slightly different assumptions.

Proposition 3.3. *Let S and T be strongly Morita equivalent posemigroups with common local units. If S satisfies an inequality, then T also satisfies the same inequality.*

Proof. Suppose that S and T are strongly Morita equivalent posemigroups with common local units. Moreover, let S satisfy an inequality $w \leq w'$, where $w(x_1,\ldots,x_n) = x_{i_1}\ldots x_{i_k}$ and $w'(x_1,\ldots,x_n) = x_{j_1}\ldots x_{j_l}$. By Corollary 2.1, T satisfies $w \leq w'$ locally. Take $t_1,\ldots,t_n \in T$ and $e,f \in E(T)$ such that $t_i = et_if$ for all $i = 1,\ldots,n$. Then $t_i \in eTf$ and thus

$$w(t_1,\ldots,t_n) \leq w'(t_1,\ldots,t_n).$$ □

Remark 3.1. Since an identity is equivalent to two inequalities, the above results hold for identities as well.

Corollary 3.2. *Commutativity and being a band or a semilattice are invariant properties of Morita equivalent posemigroups with common* (*two-sided weak*) *local units*.

Actually, since idempotence involves an identity with only one variable, we can easily derive the following version of Theorem 3.1.

Proposition 3.4. *Let S and T be strongly Morita equivalent posemigroups with* (*two-sided weak*) *local units. If S is a band*, *then so is T*.

A more in-depth result on commutativity is as follows.

Theorem 3.2 (cf. Th. 4 of [12])**.** *Let S and T be strongly Morita equivalent posemigroups with common two-sided weak local units. Then their greatest commutative images are isomorphic posemigroups.*

Proof. For any posemigroup S, let α_S^1 be the binary relation

$$\alpha_S^1 = \{(cabd, cbad) \,|\, a,b \in S, c,d \in S^1\} \subseteq S \times S$$

and let $\alpha_S = \leq_{\alpha_S^1} \cap \geq_{\alpha_S^1}$. Then α_S is a posemigroup congruence and S/α_S is the greatest commutative image of S. Let $\pi_S : S \to S/\alpha_S$ be the canonical projection. Now, suppose that S and T are strongly Morita equivalent posemigroups with common two-sided weak local units. Define two mappings $f : S/\alpha_S \to T/\alpha_T$ and $g : T/\alpha_T \to S/\alpha_S$ by

$$\begin{aligned} f(\pi_S(s)) &= \pi_T([q,sp]), \text{ where } s = us = su \text{ and } u = \langle p,q\rangle, \\ g(\pi_T(t)) &= \pi_S(\langle p,tq\rangle), \text{ where } t = vt = tv \text{ and } v = [q,p]. \end{aligned}$$

We will show that f is a posemigroup homomorphism. Then g will also be a homomorphism by symmetry. First, we check that the choice of u does not influence the definition of f. For this suppose that $s = us = su = u's = su'$ for $u = \langle p,q\rangle$ and $u' = \langle p',q'\rangle$. Then $su = su' = us$ yields $\langle sp,q\rangle = \langle sp',q'\rangle = \langle p,qs\rangle$. Therefore

$$\begin{aligned} [q,sp] &= [q,su'p] = [q,\langle sp',q'\rangle p] = [q,sp'][q',p] \\ &\alpha_T \; [q',p][q,sp'] = [q',\langle p,q\rangle sp'] = [q',usp'] = [q',sp'] \end{aligned}$$

and thus $\pi_T([q,sp]) = \pi_T([q',sp'])$, as required.

Moreover, take $(s,s') = (cabd,cbad) \in \alpha_S^1$. We only consider the case where $c,d \in S$, the others can be proved in a similar way. So let $u = \langle p,q\rangle \in S$ be such that $a = ua = au$, $b = ub = bu$, $c = uc = cu$, and $d = ud = du$. Then

$$\begin{aligned} f(\pi_S(s)) &= \pi_T([q,cabdp]) = \pi_T([q,cuaubudp]) \\ &= \pi_T([q,c\langle p,q\rangle\langle ap,q\rangle\langle bp,q\rangle dp]) \\ &= \pi_T([q,cp][qa,p][qb,p][qd,p]) \\ &= \pi_T([q,cp][qb,p][qa,p][qd,p]) \\ &= \pi_T([q,c\langle p,q\rangle\langle bp,q\rangle\langle ap,q\rangle dp]) \\ &= \pi_T([q,cubuaudp]) = \pi_T([q,cbadp]) = f(\pi_S(s')). \end{aligned}$$

Additionally, if $s \leq s'$, we can again take $u = \langle p,q\rangle \in S$ such that $s = su = us$ and $s' = s'u = us'$ and get $f(\pi_S(s)) = \pi_T([q,sp]) \leq \pi_T([q,s'p]) = f(\pi_S(s'))$.

To verify that f is monotone, we need to show that if $s \leq_{\alpha_S^1} s'$, then $f(\pi_S(s)) \leq f(\pi_S(s'))$. Let $n \in \mathbb{N}$, $s_i,t_i \in S$, $1 \leq i \leq n$ be such that $s \leq s_1 \,\alpha_S^1\, t_1 \leq s_2 \,\alpha_S^1\, t_2 \leq \ldots \leq s_n \,\alpha_S^1\, t_n \leq s'$. By the above,

$$\begin{aligned} f(\pi_S(s)) &\leq f(\pi_S(s_1)) = f(\pi_S(t_1)) \leq f(\pi_S(s_2)) = f(\pi_S(t_2)) \\ &\leq \ldots \leq f(\pi_S(s_n)) = f(\pi_S(t_n)) \leq f(\pi_S(s')). \end{aligned}$$

So f is monotone and consequently well-defined.

To see that f is a homomorphism, take $s, s' \in S$ and let $u = \langle p, q\rangle \in S$ be such that $s = su = us$ and $s' = s'u = us'$. Then $ss' = uss' = ss'u$ and

$$\begin{aligned} f(\pi_S(ss')) &= \pi_T([q, ss'p]) = \pi_T([q, sus'p]) = \pi_T([q, s\langle p, q\rangle s'p]) \\ &= \pi_T([q, sp[q, s'p]]) = \pi_T([q, sp][q, s'p]) \\ &= \pi_T([q, sp])\pi_T([q, s'p]) = f(\pi_S(s))f(\pi_S(s')). \end{aligned}$$

Finally, take $s \in S$ and $u = \langle p, q\rangle \in S$ such that $s = su = us$. Then

$$[q, sp] = [q, sup] = [q, s\langle p, q\rangle p] = [q, sp][q, p],$$

similarly $[q, sp] = [q, p][q, sp]$ and therefore

$$\begin{aligned} (gf)(\pi_S(s)) &= g(\pi_T([q, sp])) = \pi_S(\langle p, [q, sp]q\rangle) \\ &= \pi_S(\langle p, q\rangle\langle sp, q\rangle) = \pi_S(usu) = \pi_S(s). \end{aligned}$$

So $gf = 1_{S/\alpha_S}$ and similarly $fg = 1_{T/\alpha_T}$. □

Corollary 3.3 (cf. Cor. 1 of [12], Cor. 6.2 of [10])**.** *Let S and T be two strongly Morita equivalent commutative semigroups with common two-sided weak local units. Then S and T are isomorphic.*

4. IDEAL LATTICES

We now proceed to show that the lattices of various kinds of ideals of strongly Morita equivalent posemigroups are isomorphic. This allows us to make several further observations on strong Morita equivalence. Note that we allow empty ideals so that the posets of ideals are proper lattices.

By $\Downarrow I$, $\Uparrow I$, and $\mathrm{Conv}(I)$ we will denote the down-set, up-set, and convex poset generated by a fixed subposet $I \subseteq S$, i.e.,

$$\begin{gathered} \Downarrow I = \{x \in S \mid \exists y \in I \text{ such that } x \le y\}, \\ \Uparrow I = \{x \in S \mid \exists y \in I \text{ such that } y \le x\}, \\ \mathrm{Conv}(I) = \{x \in S \mid \exists y, z \in I \text{ such that } y \le x \le z\}. \end{gathered}$$

We will use the notation $\mathrm{Id}(S)$, $\mathrm{DId}(S)$, $\mathrm{UId}(S)$, and $\mathrm{CId}(S)$ for the lattices of (two-sided) ideals, downwards closed ideals, upwards closed ideals, and convex ideals of S. The lattice operations on the ideal posets with respect to inclusion are the usual intersection and union operations, except for convex ideals where the join of two convex ideals is the convex ideal generated by their union.

Assume that there is a Morita context $(S, T, P, Q, \langle -, -\rangle, [-, -])$ with posemigroups S and T. We can define the following mappings

$$\begin{gathered} \Phi : \mathrm{Id}(S) \to \mathrm{Id}(T), \quad \Theta : \mathrm{Id}(T) \to \mathrm{Id}(S), \\ \Phi_\downarrow : \mathrm{DId}(S) \to \mathrm{DId}(T), \quad \Phi_\uparrow : \mathrm{UId}(S) \to \mathrm{UId}(T), \quad \Phi_\updownarrow : \mathrm{CId}(S) \to \mathrm{CId}(T), \\ \Theta_\downarrow : \mathrm{DId}(T) \to \mathrm{DId}(S), \quad \Theta_\uparrow : \mathrm{UId}(T) \to \mathrm{UId}(S), \quad \Theta_\updownarrow : \mathrm{CId}(T) \to \mathrm{CId}(S) \end{gathered}$$

by

$$\begin{gathered} \Phi(I) = [QI, P] = \{[qi, p] \mid p \in P, q \in Q, i \in I\}, \\ \Theta(J) = \langle PJ, Q\rangle = \{\langle pj, q\rangle \mid p \in P, q \in Q, j \in J\}, \\ \Phi_\downarrow(I) = \Downarrow[QI, P], \quad \Phi_\uparrow(I) = \Uparrow[QI, P], \quad \Phi_\updownarrow(I) = \mathrm{Conv}[QI, P], \\ \Theta_\downarrow(I) = \Downarrow\langle PJ, Q\rangle, \quad \Theta_\uparrow(I) = \Uparrow\langle PJ, Q\rangle, \quad \Theta_\updownarrow(I) = \mathrm{Conv}\langle PJ, Q\rangle. \end{gathered}$$

Lemma 4.1. *Let* $(S,T,P,Q,\langle -,-\rangle,[-,-])$ *be a Morita context with surjective mappings. Then*

$$\begin{aligned}
&\mathrm{Conv}\langle P\mathrm{Conv}X,Q\rangle = \mathrm{Conv}\langle PX,Q\rangle,\\
&\downarrow\langle P\downarrow X,Q\rangle = \downarrow\langle PX,Q\rangle,\ \uparrow\langle P\uparrow X,Q\rangle = \uparrow\langle PX,Q\rangle,\\
&\mathrm{Conv}[Q\mathrm{Conv}Y,P] = \mathrm{Conv}[QY,P],\\
&\downarrow[Q\downarrow Y,P] = \downarrow[QY,P],\ \text{and}\ \uparrow[Q\uparrow Y,P] = \uparrow[QY,P]
\end{aligned}$$

for any subsets $X \subseteq T$, $Y \subseteq S$.

Proof. We will only prove the first equality. Take $x \in \mathrm{Conv}\langle P\mathrm{Conv}X,Q\rangle$, i.e. $\langle py,q\rangle \le x \le \langle p'z,q'\rangle$ and $y_1 \le y \le y_2$, $z_1 \le z \le z_2$ for some $p,p' \in P$, $q,q' \in Q$, $y,z \in T$, $y_1,y_2,z_1,z_2 \in X$. Then $\langle py_1,q\rangle \le x \le \langle p'z_2,q'\rangle$, i.e. $x \in \mathrm{Conv}\langle PX,Q\rangle$. The converse is obvious since $X \subseteq \mathrm{Conv}X$. □

The following theorem is a direct generalization of Theorem 3 of [12], which in turn ultimately arises from a similar ring-theoretic result (e.g. Proposition 21.11 of [2]).

Theorem 4.1. *Let S and T be strongly Morita equivalent posemigroups with weak local units. Then*

(1) $\Phi : \mathrm{Id}(S) \to \mathrm{Id}(T)$ *is a lattice isomorphism that takes finitely generated ideals to finitely generated ideals and principal ideals to principal ideals*;

(2) $\Phi_\downarrow : \mathrm{DId}(S) \to \mathrm{DId}(T)$ *is a lattice isomorphism that takes down-sets of finitely generated ideals to down-sets of finitely generated ideals and down-sets of principal ideals to down-sets of principal ideals*;

(3) $\Phi_\uparrow : \mathrm{UId}(S) \to \mathrm{UId}(T)$ *is a lattice isomorphism that takes up-sets of finitely generated ideals to up-sets of finitely generated ideals and up-sets of principal ideals to up-sets of principal ideals*;

(4) $\Phi_\updownarrow : \mathrm{CId}(S) \to \mathrm{CId}(T)$ *is a lattice isomorphism that takes convex subsets generated by finitely generated ideals to convex subsets generated by finitely generated ideals and convex subsets generated by principal ideals to convex subsets generated by principal ideals.*

Proof. (1) This can be proved in exactly the same way as Theorem 3 of [12]. The proofs of (2)–(4) are in fact extensions of that proof.

(2) First, $\mathrm{DId}(S)$ and $\mathrm{DId}(T)$ are indeed sublattices of $\mathrm{Id}(S)$ and $\mathrm{Id}(T)$, respectively, since intersections and unions of down-sets are again down-sets. Second, the mappings Φ and Θ clearly preserve subset inclusion, i.e. are monotone. Moreover, take $I \in \mathrm{DId}(S)$, $x \in \Phi_\downarrow(I)$, and $t,t' \in T$. Because $x \le [qi,p]$ for some $p \in P$, $q \in Q$, $i \in I$, we also have $txt' \le t[qi,p]t' = [(tq)i,pt']$, so $txt' \in \downarrow[QI,P] = \Phi_\downarrow(I)$. Therefore $\Phi_\downarrow(I) \in \mathrm{DId}(T)$. By Lemma 4.1, we get for any $I \in \mathrm{DId}(S)$ that

$$\begin{aligned}
\Theta_\downarrow(\Phi_\downarrow(I)) &= \downarrow\langle P\downarrow[QI,P],Q\rangle = \downarrow\langle P[QI,P],Q\rangle\\
&= \downarrow\langle\langle P,QI\rangle P,Q\rangle = \downarrow(\langle P,Q\rangle I\langle P,Q\rangle) = \downarrow(SIS) = \downarrow I = I.
\end{aligned}$$

Symmetrically $\Phi_\downarrow(\Theta_\downarrow(J)) = J$ for all $J \in \mathrm{DId}(T)$ and thus $\Theta_\downarrow$ is also a lattice isomorphism.

Fix a finitely generated ideal $I = \bigcup_{i=1}^n Ss_iS \in \mathrm{Id}(S)$. Then $\downarrow I \in \mathrm{DId}(S)$, so we can consider $\Phi_\downarrow(\downarrow I)$. Take any $s \in I$. Since $\langle -,-\rangle$ is surjective, there exist $i \in \{1,\dots,n\}$ and $p,p' \in P$, $q,q' \in Q$ such that $s = \langle p,q\rangle s_i\langle p',q'\rangle$. Because S has weak local units, we can put $s_i = \langle p_i,q_i\rangle s_i\langle p_i',q_i'\rangle$ for some $p_i,p_i' \in P$, $q_i,q_i' \in Q$. So $s = \langle p,q\rangle\langle p_i,q_i\rangle s_i\langle p_i',q_i'\rangle\langle p',q'\rangle$. Thus we get for any $p'' \in P$, $q'' \in Q$ and $x \le [q''y,p'']$, $y \in \downarrow s$ that

$$x \le [q''s,p''] = [q'',p][q,p_i][q_is_i,p_i'][q_i',p'][q',p''],$$

so

$$x \in \downarrow(T[q_is_i,p_i']T),\ \text{i.e.}\ \downarrow[Q\downarrow I,P] \subseteq \bigcup_{j=1}^n \downarrow(T[q_js_j,p_j']T).$$

And if $y \le t[q_j s_j, p'_j]t'$, then $y \le [(tq_j)s_j, p'_j t'] \in [Q \Downarrow I, P]$, so $y \in \Downarrow [Q \Downarrow I, P] = \Phi(\Downarrow I)$, whence

$$\Phi(\Downarrow \bigcup_{i=1}^{n} Ss_iS) = \bigcup_{j=1}^{n} \Downarrow (T[q_js_j, p'_j]T) = \Downarrow \bigcup_{j=1}^{n} (T[q_js_j, p'_j]T).$$

(3) Dual to (2).

(4) $\mathrm{CId}(S)$ and $\mathrm{CId}(T)$ are lattices, but not necessarily sublattices of $\mathrm{Id}(S)$ and $\mathrm{Id}(T)$. The mappings $\Theta_\updownarrow$ and $\Phi_\updownarrow$ are again monotone and yield two-sided ideals. Once more, $\Theta_\updownarrow$ is a poset isomorphism, because Lemma 4.1 implies for any $I \in \mathrm{CId}(S)$ that

$$\begin{aligned} \Theta_\updownarrow(\Phi_\updownarrow(I)) &= \mathrm{Conv}\langle P\mathrm{Conv}[QI,P], Q\rangle = \mathrm{Conv}\langle P[QI,P], Q\rangle \\ &= \mathrm{Conv}\langle \langle P, QI\rangle P, Q\rangle = \mathrm{Conv}(\langle P,Q\rangle I \langle P,Q\rangle) \\ &= \mathrm{Conv}(SIS) = \mathrm{Conv}(I) = I \end{aligned}$$

and likewise $\Phi_\updownarrow(\Theta_\updownarrow(J)) = J$ for all $J \in \mathrm{CId}(T)$.

Take $I = \bigcup_{i=1}^{n} Ss_iS \in \mathrm{Id}(S)$. Clearly $\mathrm{Conv}(I) \in \mathrm{CId}(S)$, so we can consider $\Phi_\updownarrow(\mathrm{Conv}(I))$. Fix any $x \in \Phi_\updownarrow(\mathrm{Conv}(I))$. Then $[qs,p] \le x \le [q's',p']$ for some $s,s' \in \mathrm{Conv}(I)$. We can again write $s \ge \langle p_1,q_1\rangle\langle p_{i_1},q_{i_1}\rangle s_{i_1}\langle p'_{i_1},q'_{i_1}\rangle\langle p'_1,q'_1\rangle$ and $s' \le \langle p_2,q_2\rangle\langle p_{i_2},q_{i_2}\rangle s_{i_2}\langle p'_{i_2},q'_{i_2}\rangle\langle p'_2,q'_2\rangle$. Thus

$$[q,p_1][q_1,p_{i_1}][q_{i_1}s_{i_1},p'_{i_1}][q'_{i_1},p'_1][q'_1,p] \le x,$$
$$x \le [q',p_2][q_2,p_{i_2}][q_{i_2}s_{i_2},p'_{i_2}][q'_{i_2},p'_2][q'_2,p'],$$

implying that

$$x \in \mathrm{Conv}\left(T[q_{i_1}s_{i_1},p'_{i_1}]T \cup T[q_{i_2}s_{i_2},p'_{i_2}]T\right) \subseteq \mathrm{Conv}\bigcup_{j=1}^{n} T[q_js_j,p'_j]T.$$

Finally, if $t_1[q_{j_1}s_{j_1},p'_{j_1}]t'_1 \le y \le t_2[q_{j_2}s_{j_2},p'_{j_2}]t'_2$, then

$$[(t_1q_{j_1})s_{j_1},p'_{j_1}t'] \le y \le [(t_2q_{j_2})s_{j_2},p'_{j_2}t'],$$

i.e. $y \in \mathrm{Conv}[QI,P] \subseteq \mathrm{Conv}[Q\mathrm{Conv}(I),P] = \Phi_\updownarrow(\mathrm{Conv}(I))$, whence

$$\Phi_\updownarrow\left(\mathrm{Conv}\left(\bigcup_{i=1}^{n} Ss_iS\right)\right) = \mathrm{Conv}\bigcup_{j=1}^{n}(T[q_js_j,p'_j]T). \qquad \square$$

Every commutative band (semilattice) has a compatible *natural* order ω defined by $a\omega b \Leftrightarrow ab = a$. Yet, there can be other compatible partial orders on that semilattice as well. We call a commutative band with any kind of partial order compatible with its multiplication a *posemilattice*.

Proposition 4.1. *Two posemilattices that have common weak local units are strongly Morita equivalent if and only if they are isomorphic.*

Proof. Sufficiency is obvious. Let

$$(S,T,P,Q,\langle -,-\rangle,[-,-])$$

be a Morita context. Define an additional order $\preceq$ on $\mathrm{PrId}(S)$, the poset of principal ideals of S, by $SsS \preceq StS$ iff $s \le t$. Since S is a semilattice, every principal ideal SsS has a unique generator $s \in S$, so this order is well defined. We consider the posemilattices $(S,\cdot,\le)$ and $(\mathrm{PrId},\cap,\preceq)$. First, $S \cong \mathrm{PrId}(S)$ as semigroups. To see this, we observe that $SsS \cap StS = SstS$ for all $s,t \in S$. Obviously $SsS \cap StS \supseteq SstS$. For the converse, take $z = s_1ss_2 = t_1tt_2 \in SsS \cap StS$ for some $s_1,s_2,t_1,t_2 \in S$. Then $z = z^2 = (s_1ss_2)(t_1tt_2) = s_1(st)(s_2t_1t_2) \in SstS$, because S is a commutative band. From our definition it is clear that now $(S,\cdot,\le) \cong (\mathrm{PrId},\cap,\preceq)$ as posemigroups. The lattice isomorphism Φ from Theorem 4.1 is a posemigroup isomorphism between the posets of principal ideals. To conclude the proof, it now suffices to show that Φ preserves and reflects the above order $\preceq$ on $\mathrm{PrId}(S)$ and $\mathrm{PrId}(T)$. We do this via generators, so take $s, s' \in S$ and fix $e, f \in S$ such that $es = es'$ and $sf = s'f$. Since we have surjective $[-,-]$, there exist $p, p' \in P$ and $q, q' \in Q$ such that $s = \langle p,q\rangle s\langle p',q'\rangle$ and $s' = \langle p,q\rangle s'\langle p',q'\rangle$. If $s \leq s'$, then clearly $[qs,p'] \leq [qs',p']$. Conversely, if $[qs,p'] \leq [qs',p']$, then

$$s = \langle p,q\rangle s\langle p',q'\rangle = \langle p,[qs,p']q'\rangle \leq \langle p,[qs',p']q'\rangle = \langle p,q\rangle s'\langle p',q'\rangle = s'. \qquad \square$$

A polattice or directed posemilattice means a (lower) posemilattice where the natural order ω is a lattice order or (upwards) directed order.

The preceding result can thus be reformulated as

Corollary 4.1. *Two directed posemilattices are strongly Morita equivalent if and only if they are isomorphic.*

Corollary 4.2. *Two polattices are strongly Morita equivalent if and only if they are isomorphic.*

The next claim holds for unordered semigroups (see Proposition 15 of [11]), but is an open question for posemigroups.

Conjecture 4.1. Two posemilattices with weak local units are strongly Morita equivalent if and only if they are isomorphic.

5. CONCLUSION

We established that order-related properties are not Morita invariants for general posemigroups, but at least a few simpler ones are for those posemigroups that have common local units (or common two-sided weak local units). Those invariants are being a chain, antichain, directed poset, semiorder and satisfying a given inequality. We also proved that some Morita invariants of semigroups are also Morita invariants of posemigroups, namely the greatest commutative images and the lattice of ideals. The latter result was expanded to include lattices of various order-related ideals. One point of interest for further work is to prove or refute Conjecture 4.1, which could possibly lead to better techniques for handling order-related properties via Morita contexts. A more general direction is to see what other kinds of order-theoretic properties (some immediate candidates include lattices or relatively complemented lattices) are Morita invariants under some (possibly stronger) conditions.

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Lokaalsete ühikutega osaliselt järjestatud poolrühmade Morita invariandid

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On uuritud, millised omadused on erinevate lokaalsete ühikutega osaliselt järjestatud poolrühmade jaoks Morita invariandid, st kanduvad üle antud osaliselt järjestatud poolrühmaga tugevalt Morita ekvivalentsetele osaliselt järjestatud poolrühmadele. Osutub, et puhtalt järjestusega seotud omadused ei ole üldiselt invariandid, aga kui eeldada ühiste lokaalsete ühikute olemasolu, siis mõned lihtsamad omadused (ahelaks olek, suunatus) siiski säilivad. On näidatud, et teatud liiki lokaalsete ühikute olemasolu korral on tugevalt Morita ekvivalentsete osaliselt järjestatud poolrühmade suurimad kommutatiivsed kujutised isomorfsed, need rahuldavad samu võrdusi ja võrratusi ning nende (alla- või ülespoole kinniste, kumerate) ideaalide võred on isomorfsed. On järeldatud, et vastavate ühikute olemasolul on tugevalt Morita ekvivalentsed kommutatiivsed osaliselt järjestatud poolrühmad või osaliselt järjestatud poolvõred isomorfsed.