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MATHEMATICS

Some characterizations of null osculating curves in the Minkowski space-time

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Abstract. In this paper we give the necessary and sufficient conditions for null curves in $\mathbb{E}_1^4$ to be osculating curves in terms of their curvature functions. In particular, we obtain some relations between null normal curves and null osculating curves as well as between null rectifying curves and null osculating curves. Finally, we give some examples of the null osculating curves in $\mathbb{E}_1^4$.

Key words: Minkowski space-time, null curve, osculating curve, curvature.

1. INTRODUCTION

In the Euclidean space $\mathbb{E}^3$ there exist three classes of curves, so-called *rectifying*, *normal*, and *osculating curves*, which satisfy *Cesàro's fixed point condition* [3]. Namely, rectifying, normal, and osculating planes of such curves always contain a particular point. It is well known that if all the normal or osculating planes of a curve in $\mathbb{E}^3$ pass through a particular point, then the curve lies in a sphere or is a planar curve, respectively. It is also known that if all rectifying planes of the curve in $\mathbb{E}^3$ pass through a particular point, then the ratio of torsion and curvature of the curve is a non-constant linear function [2]. Rectifying curves in the Minkowski space-time $\mathbb{E}_1^4$ are defined and studied in [4].

Normal curves in the Minkowski space-time are defined in [5] as the space curves whose position vector (with respect to some chosen origin) always lies in its normal space $T^{\perp}$, which represents the orthogonal complement of the tangent vector field of the curve.

Analogously, osculating curves in the Minkowski space-time are defined in [6] as the space curves whose position vector (with respect to some chosen origin) always lies in its osculating space, which represents the orthogonal complement of the first binormal or second binormal vector field of the curve. Timelike osculating curves as well as spacelike osculating curves, whose Frénet frame contains only non-null vector fields, are characterized in [6].

In this paper we give the necessary and sufficient conditions for null curves in $\mathbb{E}_1^4$ to be osculating curves in terms of their curvature functions. In particular, we obtain some relations between null normal curves and null osculating curves as well as between null rectifying curves and null osculating curves. Finally, we give some examples of the null osculating curves in $\mathbb{E}_1^4$.

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2. PRELIMINARIES

The Minkowski space-time $\mathbb{E}_1^4$ is the Euclidean 4-space $\mathbb{E}^4$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}_1^4$. Recall that a vector $v \in \mathbb{E}_1^4$ can be *spacelike* if $g(v,v) > 0$ or $v = 0$, *timelike* if $g(v,v) < 0$, and *null* (*lightlike*) if $g(v,v) = 0$ and $v \neq 0$. The norm of a vector v is given by $||v|| = \sqrt{|g(v,v)|}$, and two vectors v and w are said to be orthogonal if $g(v,w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^4$ can locally be *spacelike*, *timelike*, or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike, or null [7]. A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$ [1].

Let $\{T, N, B_1, B_2\}$ be the moving Frénet frame along a curve α in $\mathbb{E}_1^4$, consisting of the tangent, principal normal, first binormal vector field, and second binormal vector field, respectively. If α is a null Cartan curve, the Frénet equations are given by [1,8]

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_2 & 0 & -\kappa_1 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ -\kappa_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{1}$$

where the first curvature $\kappa_1(s) = 0$ if α is a straight line, in all other cases $\kappa_1(s) = 1$. Therefore, such curve has two curvatures $\kappa_2(s)$ and $\kappa_3(s)$ and the following conditions hold

$$g(T,T) = g(B_1,B_1) = 0, \quad g(N,N) = g(B_2,B_2) = 1,$$

$$g(T,N) = g(T,B_2) = g(N,B_1) = g(N,B_2) = g(B_1,B_2) = 0, \quad g(T,B_1) = 1.$$

Recall that arbitrary curve α in $\mathbb{E}_1^4$ is called *osculating curve of the first or second kind* if its position vector (with respect to some chosen origin) always lies in the orthogonal complement $B_2^\perp$ or $B_1^\perp$, respectively [6].

Consequently, the position vector of the null osculating curve of the first and second kind satisfies respectively the equations

$$\alpha(s) = \lambda(s)T(s) + \mu(s)N(s) + \nu(s)B_1(s), \tag{2}$$

$$\alpha(s) = \lambda(s)N(s) + \mu(s)B_1(s) + \nu(s)B_2(s), \tag{3}$$

for some differentiable functions $\lambda(s)$, $\mu(s)$, and $\nu(s)$ in pseudo-arc function s.

3. NULL OSCULATING CURVES OF THE FIRST KIND IN $\mathbb{E}_1^4$

In this section we characterize null osculating curves of the first kind in the Minkowski space-time. We show that a null curve in $\mathbb{E}_1^4$ is an osculating curve of the first kind if and only if it lies fully in the timelike hyperplane of $\mathbb{E}_1^4$. In relation to that, we give the following theorem.

Theorem 3.1. *Let α be a null Cartan curve in $\mathbb{E}_1^4$. Then α is congruent to an osculating curve of the first kind if and only if its third curvature $\kappa_3(s)$ is equal to zero for each s.*

Proof. First assume that α is an osculating curve of the first kind in $\mathbb{E}_1^4$. Then its position vector satisfies relation (2). Differentiating relation (2) with respect to s and using (1), we easily find that $\kappa_3(s) = 0$.

Conversely, assume that the third curvature $\kappa_3(s) = 0$ for each s. Then relation (1) implies that $B_2(s)$ is a constant vector and hence $g(\alpha, B_2)$ is a constant function. Since the position vector of α can be decomposed as

$$\alpha = g(\alpha, B_1)T + g(\alpha, N)N + g(\alpha, T)B_1 + g(\alpha, B_2)B_2,$$

and $g(\alpha,B_2)B_2$ is a constant vector, it follows that α is congruent to an osculating curve of the first kind. This completes the proof of the theorem. □

As a consequence, we easily get the following two statements.

Corollary 3.1. *Let α be a null Cartan curve in $\mathbb{E}^4_1$. Then α lies fully in the timelike hyperplane of $\mathbb{E}^4_1$ if and only if it is an osculating curve of the first kind.*

Corollary 3.2. *Let α be the null Cartan curve in $\mathbb{E}^4_1$ parameterized by pseudo-arc s. Then α is an osculating curve of the first kind if and only if its position vector has the following form*

$$\alpha(s)=(\nu(s)k_2(s)-\nu''(s))T(s)+\nu'(s)N(s)+\nu(s)B_1(s),$$

where $\nu(s)=g(\alpha(s),T(s))\neq 0$ is an arbitrary differentiable function satisfying the differential equation

$$\nu'''(s)-2\nu'(s)k_2(s)-\nu(s)k_2'(s)+1=0.$$

4. NULL OSCULATING CURVES OF THE SECOND KIND IN $\mathbb{E}^4_1$

In this section, we characterize null osculating curves of the second kind lying fully in $\mathbb{E}^4_1$ by using the components of their position vectors and the curvature functions. Let α be a null osculating curve of the second kind in $\mathbb{E}^4_1$ parameterized by pseudo-arc s and with the third curvature $k_3(s)\neq 0$ for each s. Then its second curvature $k_2(s)$ can be equal to zero or different from zero. Thus we consider two cases: (A) $k_2(s)=0$ and (B) $k_2(s)\neq 0$.

Case (A) $k_2(s)=0$.

Differentiating relation (3) with respect to s and by using (1), we obtain the system of equations

$$\begin{aligned}\mu'(s)-\lambda(s)&=0,\\ \lambda'(s)&=0,\\ -\nu(s)\kappa_3(s)&=1,\\ \nu'(s)+\mu(s)\kappa_3(s)&=0,\end{aligned}\tag{4}$$

where $\lambda=g(\alpha,N)$, $\mu=g(\alpha,T)$, and $\nu=g(\alpha,B_2)$ are the principal normal, the tangential, and the second binormal component of the position vector of the curve, respectively. From the second equation of relation (4) we get $\lambda(s)=c\in\mathbb{R}$, so we may distinguish two subcases: **(A.1)** $\lambda(s)=0$ and **(A.2)** $\lambda(s)=c\in\mathbb{R}_0$.

(A.1) $\lambda(s)=0$. In this subcase, the first equation of (4) implies that the tangential component $\mu=g(\alpha,T)$ is a real constant, so again we may distinguish two subcases: **(A.1.1)** $\mu(s)=0$ and **(A.1.2)** $\mu(s)=a$, $a\in\mathbb{R}_0$.

(A.1.1) $\mu(s)=0$. In this subcase, we obtain the following theorem which gives the necessary and sufficient conditions for the null curve in $\mathbb{E}^4_1$ with the zero tangential component of the position vector to be an osculating curve of the second kind.

Theorem 4.1. *Let α be a null Cartan curve in $\mathbb{E}^4_1$ parameterized by pseudo-arc s with curvatures $k_1(s)=1$, $k_2(s)=0$, and $k_3(s)\neq 0$. If α is an osculating curve of the second kind with tangential component $g(\alpha,T)=0$, then the following statements hold*:
(i) *the third curvature $k_3(s)$ is a non-zero constant*;
(ii) *α lies in a pseudosphere $S^3_1(r)$, $r\in\mathbb{R}^+_0$*;
(iii) *the second binormal component $g(\alpha,B_2)$ of the position vector of the curve is a non-zero constant.*

Conversely, if α is a null Cartan curve in $\mathbb{E}^4_1$ parameterized by pseudo-arc s, with curvatures $k_1(s)=1$, $k_2(s)=0$, $k_3(s)\neq 0$ and one of the statements (i), (ii), *or* (iii) *holds, then α is an osculating curve of the second kind.*

Proof. First assume that $\alpha(s)$ is an osculating curve of the second kind with tangential component $g(\alpha,T)=0$. By using the system of equations (4) we get

$$\lambda(s)=0, \quad -\nu(s)k_3(s)=1, \quad \nu'(s)=0.$$

It follows that $\nu(s)=c\in\mathbb{R}_0$ and thus $k_3(s)=-1/c=\text{constant}$, which proves statement (i). Since the position vector of the curve is given by

$$\alpha(s)=cB_2(s), \quad c\in\mathbb{R}_0,$$

we easily get $g(\alpha,\alpha)=c^2$, $g(\alpha,B_2)=c$, which proves statements (ii) and (iii).

Conversely, assume that statement (i) holds. Putting $k_3(s)=c$, $c\in\mathbb{R}_0$ and using relation (1), we find

$$\frac{d}{ds}(\alpha(s)+\frac{1}{c}B_2(s))=0.$$

Thus α is congruent to the second kind null osculating curve. If statement (ii) holds, differentiating the equation $g(\alpha,\alpha)=r^2$, $r\in\mathbb{R}_0^+$, three times with respect to s and using (1), we obtain $g(\alpha,B_1)=0$, which means that α is a second kind osculating curve. If statement (iii) holds, by taking the derivative of the equation $g(\alpha,B_2)=\text{constant}\neq 0$ with respect to s and by applying (1), we get $g(\alpha,B_1)=0$, which proves the theorem. $\square$

The next statement is an easy consequence of Theorem 4.1. It gives the simple relationship between null normal curves and null osculating curves of the second kind.

Corollary 4.1. *Every null normal curve lying fully in $\mathbb{E}_1^4$ is a second kind null osculating curve. Conversely, every second kind null osculating curve in $\mathbb{E}_1^4$ with zero tangential component $g(\alpha,T)$ is a null normal curve.*

(A.1.2) $\mu(s)=a\in\mathbb{R}_0$. In this subcase the following theorem, which can be proved in a similar way as Theorem 4.1, holds.

Theorem 4.2. *Let α be a null Cartan curve in $\mathbb{E}_1^4$ parameterized by pseudo-arc s with curvatures $k_1(s)=1$, $k_2(s)=0$, and $k_3(s)\neq 0$. If α is an osculating curve of the second kind with tangential component $g(\alpha,T)=a\in\mathbb{R}_0$, then the following statements hold*:
(i) *the third curvature $k_3(s)$ is given by*

$$k_3(s)=\frac{1}{\sqrt{2as+b}}, \quad a\in\mathbb{R}_0, \quad b\in\mathbb{R};$$

(ii) *the principal normal component of the position vector of the curve is zero, i.e. $g(\alpha,N)=0$;*
(iii) *the distance function $\rho=||\alpha||$ satisfies $\rho^2=|c_1s+c_2|$, $c_1\in\mathbb{R}_0$, $c_2\in\mathbb{R}$.*

Conversely, if α is a null Cartan curve in $\mathbb{E}_1^4$ parameterized by pseudo-arc s with curvatures $k_1(s)=1$, $k_2(s)=0$, $k_3(s)\neq 0$ and one of the statements (i), (ii), *or* (iii) *holds, then α is an osculating curve of the second kind.*

As a consequence of Theorem 4.2, we obtain the next statement which gives a simple relationship between null rectifying curves and null osculating curves of the second kind.

Corollary 4.2. *Every null rectifying curve in $\mathbb{E}_1^4$ with the second curvature $k_2(s)=0$ is an osculating curve of the second kind. Conversely, every osculating curve α of the second kind in $\mathbb{E}_1^4$ with principal normal component $g(\alpha,N)=0$ is a null rectifying curve.*

(A.2) $\lambda(s)=c\in\mathbb{R}_0$. In this subcase we obtain the next theorem which gives the necessary and sufficient conditions for the null curve in $\mathbb{E}_1^4$ with the constant principal normal component of the position vector to be an osculating curve of the second kind.

Theorem 4.3. *Let α be a null Cartan curve in $\mathbb{E}_1^4$ parameterized by pseudo-arc s with curvatures $k_1(s)=1$, $k_2(s)=0$, and $k_3(s)\neq 0$. If α is an osculating curve of the second kind with principal normal component $g(\alpha,N)=c\in\mathbb{R}_0$, then the following statements hold*:
(i) *the third curvature $k_3(s)$ is given by*

$$k_3(s)=\frac{1}{\sqrt{as^2+bs+c}},\quad a\in\mathbb{R}_0, b,c\in\mathbb{R};$$

(ii) *the tangential component of the position vector of the curve is given by $g(\alpha,T)=as+b$, $a\in\mathbb{R}_0$, $b\in\mathbb{R}$;*
(iii) the distance function $\rho(s)=||\alpha(s)||$ satisfies $\rho^2(s)=|ds^2+es+f|$, $d\in\mathbb{R}_0$, $e,f\in\mathbb{R}$.

Conversely, if α is a null Cartan curve in $\mathbb{E}_1^4$ parameterized by pseudo-arc s, with curvatures $k_1(s)=1$, $k_2(s)=0$, $k_3(s)\neq 0$ and one of the statements (i), (ii), *or* (iii) *holds, then α is an osculating curve of the second kind.*

The proof of Theorem 4.3 is similar to the proof of Theorem 4.1.

Case (B) $k_2(s)\neq 0$.

Differentiating relation (3) with respect to s and using (1), we obtain the system of equations

$$\begin{aligned} \lambda\kappa_2-\nu\kappa_3&=1,\\ \lambda'&=\mu\kappa_2,\\ \mu'&=\lambda,\\ \nu'+\mu\kappa_3&=0. \end{aligned}\tag{5}$$

By using the first and last equation of relation (5), we get

$$\left(\frac{\lambda\kappa_2-1}{\kappa_3}\right)'+\mu\kappa_3=0.$$

By using the second and the third equation of relation (5), a straightforward calculation leads to the differential equation

$$\mu'\left(\frac{k_2}{k_3}\right)'+\frac{\mu(k_2^2+k_3^2)}{k_3}-\left(\frac{1}{k_3}\right)'=0.\tag{6}$$

Then we may distinguish two subcases: **(B.1)** $(k_2/k_3)'=0$ and **(B.2)** $(k_2/k_3)'\neq 0$.

(B.1) $(k_2/k_3)'=0$. It follows that the ratio of the curvature functions k_2 and k_3 is a constant, i.e. $k_2(s)/k_3(s)=a\in R_0$, so we again may distinguish two subcases: **(B.1.1)** $k_3(s)=c\in\mathbb{R}_0$ and **(B.1.2)** $k_3(s)\neq$ constant.

(B.1.1) $k_3(s)=c\in\mathbb{R}_0$. Then $k_2(s)=ac\in\mathbb{R}_0$, which means that α is a null helix. Moreover, relation (6) becomes

$$c\mu(s)(a^2+1)=0.$$

The last equation implies that tangential component $\mu(s)=0$. In this way, we obtain the next characterization of null osculating helices of the second kind.

Theorem 4.4. *Let α be a null Cartan helix in $\mathbb{E}_1^4$. Then α is congruent to an osculating curve of the second kind with tangential component $g(\alpha,T)=0$ if and only if α lies in a pseudosphere $S_1^3(r)$ in $\mathbb{E}_1^4$.*

We omit the proof of the previous theorem since it is analogous to the proof of Theorem 4.1.

(B.1.2) $k_3(s) \neq$constant. In this subcase the following theorem holds.

Theorem 4.5. *Let α be a null Cartan curve in $\mathbb{E}_1^4$ parameterized by pseudo-arc s and with curvatures $k_1(s) = 1$, $k_2(s) = ak_3(s)$, $a \in \mathbb{R}_0$, $k_3(s) \neq$ constant. Then α is congruent to an osculating curve of the second kind if and only if the third curvature $k_3(s)$ satisfies the differential equation*

$$9k_3''(s)k_3'(s)k_3(s) - k_3'''(s)k_3^2(s) - 12k_3'^3(s) + ak_3'(s)k_3^3(s) = 0, \quad a \in \mathbb{R}_0. \tag{7}$$

Proof. First assume that α is congruent to the second kind osculating curve. By using relation (6) we get

$$\mu = -\frac{k_3'}{k_3^3(a^2+1)}. \tag{8}$$

Next, relation (8) and the third equation of relation (5) imply

$$\lambda = \frac{3k_3'^2 - k_3''k_3}{k_3^4(a^2+1)}. \tag{9}$$

Substituting (9) in the first equation of relation (5), we obtain

$$\nu = \frac{a(3k_3'^2 - k_3''k_3) - k_3^3(a^2+1)}{k_3^4(a^2+1)}. \tag{10}$$

In this way, all components of the position vector α are expressed in terms of the third curvature k_3. Moreover, by using (8), (9), and the second equation of relation (5) we find that k_3 satisfies relation (7).

Conversely, assume that the third curvature k_3 satisfies relation (7). Let us consider the vector $X \in \mathbb{E}_1^4$ given by

$$X = \alpha - \frac{3k_3'^2 - k_3''k_3}{k_3^4(a^2+1)}N + \frac{k_3'}{k_3^3(a^2+1)}B_1 - \frac{a(3k_3'^2 - k_3''k_3) - k_3^3(a^2+1)}{k_3^4(a^2+1)}B_2.$$

By using (1) and (7), we find that $X' = 0$, which means that X is a constant vector. Consequently, α is congruent to an osculating curve of the second kind. This completes the proof of the theorem. $\square$

(B.2) $(k_2/k_3)' \neq 0$. It follows that $k_2(s)/k_3(s) \neq$ constant. Then we may distinguish two subcases: **(B.2.1)** $k_2(s) \neq$ constant, $k_3(s) =$ constant $\neq 0$; **(B.2.2)** $k_2(s) \neq 0$, $k_3(s) \neq$ constant;

(B.2.1) $k_2(s) \neq$ constant, $k_3(s) =$ constant $\neq 0$. In this subcase, we obtain the next theorem which can be proved in a similar way as Theorem 4.5.

Theorem 4.6. *Let α be a null Cartan curve in $\mathbb{E}_1^4$ parameterized by pseudo-arc s with curvatures $k_1(s) = 1$, $k_2(s) \neq$ constant, and $k_3(s) = a \in \mathbb{R}_0$. Then α is congruent to an osculating curve of the second kind if and only if the second curvature $k_2(s)$ satisfies the differential equation*

$$(k_2^2(s) + a^2)k_2''(s) - 3k_2(s)k_2'^2(s) + (k_2^2(s) + a^2)^2 = 0, \quad a \in \mathbb{R}_0. \tag{11}$$

(B.2.2) $k_2(s) \neq 0$, $k_3(s) \neq$ constant. In the last subcase, the following theorem holds.

Theorem 4.7. *Let α be a null Cartan curve in $\mathbb{E}_1^4$ parameterized by pseudo-arc s with curvatures $k_1(s) = 1$, $k_2(s) \neq 0$, and $k_3(s) \neq$ constant. Then α is congruent to an osculating curve of the second kind if and only if the curvature functions $k_2(s)$ and $k_3(s)$ satisfy the relation*

$$e^{-\int p(s)ds}\left(c + \int q(s)e^{\int p(s)ds}ds\right) = \frac{p(s)q(s) - q'(s)}{p^2(s) - p'(s) - k_2(s)}, \quad c \in \mathbb{R},$$

where

$$p(s) = \frac{k_3(s)(k_2^2(s) + k_3^2(s))}{k_2'(s)k_3(s) - k_2(s)k_3'(s)}, \quad q(s) = -\frac{k_3'(s)}{k_2'(s)k_3(s) - k_2(s)k_3'(s)}.$$

5. SOME EXAMPLES OF NULL OSCULATING CURVES IN $\mathbb{E}_1^4$

Let us consider the null cubic in $\mathbb{E}_1^4$ given by

$$\alpha(s) = \left(\frac{1}{6}s^3 + s, \frac{1}{2}s^2, \frac{1}{6}s^3, s\right).$$

Since its curvature functions have the form

$$\kappa_1(s) = 1, \quad \kappa_2(s) = 0, \quad \kappa_3(s) = 0,$$

Theorem 3.1 implies that α is an osculating curve of the first kind.

As the second example of a null osculating curve, let us consider the curve α in $\mathbb{E}_1^4$ with the equation

$$\alpha(s) = \left(\frac{1}{\sqrt{2}}\sinh s, \frac{1}{\sqrt{2}}\cosh s, \frac{1}{\sqrt{2}}\sin s, \frac{1}{\sqrt{2}}\cos s\right).$$

It can be easily verified that curvature functions of α are given by

$$\kappa_1(s) = 1, \quad \kappa_2(s) = 0, \quad \kappa_3(s) = -1.$$

Hence α is a null helix lying in the pseudosphere. According to Theorem 4.4, α is an osculating curve of the second kind.

Next, let us consider the null curve in $\mathbb{E}_1^4$ given by

$$\alpha(s) = \left(e^{as} + \frac{1}{c}\cos(cs) - \frac{1}{2a^2}e^{-as}, e^{as} + \frac{1}{c}\cos(cs), \frac{1}{2a^2}e^{-as} - \frac{1}{c}\cos(cs), \frac{1}{c}\sin(cs)\right),$$

where a, $c \in \mathbb{R}_0$. By using a straightforward calculation it can be shown that tangent vector $T(s) = \alpha'(s)$ satisfies the equation $g(\alpha, T) = 0$. Hence Theorem 4.2 implies that α is an example of an osculating curve of the second kind.

Finally, let us consider the null curve in $\mathbb{E}_1^4$ with the equation

$$\begin{aligned}\alpha(s) = \quad & \left(\frac{1}{\sqrt{56}}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{2+\frac{3\sqrt{6}}{2}} + \frac{s^{2-\frac{3\sqrt{6}}{2}}}{2-\frac{3\sqrt{6}}{2}}\right), \frac{1}{\sqrt{56}}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{2+\frac{3\sqrt{6}}{2}} - \frac{s^{2-\frac{3\sqrt{6}}{2}}}{2-\frac{3\sqrt{6}}{2}}\right),\right.\\ & \frac{2s^2}{9\sqrt{14}}\left(2\cos\left(\frac{\sqrt{2}}{2}\ln(s)\right) + \frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}\ln(s)\right)\right),\\ & \left.\frac{2s^2}{9\sqrt{14}}\left(2\sin\left(\frac{\sqrt{2}}{2}\ln(s)\right) - \frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}\ln(s)\right)\right)\right),\end{aligned}$$

where s is a pseudo-arc parameter. A straightforward calculation shows that the curvature functions of α are given by

$$k_1(s) = 1, \quad k_2(s) = 6/s^2, \quad k_3(s) = 6/as^2,$$

where $a \in \mathbf{R}_0$. Moreover, it can be easily verified that the curvature functions satisfy relation (7). According to Theorem 4.7, it follows that α is congruent to an osculating curve of the second kind.

6. CONCLUSION AND FURTHER REMARKS

Many classical results from Riemannian geometry have Lorentz counterparts. In fact, spacelike and timelike curves can be studied by a similar approach as in positive definite Riemannian geometry. However, null curves have some different properties compared to spacelike and timelike curves. In other words, the results of the null curve theory have no Riemannian analogues, because the arc length of the null curve vanishes, so that it is not possible to normalize the tangent vector in the usual way.

In this paper, we introduce the notions of the first kind and the second kind null osculating curves in the Minkowski space-time $\mathbb{E}_1^4$. Also we give the necessary and sufficient conditions for the null curves in $\mathbb{E}_1^4$ to be osculating curves in terms of their curvature functions. Relevant examples are given. In the light of the obtained results, we also emphasize that the same problem can be studied for null, pseudo null, partially null, spacelike, and timelike curves in the semi-Riemannian space $\mathbb{E}_2^4$.

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Nullkooldumiskõverate iseloomustusest Minkowski aegruumis

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On esitatud kõverusfunktsioonide abil tarvilikud ja piisavad tingimused selleks, et nullkõverad Minkowski ruumis $\mathbb{E}_1^4$ oleksid kooldumiskõverateks. Lähemalt vaadeldes leiame seosed nullnormaalkõverate ja nullkooldumiskõverate, samuti nullsirgestuvate ning nullkooldumiskõverate vahel. Kokkuvõttes on esitatud mõned nullkooldumiskõverate näited Minkowski ruumis $\mathbb{E}_1^4$.