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MATHEMATICS

# Necessary conditions for inclusion relations for double absolute summability

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**Abstract.** We establish necessary conditions for a general inclusion theorem involving a pair of doubly triangular matrices. As corollaries we obtain inclusion results for some special classes of doubly triangular matrices.

**Key words:** absolute summability factors, doubly triangular summability.

## 1. INTRODUCTION

A doubly infinite matrix $A = (a_{mnij})$ is said to be doubly triangular if $a_{mnij} = 0$ for $i > m$ and $j > n$. The $mn$th term of the $A$-transform of a double sequence $\{s_{mn}\}$ is defined by

$$T_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mnij} s_{ij}.$$

A series $\sum\sum c_{mn}$, with partial sums $s_{mn}$ is said to be absolutely $A$-summable, of order $k \geq 1$, if

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (mn)^{k-1} |\Delta_{11} T_{m-1,n-1}|^k < \infty, \tag{1}$$

where, for any double sequence $\{u_{mn}\}$, and for any fourfold sequence $\{a_{mnij}\}$, we define

$$\begin{aligned} \Delta_{11} u_{mn} &= u_{mn} - u_{m+1,n} - u_{m,n+1} + u_{m+1,n+1}, \\ \Delta_{11} a_{mnij} &= a_{mnij} - a_{m+1,n,i,j} - a_{m,n+1,i,j} + a_{m+1,n+1,i,j}, \\ \Delta_{ij} a_{mnij} &= a_{mnij} - a_{m,n,i+1,j} - a_{m,n,i,j+1} + a_{m,n,i+1,j+1}, \\ \Delta_{i0} a_{mnij} &= a_{mnij} - a_{m,n,i+1,j}, \\ \Delta_{0j} a_{mnij} &= a_{mnij} - a_{m,n,i,j+1}. \end{aligned} \tag{2}$$

The one-dimensional version of (1) appears in [1].

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Associated with $A$ are two matrices $\overline{A}$ and $\hat{A}$ defined by

$$\bar{a}_{mnij} = \sum_{\mu=i}^{m} \sum_{\nu=j}^{n} a_{mn\mu\nu}, \ \ 0 \le i \le m, \ \ 0 \le j \le n, \ \ m,n = 0,1,\ldots,$$

and

$$\hat{a}_{mnij} = \Delta_{11} \bar{a}_{m-1,n-1,i,j}, \ \ 0 \le i \le m, \ \ 0 \le j \le n, \ \ m,n = 1,2,\ldots.$$

It is easily verified that $\hat{a}_{0000} = \bar{a}_{0000} = a_{0000}$. In [3] it is shown that

$$\hat{a}_{mnij} = \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} \Delta_{11} a_{m-1,n-1,\mu,\nu}.$$

Thus $\hat{a}_{mni0} = \hat{a}_{mn0j} = 0$.

Let $x_{mn}$ denote the $mn$th term of the $A$-transform of the sequence of partial sums $\{s_{mn}\}$ of the series $\sum\sum c_{mn}$. Then

$$\begin{aligned} x_{mn} &= \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mnij} s_{ij} = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{\mu=0}^{i} \sum_{\nu=0}^{j} a_{mnij} c_{\mu\nu} \\ &= \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \sum_{i=\mu}^{m} \sum_{j=\nu}^{n} a_{mn\mu\nu} c_{\mu\nu} \\ &= \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \bar{a}_{mn\mu\nu} c_{\mu\nu}, \end{aligned}$$

and a direct calculation verifies that

$$X_{mn} := \Delta_{11} x_{m-1,n-1} = \sum_{i=1}^{m} \sum_{j=1}^{n} \hat{a}_{mnij} c_{ij},$$

since

$$\bar{a}_{m-1,n-1,m,j} = a_{m-1,n-1,i,n} = \hat{a}_{m,n-1,i,n} = \hat{a}_{m-1,n,m,n} = 0.$$

## 2. MAIN RESULT

We have the following theorem

**Theorem 1.** *Let $1 < k \le s < \infty$, A and B be doubly triangular matrices with A satisfying*

$$\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\Delta_{uv} \hat{a}_{mnuv}|^k = O(M^k(\hat{a}_{uvuv})), \tag{3}$$

*where*

$$M(\hat{a}_{uvuv}) := \max\{|\hat{a}_{uvuv}|, |\Delta_{u0} \hat{a}_{u+1v,u,v}|, |\Delta_{0v} \hat{a}_{uv+1,u,v}|\}.$$

*Then necessary conditions for $\sum\sum c_{mn}$ summable $|A|_k$ to imply that $\sum\sum c_{mn}$ is summable $|B|_s$ are:*

(i) $|\hat{b}_{uvuv}| = O((uv)^{1/s-1/k} M(\hat{a}_{uvuv}))$,

(ii) $|\Delta_{u0} \hat{b}_{u+1,v,u,v}| = O((uv)^{1/s-1/k} M(\hat{a}_{uvuv}))$,

(iii) $|\Delta_{0v} \hat{b}_{u,v+1,u,v}| = O((uv)^{1/s-1/k} M(\hat{a}_{uvuv}))$,

(iv) $\displaystyle\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{s-1} |\Delta_{uv} \hat{b}_{mnuv}|^s = O((uv)^{s-s/k} M^s(\hat{a}_{uvuv}))$,

(v) $\displaystyle\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{s-1} |\hat{b}_{m,n,u+1,v+1}|^s = O\Big( \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{a}_{m,n,u+1,v+1}|^k \Big)^{s/k}$.

*Proof.* We are given that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{s-1}|Y_{mn}|^{s}<\infty, \tag{4}$$

whenever

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|X_{mn}|^{k}<\infty, \tag{5}$$

where

$$Y_{mn}=\Delta_{11}y_{m-1,n-1},$$

$$y_{mn}=\sum_{i=0}^{m}\sum_{j=0}^{n}\bar{b}_{mnij}c_{ij}.$$

The space of sequences satisfying (5) is a Banach space if normed by

$$\|X\|=\Big(|X_{00}|^{k}+|X_{01}|^{k}+|X_{10}|^{k}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|X_{mn}|^{k}\Big)^{1/k}. \tag{6}$$

We shall also consider the space of sequences $\{Y_{mn}\}$ that satisfy (4). This space is also a BK-space with respect to the norm

$$\|Y\|=\Big(|Y_{00}|^{s}+|Y_{01}|^{s}+|Y_{10}|^{s}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{s-1}|Y_{mn}|^{s}\Big)^{1/s}. \tag{7}$$

The transformation $y_{mn}=\sum_{i=0}^{m}\sum_{j=0}^{n}\bar{b}_{mnij}c_{ij}$ maps sequences satisfying (5) into sequence spaces satisfying (4). By the Banach–Steinhaus Theorem there exists a constant $K>0$ such that

$$\|Y\|\le K\|X\|. \tag{8}$$

For fixed $u,v$, the sequence $\{c_{ij}\}$ is defined by $c_{uv}=c_{u+1,v+1}=1, c_{u+1,v}=c_{u,v+1}=-1, c_{ij}=0$, otherwise, gives

$$X_{mn}=\begin{cases}0, & m\le u,\ n<v,\\ 0, & m<u,\ n\le v,\\ \hat{a}_{mnuv}, & m=u,\ n=v,\\ \Delta_{u0}\hat{a}_{mnuv}, & m=u+1,\ n=v,\\ \Delta_{0v}\hat{a}_{mnuv} & m=u,\ n=v+1,\\ \Delta_{uv}\hat{a}_{mnuv}, & m>u,\ n>v\end{cases}$$

and

$$Y_{mn}=\begin{cases}0, & m\le u,\ n<v,\\ 0, & m<u,\ n\le v,\\ \hat{b}_{mnuv}, & m=u,\ n=v,\\ \Delta_{u0}\hat{b}_{mnuv}, & m=u+1,\ n=v,\\ \Delta_{0v}\hat{b}_{mnuv}, & m=u,\ n=v+1,\\ \Delta_{uv}\hat{b}_{mnuv}, & m>u,\ n>v.\end{cases}$$

From (6) and (7) it follows that

$$\begin{aligned}\|X\|=&\Big\{(uv)^{k-1}|\hat{a}_{uvuv}|^{k}+((u+1)v)^{k-1}|\Delta_{u0}a_{u+1,v,u,v}|^{k}\\ &+(u(v+1))^{k-1}|\Delta_{0v}a_{u,v+1,u,v}|^{k}+\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{a}_{mnuv}|^{k}\Big\}^{1/k}\end{aligned} \tag{9}$$

and

$$\|Y\| = \Big\{(uv)^{s-1}|\hat{b}_{uvuv}|^s + ((u+1)v)^{s-1}|\Delta_{u0}\hat{b}_{u+1,v,u,v}|^s + (u(v+1))^{s-1}|\Delta_{0v}\hat{b}_{u,v+1,u,v}|^s + \sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{s-1}|\Delta_{uv}\hat{b}_{mnuv}|^s\Big\}^{1/s}. \tag{10}$$

Substituting (9) and (10) into (8), along with (3), gives

$$\begin{aligned}(uv)^{s-1}|\hat{b}_{uvuv}|^s &+ ((u+1)v)^{s-1}|\Delta_{u0}\hat{b}_{u+1,v,u,v}|^s + (u(v+1))^{s-1}|\Delta_{0v}\hat{b}_{u,v+1,u,v}|^s \\ &+ \sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{s-1}|\Delta_{uv}\hat{b}_{mnuv}|^s \le K^s\Big\{(uv)^{k-1}|\hat{a}_{uvuv}|^k \\ &+ ((u+1)v)^{k-1}|\Delta_{u0}\hat{a}_{u+1,v,u,v}|^k + (u(v+1))^{k-1}|\Delta_{i0}\hat{a}_{u,v+1,u,v}|^k \\ &+ \sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{a}_{mnuv}|^k\Big\}^{s/k} \\ &= K^s\{O(1)(uv)^{k-1}M^k(\hat{a}_{uvuv})\}^{s/k}.\end{aligned}$$

The above inequality implies that each term of the left-hand side is $O(\{(uv)^{k-1}M^k(a_{uvuv})\}^{s/k})$. Using the first term, one obtains

$$(uv)^{s-1}|\hat{b}_{uvuv}|^s = O(\{(uv)^{k-1}M^k(\hat{a}_{uvuv})\}^{s/k}),$$

or

$$|\hat{b}_{uvuv}|^s = O((uv)^{s-s/k-s+1}M^s(\hat{a}_{uvuv})).$$

Thus

$$|\hat{b}_{uvuv}| = O((uv)^{1/s-1/k}M(\hat{a}_{uvuv})),$$

which is condition (i).

In a similar manner one obtains conditions (ii)–(iv).

Using the sequence defined by $c_{u+1,v+1} = 1$, and $c_{ij} = 0$ otherwise yields

$$X_{mn} = \begin{cases} 0, & m \le u+1, \quad n \le v, \\ 0, & m \le u, \quad n \le v+1, \\ \hat{a}_{m,n,u+1,v+1}, & m \ge u+1, \quad n \ge v+1 \end{cases}$$

and

$$Y_{mn} = \begin{cases} 0, & m \le u+1, \ n \le v, \\ 0, & m \le u, \ n \le v+1, \\ \hat{b}_{m,n,u+1,v+1}, & m \ge u+1, \ n \ge v+1. \end{cases}$$

The corresponding norms are

$$\|X\| = \Big\{\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\hat{a}_{m,n,u+1,v+1}|^k\Big\}^{1/k}$$

and

$$\|Y\| = \Big\{\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{s-1}|\hat{b}_{m,n,u+1,v+1}|^s\Big\}^{1/s}.$$

Applying (8), one obtains

$$\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{s-1}|\hat{b}_{m,n,u+1,v+1}|^s \le K^s\Big\{\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\hat{a}_{m,n,u+1,v+1}|^k\Big\}^{s/k},$$

which is equivalent to (v). □

**Corollary 1.** *Let $1 \leq k < \infty$, $A$ and $B$ be two doubly triangular matrices, $A$ satisfying* (3). *Then necessary conditions for $\sum\sum c_{mn}$ summable $|A|_k$ to imply that $\sum\sum c_{mn}$ is summable $|B|_k$ are*

(i) $|\hat{b}_{uvuv}| = O(M(\hat{a}_{uvuv}))$,

(ii) $|\Delta_{u0}\hat{b}_{u+1,v,u,v}| = O(M(\hat{a}_{uvuv}))$,

(iii) $|\Delta_{0v}\hat{b}_{u,v+1,u,v}| = O(M(\hat{a}_{uvuv}))$,

(iv) $\displaystyle\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{b}_{mnuv}|^k = O(uv^{k-1}M^k(\hat{a}_{uvuv}))$, *and*

(v) $\displaystyle\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\hat{b}_{m,n,u+1,v+1}|^k = O\Big(\Big\{\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\hat{a}_{m,n,u+1,v+1}|^k\Big\}\Big)$.

*Proof.* To prove Corollary 1, simply set $s = k$ in Theorem 1. □

We shall call a doubly infinite matrix a product matrix if it can be written as the termwise product of two singly infinite matrices $F$ and $G$; i.e., $a_{mnij} = f_{mi}g_{nj}$ for each $i, j, m, n$.

A doubly infinite weighted mean matrix $P$ has nonzero entries $p_{ij}/P_{mn}$, where $p_{00}$ is positive and all of the other $p_{ij}$ are nonnegative, and $P_{mn} := \sum_{i=0}^{m}\sum_{j=0}^{n} p_{ij}$. If $P$ is a product matrix, then the nonzero entries are $p_iq_j/P_mQ_n$, where $p_0 > 0,\ p_i > 0$ for $i > 0,\ q_0 > 0,\ q_j \geq 0$ for $j > 0$ and $P_m := \sum_{i=0}^{m} p_i,\ Q_n := \sum_{j=0}^{n} q_j$.

**Corollary 2.** *Let $1 \leq k < \infty$, $P$ be a product weighted mean matrix, $B$ be a doubly triangular matrix with $P$ satisfying*

$$\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}\Big|\Delta_{uv}\Big(\frac{p_mq_nP_{u-1}Q_{v-1}}{P_mP_{m-1}Q_nQ_{n-1}}\Big)\Big|^k = O\Big(\frac{p_uq_v}{P_uQ_v}\Big). \tag{11}$$

*Then necessary conditions for $\sum\sum c_{mn}$ summable $|P|_k$ to imply that $\sum\sum c_{mn}$ is summable $|B|_s$ are*:

(i) $|\hat{b}_{uvuv}| = O\Big((uv)^{1/s-1/k}\dfrac{p_uq_v}{P_uQ_v}\Big)$,

(ii) $|\Delta_{u0}\hat{b}_{u+1,v,u,v}| = O\Big((uv)^{1/s-1/k}\dfrac{p_uq_v}{P_uQ_v}\Big)$,

(iii) $|\Delta_{0v}\hat{b}_{u,v+1,u,v}| = O\Big((uv)^{1/s-1/k}\dfrac{p_uq_v}{P_uQ_v}\Big)$,

(iv) $\displaystyle\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{s-1}|\Delta_{uv}\hat{b}_{mnuv}|^s = O\Big((uv)^{s-s/k}\Big(\frac{p_uq_v}{PuQ_v}\Big)^s\Big)$, *and*

(v) $\displaystyle\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{s-1}|\hat{b}_{m,n,u+1,v+1}|^s = O(1)$.

*Proof.* From [3]

$$\hat{p}_{uvuv} = \sum_{i=0}^{u-1}\sum_{j=0}^{v-1}\Delta_{11}p_{u-1,v-1,i,j}. \tag{12}$$

Note that

$$\begin{aligned}\Delta_{11}p_{u-1,v-1,i,j} &= p_{u-1,v-1,i,j} - p_{u,v-1,i,j} - p_{u-1,v,i,j} + p_{uvij}\\ &= \frac{p_iq_j}{P_{u-1}q_{v-1}} - \frac{p_iq_j}{P_uQ_{v-1}} - \frac{p_iq_j}{P_{u-1}Q_v} + \frac{p_iq_j}{P_uQ_v}\\ &= \frac{p_iq_jp_uq_v}{P_{u-1}P_uQ_{v-1}Q_v}.\end{aligned} \tag{13}$$

Therefore

$$\hat{p}_{uvuv} = \sum_{i=0}^{u-1}\sum_{j=0}^{v-1} \frac{p_i q_j p_u q_v}{P_{u-1}P_u Q_{v-1}Q_v} = \frac{p_u q_v}{P_u Q_v} = p_{uvuv}. \tag{14}$$

From (12) and (13),

$$\hat{p}_{u+1,v,u,v} = \sum_{i=0}^{u}\sum_{j=0}^{v-1} \Delta_{11} p_{u,v-1.i.j} = \sum_{i=0}^{u}\sum_{j=0}^{v-1} \frac{p_i q_j p_{u+1} q_v}{P_u P_{u+1} Q_{v-1} Q_v} = \frac{p_{u+1} q_v}{P_{u+1} Q_v}.$$

Using (2) and (14),

$$\Delta_{u0}\hat{p}_{u+1,v,u,v} = \hat{p}_{u+1,v,u,v} - \hat{p}_{u+1,v,u+1,v} = \frac{p_{u+1}q_v}{P_{u+1}Q_v} - \frac{p_{u+1}q_v}{P_{u+1}Q_v} = 0.$$

Similarly, $\Delta_{0v} p_{u,v+1,u,v} = 0$. Thus

$$M(\hat{p}_{uvuv}) = \frac{p_u q_v}{P_u Q_v},$$

and conditions (i)–(v) take the form represented. □

**Corollary 3.** *Let $B$ be a doubly triangular matrix, $P$ a product weighted mean matrix satisfying (11). Then necessary conditions for $\sum\sum c_{mn}$ summable $|P|_k$ to imply that $\sum\sum c_{mn}$ is summable $|B|_k$ are*

(i) $|\hat{b}_{uvuv}| = O\left(\frac{p_u q_v}{P_u Q_v}\right)$,

(ii) $|\Delta_{u0}\hat{b}_{u+1,v,u,v}| = O\left(\frac{p_u q_v}{P_u Q_v}\right)$,

(iii) $|\Delta_{0v}\hat{b}_{u,v+1,u,v}| = O\left(\frac{p_u q_v}{P_u Q_v}\right)$,

(iv) $\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty} (mn)^{k-1}|\Delta_{uv}\hat{b}_{uvuv}|^k = O\left(\left(\frac{p_u q_v}{P_u Q_v}\right)^k\right)$, *and*

(v) $\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty} (mn)^{k-1}|\hat{b}_{m,n,u+1,v+1}|^k = O(1)$.

*Proof.* In Corollary 2 set $s = k$. □

The results of this paper for single summability are available in [2].

## 3. CONCLUSION

Let $\sum a_v$ denote a series with partial sums $s_n$. For an infinite matrix $A$, the $n$th term of the $A$-transform of $\{s_n\}$ is denoted by

$$t_n = \sum_{v=0}^{\infty} t_{nv} s_v.$$

Recently, Savas [2] established a general absolute inclusion theorem involving a pair of triangles. But the necessary conditions for a general inclusion theorem involving a pair of doubly triangular matrices has not been studied so far. The present paper has filled in a gap in the existing literature.

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## Kahekordsete ridade maatriksmenetluste absoluutse sisalduvuse tarvilikud tingimused

Ekrem Savaş ja B. E. Rhoades

Olgu $A$ kahekordsete ridade kolmnurkne maatriksmenetlus ja $k \geq 1$. Artiklis on defineeritud maatriksmenetlusega $A$ $k$-järku absoluutse summeeruvuse ehk $|A|_k$-summeeruvuse mõiste ja tõestatud teoreem, mis annab tarvilikud tingimused selleks, et kahekordse rea $|A|_k$-summeeruvusest järelduks selle rea $|B|_s$-summeeruvus, kus $B$ on samuti mingi kahekordsete ridade kolmnurkne maatriksmenetlus ning $s \geq k$. Seejuures $s = k$ korral saadakse nimetatud teoreemist efektiivsemad tarvilikud tingimused. Erijuhuna on vaadeldud veel juhtumit, kus $A$ on kahekordsete ridade Rieszi kaalutud keskmiste menetlus. Antud artikli tulemused üldistavad E. Savaşe varasemaid tulemusi (vt *Appl. Math. Comp.*, 2004, **15**, 523–531).