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MATHEMATICS

Retraction: Tangent structures and analytical mechanics

Retraction note: The article "Tangent structures and analytical mechanics" by Maido Rahula published in Proc. Estonian Acad. Sci., 2011, vol. 60, no. 2, 98-103 was retracted by the Estonian Academy Publishers because it duplicated the article with the same title and by the same author in the Balkan Journal of Geometry and its Applications, 2011, vol. 16, 122-127.

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Abstract. We establish a link between the sector-forms of White and the exterior forms of Cartan. We show that the Hamiltonian system on T^2M reduces to Lagrange's equations on the osculating bundle Osc$M$. The structures T^kM and $\mathrm{Osc}^{k-1}M$ are presented explicitly.

Key words: Hamiltonian mechanics, Lagrangian mechanics, differentiable manifolds, tangent structures.

1. INTRODUCTION

Tangent and osculating bundles of smooth manifolds are of fundamental significance. While the osculating bundles correspond to the usual differential calculus (local analysis), the tangent bundles form a basis for the description of higher order motion. This is not a repetition of a differential operator (vector field) $X, X^2, \ldots$, but an iterative process in which the flow of a vector field is exposed to transformation by the flow of another vector field which is influenced by the flow of a third vector field, etc. It is shown that classical Lagrangian mechanics is constructed completely on osculating bundles while the levels (higher order tangent bundles) provide a setting for Hamiltonian mechanics.

2. TANGENT BUNDLES AND OSCULATORS

The tangent functor T iterated k times associates to a smooth manifold M its k-fold tangent bundle T^kM (the kth *level* of M) and associates to a smooth map $\varphi : M_1 \to M_2$ the graded morphism $T^k\varphi : T^kM_1 \to T^kM_2$, the kth *derivative* of φ. The level T^kM has a multiple vector bundle structure with k projections onto $T^{k-1}M$

$$\rho_s \doteq T^{k-s}\pi_s : T^kM \to T^{k-1}M, \quad s = 1,2,\ldots,k,$$

where π_s is the natural projection $T^sM \to T^{s-1}M$.

Local coordinates in neighbourhoods

$$T^sU \subset T^sM,\ s = 1,2,\ldots,k, \quad \text{where } T^{s-1}U = \pi_s(T^sU),$$

are determined automatically by those in the neighbourhood $U \subset M$, the quantities (u^i) being regarded either as coordinate functions on U or as the coordinate components of the point $u \in U$:

$$\begin{aligned} &U: && (u^i),\ i=1,2,\dots,n=\dim M,\\ &TU: && (u^i,u^i_1),\quad \text{with}\ \ u^i \doteq u^i\circ\pi_1,\ u^i_1 \doteq du^i,\\ &T^2U: && (u^i,u^i_1,u^i_2,u^i_{12}),\end{aligned}$$

with $u^i \doteq u^i\circ\pi_1\pi_2,\ u^i_1 \doteq du^i\circ\pi_2,\ u^i_2 \doteq d(u^i\circ\pi_1),\ u^i_{12} \doteq d(du^i)$, etc.

We set up the following convention: *to introduce coordinates on T^kU we take the coordinates on $T^{k-1}U$ and repeat them with an additional index k* – so that a tangent vector is preceded by its point of origin. This indexing is convenient since the symbols with index s thereby become coordinates in the fibre of the projection $\rho_s,\ s=1,2,\dots,k$.

Thus, for example, under the projections $\rho_s : T^3U \to T^2U,\ s=1,2,3$, the coordinates with indices 1, 2, and 3 are each suppressed in turn:

$$\begin{tikzcd} & (u^i\,u^i_1\,u^i_2\,u^i_{12}\,u^i_3\,u^i_{13}\,u^i_{23}\,u^i_{123}) \arrow[dl, "\rho_1"'] \arrow[d, "\rho_2"] \arrow[dr, "\rho_3"] & \\ (u^i\,u^i_2\,u^i_3\,u^i_{23}) & (u^i\,u^i_1\,u^i_3\,u^i_{13}) & (u^i\,u^i_1\,u^i_2\,u^i_{12}). \end{tikzcd}$$

The level T^kM is a smooth manifold of dimension 2^kn and admits an important subspace of dimension $(k+1)n$, called the *osculating bundle* of M of order $k-1$ and denoted $\mathrm{Osc}^{k-1}M$. The bundle $\mathrm{Osc}^{k-1}M$ is determined by the equality of the projections

$$\rho_1=\rho_2=\ldots=\rho_k\,,$$

meaning that an element of T^kM belongs to the bundle $\mathrm{Osc}^{k-1}M$ precisely when all its k projections into $T^{k-1}M$ coincide. In this case all coordinates with the same number of lower indices coincide. For example, the first bundle $\mathrm{Osc}M$ is determined in $T^2U\subset T^2M$ by the equations $u^i_1=u^i_2$, the second bundle Osc^2M in $T^3U\subset T^3M$ by $u^i_1=u^i_2=u^i_3,\ u^i_{12}=u^i_{13}=u^i_{23}$, etc. The coordinates in $\mathrm{Osc}^{k-1}M$ will be denoted by the derivatives of the coordinate functions on U, that is to say $(u^i,du^i,d^2u^i,\dots,d^ku^i)$.

The immersion $\zeta : \mathrm{Osc}M \hookrightarrow T^2M$ and its derivative $T\zeta$ are determined in coordinates by matrix formulae:

$$\begin{pmatrix} u^i\\ u^i_1\\ u^i_2\\ u^i_{12}\end{pmatrix}\circ\zeta=\begin{pmatrix} u^i\\ du^i\\ du^i\\ d^2u^i\end{pmatrix},\qquad \begin{pmatrix} u^i_3\\ u^i_{13}\\ u^i_{23}\\ u^i_{123}\end{pmatrix}\circ T\zeta=\begin{pmatrix} du^i\\ d^2u^i\\ d^2u^i\\ d^3u^i\end{pmatrix},$$

$$T\zeta\Big(\frac{\partial}{\partial u^i},\,\frac{\partial}{\partial(du^i)},\,\frac{\partial}{\partial(d^2u^i)}\Big)=\Big(\frac{\partial}{\partial u^i},\,\frac{\partial}{\partial u^i_1}+\frac{\partial}{\partial u^i_2},\,\frac{\partial}{\partial u^i_{12}}\Big).$$

The fibres of the bundle $\mathrm{Osc}M$ are the integral manifolds of the distribution

$$\langle\partial^1_i+\partial^2_i,\partial^{12}_i\rangle,\quad \text{with}\quad \partial^1_i+\partial^2_i\doteq\frac{\partial}{\partial u^i_1}+\frac{\partial}{\partial u^i_2},\quad \partial^{12}_i\doteq\frac{\partial}{\partial u^i_{12}}.$$

The functions $(u^i_1-u^i_2)$ vanish on $\mathrm{Osc}M$.

Historically, osculating bundles were introduced under various names long before the bundles T^kM. The systematic study was begun 60 years ago by Vagner [10] and culminated in recent times with the Miron–Atanasiu theory [2]. Meanwhile the theme of levels T^kM remained unjustly neglected for the obvious reason that the multiple fibre bundle structure demands a whole new understanding and new approach: see [5,8]. Attempts such as [11] and the so-called synthetic formulation of T^kM [3] made progress in that direction.

While an infinitesimal displacement of the point $u\in M$ is determined by a tangent vector u_1 to M, an infinitesimal displacement of the element $(u,u_1)\in TM$ is determined by the quantities (u_2,u_{12}), representing a tangent vector to TM, etc. This interpretation of the elements of T^kM allows us to develop the theory of higher order motion. Clearly the future belongs to these bundles.

White considers on the level T^kM or on a k-multiple vector bundle certain *sector-forms* which are functions simultaneously linear in all the fibres of k projections; see [11]. In particular, the sector-forms on T^2U and T^3U can be written as

$$\begin{aligned}\Phi &= \varphi_{ij}u_1^i u_2^j + \varphi_i u_{12}^i\,,\\ \Psi &= \psi_{ijk}u_1^i u_2^j u_3^k + \psi_{ij}^1 u_1^i u_{23}^j + \psi_{ij}^2 u_2^i u_{13}^j + \psi_{ij}^3 u_3^i u_{12}^j + \psi_i u_{123}^i\,,\end{aligned}$$

with coefficients in U. For example, in each term of Ψ we see the index 1 (or 2 or 3) appear exactly once. This means that the function Ψ is linear on the fibres of ρ_1 (and ρ_2 and ρ_3).

Any scalar function can be lifted from the level $T^{k-1}M$ to the level T^kM by k different projections $\rho_s : T^kM \to T^{k-1}M$. For example, for the sector-form Φ above there are three possibilities of lifting to T^3M:

$$\Phi\circ\rho_1 = \varphi_{ij}u_2^i u_3^j + \varphi_i u_{23}^i\,,\quad \Phi\circ\rho_2 = \varphi_{ij}u_1^i u_3^j + \varphi_i u_{13}^i\,,\quad \Phi\circ\rho_3 = \varphi_{ij}u_1^i u_2^j + \varphi_i u_{12}^i\,.$$

Proposition. *Every exterior k-form can be regarded as a sector-form in the sense of White, a scalar function on T^kM that is constant on the fibres of* $\mathrm{Osc}^{k-1}M$.

Proof. The sector-form Φ is constant on $\mathrm{Osc}M$ if and only if its derivatives vanish on $\mathrm{Osc}M$. Thus

$$\begin{gathered}\Phi = \varphi_{ij}u_1^i u_2^j + \varphi_i u_{12}^i \;\Rightarrow\; (\partial_i^1 + \partial_i^2)\Phi = \varphi_{ij}u_2^j + \varphi_{ji}u_1^j = (\varphi_{ij}+\varphi_{ji})u_1^j - \varphi_{ij}(u_1^j - u_2^j),\\ \partial_i^{12}\Phi = \varphi_i \;\Rightarrow\; \varphi_{(ij)} = 0,\; \varphi_i = 0\,.\end{gathered}$$

By definition Φ is an antisymmetric bilinear form and can therefore be expressed in the coordinates (u^i, du^i) as a 2-form $\Phi = \varphi_{[ij]}du^i \wedge du^j$. Thus the sector-form Φ is constant on $\mathrm{Osc}M$ if and only if it is a Cartan 2-form.

In the case $k = 3$ the fibres Osc^2M of dimension $3n$ are the integral manifolds of the distribution

$$\langle\, \partial_i^1 + \partial_i^2 + \partial_i^3,\ \partial_i^{23} + \partial_i^{13} + \partial_i^{12},\ \partial_i^{123}\rangle.$$

For the sector-form Ψ (see above) we have

$$\begin{aligned}\Psi = \psi_{ijk}u_1^i u_2^j u_3^k + \psi_{ij}^1 u_1^i u_{23}^j + \psi_{ij}^2 u_2^i u_{13}^j + \psi_{ij}^3 u_3^i u_{12}^j + \psi_i u_{123}^i \;\Rightarrow\quad&\\ (\partial_i^1 + \partial_i^2 + \partial_i^3)\Psi = \;\psi_{ijk}u_2^j u_3^k +\; \psi_{jik}u_1^j u_3^k + \psi_{jki}u_1^j u_2^k + \psi_{ij}^1 u_{23}^j + \psi_{ij}^2 u_{13}^j + \psi_{ij}^3 u_{12}^j\,,&\\ (\partial_i^{23} + \partial_i^{13} + \partial_i^{12})\Psi = \;\psi_{ji}^1 u_1^j + \psi_{ji}^2 u_2^j + \psi_{ji}^3 u_3^j\,,&\\ \partial_i^{123}\Psi = \;\psi_i\,.&\end{aligned}$$

The derivatives vanish on the fibres Osc^2M when the following conditions hold:

$$\varphi_{(ijk)} = 0,\quad \psi_{ij}^1 + \psi_{ij}^2 + \psi_{ij}^3 = 0,\quad \psi_i = 0.$$

These conditions are necessary and sufficient for the sector-form Ψ to be constant on Osc^2M, but not for Ψ to be a Cartan 3-form. However, every 3-form $\tilde{\Psi} = \varphi_{ijk}du^i \wedge du^j \wedge du^k$ can be regarded as a homogeneous sector-form that is constant on Osc^2M.

The argument extends likewise to the cases $k > 3$. □

White's theory of sector-forms is much more extensive than that of Cartan exterior forms. In particular, exterior differentiation is an operation on the set of sector-forms that are constant on the osculating bundles.

There is, however, one inconvenience: sector-forms are represented in natural coordinates in terms which are not invariant. To get rid of this one can use affine connexions and adapted coordinates. In T^2U, for example, the 'bad' coordinates u^i_{12} can be replaced by adapted coordinates $U^i_{12} = \Gamma^i_{jk} u^j_1 u^k_2 + u^i_{12}$ using the coefficients Γ^i_{jk} of the affine connexion. The sector-form Φ is represented by two invariant terms:

$$\Phi = (\varphi_{ij} - \varphi_k \Gamma^k_{ij}) u^i_1 u^j_2 + \varphi_i U^i_{12}\,.$$

In the parentheses we recognize the prototype of the covariant derivative. In fact, for the 1-form $\Theta = \theta_i u^i_1$ the ordinary differential can be written

$$d\Theta = \theta_{i,j} u^i_1 u^j_2 + \theta_i u^i_{12}, \quad \theta_{i,j} = \frac{\partial \theta_i}{\partial u^j}\,,$$

or $d\Theta = \nabla_j \theta_i u^i_1 u^j_2 + \theta_i U^i_{12}$ with the covariant derivative $\nabla_j \theta_i = \theta_{i,j} - \theta_k \Gamma^k_{ij}$.

The connections play an important role here. The local forms appear in the unified and intrinsic structures

$$\Delta_h \oplus \Delta_v \text{ on } TM, \qquad \Delta \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_{12} \text{ on } T^2M, \quad \text{etc.}$$

The theory extends by iteration to the levels T^kM: see [1,9].

3. HAMILTON, LAGRANGE, AND LEGENDRE

The essential importance of the levels TM and T^2M for analytical mechanics was first emphasized by Godbillon [4].

Specifically, Hamiltonian geometry is built on the levels TM and T^2M. Associated to a function $H = H(u, u_1)$ (called the *Hamiltonian*) is the vector field X on TM where

$$X = \sum_i H_{u^i_1} \partial_i - \sum_i H_{u^i} \partial^1_i, \quad H_i \doteq \frac{\partial H}{\partial u^i}, \quad H_{u^i_1} \doteq \frac{\partial H}{\partial u^i_1}\,,$$

for which the flow $a_t = \exp tX$ is determined by the system of differential equations (*Hamiltonian system*)

$$\begin{cases} \dot{u}^i = H_{u^i_1} \\ \dot{u}^i_1 = -H_{u^i} \end{cases}, \qquad \dot{u}^i \doteq \frac{du^i}{dt}\,,\ \dot{u}^i_1 \doteq \frac{du^i_1}{dt}\,.$$

Under the correspondence

$$(u^i, u^i_1, u^i_2, u^i_{12}) \rightsquigarrow (u^i, u^i_1, \dot{u}^i, \dot{u}^i_1)$$

we see this as a section of the bundle $\pi_2 : T^2M \to TM$, of dimension $2n$. The function H and the symplectic form $\Omega = du^i \wedge du^i_1$ [6] are invariant with respect to the vector field X:

$$XH = 0, \quad \mathscr{L}_X \Omega = 0.$$

Theorem. *The Hamiltonian system reduces to Lagrange's equations on the osculating bundle* Osc$M$.

Proof. The passage from the Hamiltonian $H = H(u, u_1)$ to the Lagrangian $L = L(u, u_2)$ ought to be realized through the equation (*Legendre transformation*)[1]

$$H(u, u_1) - \sum_i u^i_1 u^i_2 + L(u, u_2) = 0.$$

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[1] See also [7, p. 3].

However, this equation, which should hold identically on T^2M, is contradictory:

$$d(H - \sum_i u_1^i u_2^i + L) \equiv 0 \;\Rightarrow\; H_{u^i} + L_{u^i} = 0,\; H_{u_1^i} = u_2^i,\; L_{u_2^i} = u_1^i.$$

On the other hand, on Osc$M$ where $u_1^i = u_2^i = \dot{u}$, the passage $H \rightsquigarrow L$ is well determined. On Osc$M$ the Hamiltonian system can be written in Lagrangian form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^i}\right) - \frac{\partial L}{\partial u^i} = 0.$$

The Lagrangian system determines a section of the bundle Osc$M \to TM$, of the same dimension $2n$ as the Hamiltonian system on T^2M. $\square$

The Hamiltonian geometry on the levels T^kM and the Lagrangian geometry on the osculating bundles Osc^{k-1}M for $k > 2$ are structured according to an iterative scheme.

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Puutujastruktuurid ja analüütiline mehaanika

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Puutujafunktor T seab siledale muutkonnale M vastavusse puutujakihtkonna TM ja siledale kujutusele φ selle diferentsiaali $T\varphi$. Itereerides (korrates) funktorit T k-kordselt, ehitame muutkonnale M tema k-nda korruse T^kM ja kujutusele φ k-nda diferentsiaali $T^k\varphi$. Tõustes korruselt korrusele, dimensioon iga kord kahekordistub, st $\dim T^kM = 2^k \dim M$, kus $k = 1, 2, \ldots$. Laskudes aga korruselt T^kM eelmisele korrusele $T^{k-1}M$, on selleks k erinevat võimalust (projektsiooni) $\rho_s \doteq T^{k-s}\pi_s, s = 1, 2, \ldots, k$ (k.a loomulik

projektsioon $\rho_k = \pi_k$). Korruse $T^k M$ elemendid, mille kõik projektsioonid korrusel $T^{k-1}M$ langevad kokku, moodustavad alamkihtkonna – muutkonna M nn kooldumiskihtkonna $\mathrm{Osc}^{k-1}M$.

Muutkonna puutuja- ja kooldumiskihtkondadel on fundamentaalne tähendus. Kui kooldumiskihtkondadele vastab klassikaline diferentsiaalarvutus (*Analysis Situs*), on korrused vajalikud kõrgemat järku liikumiste kirjeldamisel. Sel juhul pole kõne all diferentsiaaloperaatori (vektorvälja) kordamine $X, X^2, \ldots$, vaid iteratiivne protsess, kus ühe vektorvälja voog transformeerub teise vektorvälja voos, sellele omakorda avaldab mõju kolmanda vektorvälja voog jne. Aparatuuriks on invariantne (koordinaatidevaba) Lie-Cartani tehnika.

Ilmneb, et Lagrange'i mehaanika baseerub täielikult kooldumiskihtkondadel $\mathrm{Osc}^{k-1}M$, samal ajal kui Hamiltoni mehaanika taustaks on korrused $T^k M$.