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CONTROL THEORY

# Static state feedback linearizability: relationship between two methods

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**Abstract.** The paper establishes the explicit relationship between two sets of necessary and sufficient conditions for static state feedback linearizability of a discrete-time nonlinear control system. A detailed algorithm is presented for finding the state coordinate transformation. Finally, the methods are compared from the point of view of computational complexity. Two examples illustrate the theoretical results.

**Key words:** static state feedback linearization, discrete-time control systems, nonlinear control system.

## 1. INTRODUCTION

Feedback linearization has proved to be a tremendously useful tool in nonlinear control. We will restrict ourselves to discrete-time nonlinear control systems described by state equations and to static state feedback linearization. The number of publications on static state feedback linearization of continuous-time nonlinear systems is huge, but the situation is different for the discrete-time case [1–6]. Most results, except [7], focus on smooth feedback. However, note that it is not our purpose to address the approximate solutions of the problem (see [8] and the references therein).

The main aim of the paper is to find explicit relationship between the two best-known sets of necessary and sufficient linearizability conditions. The first and the earliest set of conditions is stated in terms of a sequence of nested distributions of vector fields, associated with the control system [3]. The second set of conditions is given via a decreasing sequence of codistributions of differential one-forms [5]. Detailed algorithms are given for computation of the sequences of distributions. Moreover, the connection between the two methods for finding the state coordinate transformation, corresponding to the above linearizability conditions, is clarified. Finally, the alternative conditions and methods are compared from the point of view of computational complexity, and experience from the respective implementations in the symbolic computation system Mathematica is discussed. Two examples are given to illustrate the theoretical results.

The paper is organized as follows. Section 2 gives the statement of the static state feedback linearization problem. Proposition 1, proved in Section 3, specifies the vector fields for which the backward shift is a well-defined operator. A detailed algorithm for computation of the sequence of distributions (that relies on the backward shift) is then given and finally the linearizability conditions in terms of vector fields are recalled. Section 4 recalls the linearizability conditions formulated in terms of codistributions and Section 5 finds the explicit relationship between two sets of conditions. Section 6 relates the methods for finding the new state coordinates, corresponding to two alternative linearizability conditions. Section 7 contains the discussion and the examples, and Section 8 concludes the paper.

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## 2. STATIC STATE FEEDBACK LINEARIZABILITY

Consider a discrete-time nonlinear control system $\Sigma$ of the form

$$x^+ = f(x,u), \tag{1}$$

where $f\colon\ (M\times U)\to M^+$, and the variables $x=(x_1,\dots,x_n)$ and $u=(u_1,\dots,u_m)$ are the local coordinates of the state space $M$ and the input space $U$, respectively; $f$ is assumed to be an analytic function of its arguments, and $M^+$ is the forward shifted state space with the coordinates $x^+=(x_1^+,\dots,x_n^+)$.

In the discrete-time case the local study is useless around an arbitrary initial state, since even in one step the state can move far away from the initial state, regardless of the small control. One possibility is to work around an equilibrium point of the system. Another, more general possibility is to work around a reference trajectory that satisfies the system of equations (1). In this paper we adopt the first option.

**Definition 1.** *A point $(x_0,u_0)\in M\times U$ is called an equilibrium point of system* (1) *if $f(x_0,u_0)=x_0$.*

In the study of discrete-time nonlinear control systems the following assumption is usually made. It guarantees the forward shift operator, defined by equations (1), to be injective. Note that this assumption is not restrictive as it is always satisfied for accessible systems [9].

**Assumption 1.** *System* (1) *is, around the equilibrium point $(x_0,u_0)$, submersive, i.e. around $(x_0,u_0)$,* $\operatorname{rank}[\partial f(x,u)/\partial(x,u)]=n$.

**Definition 2.** *A regular static state feedback $u=\alpha(x,v)$ is a mapping $\alpha\colon M\times V\to U$ such that locally, around the equilibrium point,* $\operatorname{rank}[\partial\alpha(\cdot)/\partial v]=m$.

**Definition 3.** [3,4] *System* (1) *is said to be static state feedback linearizable around its equilibrium point $(x_0,u_0)$ if there exist*

1. *a state coordinate transformation $S:M\to M$ defined on a neighbourhood $X$ of $x_0$ with $S(x_0)=0$,*
2. *a regular static state feedback of the form $u=\alpha(x,v)$, satisfying the condition $\alpha(x_0,0)=u_0$ and defined on a neighbourhood $X\times\mathscr{O}$ of $(x_0,0)$ such that, in the new coordinates $z=S(x)$, the compensated system reads $z^+=Az+Bv$, where the pair $(A,B)$ is in Brunovsky canonical form.*

## 3. LINEARIZABILITY CONDITIONS IN TERMS OF VECTOR FIELDS

In [3] the local linearizability conditions for system (1) around $(x_0,u_0)$ are formulated in terms of a sequence of distributions $D_k\subset T(M\times U)$, $k=0,\dots,n$, associated with system (1). Here $T(M\times U)$ is the tangent fibre bundle of the space $M\times U$, where the fibres are the tangent vector spaces defined at each point $(x,u)\in M\times U$.

Define the distribution $K$ as the kernel of the tangent map $Tf$ of the state transition map $f\colon\ Tf(K)=0$. Under Assumption 1, $\dim K=m$. Moreover, the distribution $K$ as a kernel of a tangent map is always involutive. Therefore, one can define the commutable basis

$$K=\operatorname{span}\{K_i,\ \ i=1,\dots,m\} \tag{2}$$

such that

$$[K_i,K_j]\equiv 0, \tag{3}$$

for $i,j=1,\dots,m$. In the neighbourhood of $(x_0,u_0)$ the sequence of distributions $D_k$, $k=0,\dots,n$, is introduced:

$$\begin{aligned} D_0 &= \operatorname{span}\left\{\frac{\partial}{\partial u_1},\dots,\frac{\partial}{\partial u_m}\right\},\\ D_{k+1} &= \Delta_{k+1}+D_k,\quad k=0,\dots,n-1, \end{aligned} \tag{4}$$

where, provided the distribution $D_k + K$ is involutive and $D_k \cap K$ is constant dimensional, $\Delta_{k+1}$ can be found from

$$\Delta_{k+1}^{+} \quad = \quad Tf(D_k) \tag{5}$$

by applying the backward shift [3][1]. Note that $\Delta_{k+1}$ is defined by (5) as the span of vector fields $\{Tf(D_k)\}^{-}$; however, these vector fields do not necessarily exist, in general, since the *backward shift operator is a well-defined operation* on $TM^{+}$ iff the *basis vectors of $D_k$ respect the distribution $K$*, i.e., if for every basis element $\Xi$ of $D_k$ and for the vector field $K_i \in K$

$$[\Xi, K_i] \in K \tag{6}$$

holds.

Consider an arbitrary vector field $\Xi$, defined on the manifold $M \times U$:

$$\Xi = \sum_{I=1}^{n} \xi_I(x,u) \frac{\partial}{\partial x_I} + \sum_{i=1}^{m} \xi_i(x,u) \frac{\partial}{\partial u_i}. \tag{7}$$

Multiplying (7) with the Jacobi matrix of submersion $f$ gives the vector field

$$Tf(\Xi) = \sum_{I=1}^{n} \left( \sum_{J=1}^{n} \frac{\partial f_I}{\partial x_J} \xi_J + \sum_{j=1}^{m} \frac{\partial f_I}{\partial u_j} \xi_J \right) \frac{\partial}{\partial x_I^{+}} = \sum_{I=1}^{n} \{Tf(\Xi)\}_I \frac{\partial}{\partial x_I^{+}}. \tag{8}$$

Although this vector field belongs to $TM^{+}$, its components $\{Tf(\Xi)\}_I$ are still expressed as the functions of the coordinates $(x,u)$. The vector field $Tf(\Xi) \in TM^{+}$ is defined on the manifold $M^{+}$ iff one can, using equations (1), write its components $\{Tf(\Xi)\}_I$ in terms of the coordinates $x^{+}$ exclusively. The latter is possible if the components of the vector field (8) can be expressed as the composite functions $\{Tf(\Xi)\}_I(x,u) = \Psi_I \circ f(x,u)$ for some functions $\Psi_I$, yielding

$$Tf(\Xi) = \sum_{I=1}^{n} \Psi_I\left(x^{+}\right) \frac{\partial}{\partial x_I^{+}}. \tag{9}$$

**Proposition 1.** *The vector field* (8) *can be expressed in the form* (9) *iff* (6) *holds.*

*Proof.* The submersion $f$ defines the fibre bundle on the manifold $M \times U$. The base manifold of this fibre bundle is $M^{+}$ with coordinates $x^{+}$. The fibres are the integral surfaces of the distribution $K$. By (3) one can choose the fibre coordinates to be the canonical parameters $\chi_i = g_i(x,u)$ of the vector fields $K_j$:

$$\left\langle \mathrm{d}g_i, K_j \right\rangle \equiv 0, \quad i,j = 1,\ldots,m. \tag{10}$$

Therefore, the base coordinates $x^{+}$ and the fibre coordinates $\chi^{+}$ are defined by the following coordinate transformation, denoted by $F$:

$$\begin{aligned} x^{+} &= f(x,u), \\ \chi^{+} &= g(x,u). \end{aligned}$$

The corresponding Jacobi matrix reads

$$TF = \begin{pmatrix} \dfrac{\partial f_I}{\partial x_J} & \dfrac{\partial f_I}{\partial u_j} \\ \dfrac{\partial g_i}{\partial x_J} & \dfrac{\partial g_i}{\partial u_j} \end{pmatrix}.$$

Multiplying the vector field $\Xi$ (see (7)) by $TF$ yields

$$TF(\Xi) = Tf(\Xi) + \sum_{i=1}^{m} \left( \sum_{J=1}^{n} \frac{\partial g_i}{\partial x_J} \xi_J + \sum_{j=1}^{m} \frac{\partial g_i}{\partial u_j} \xi_j \right) \frac{\partial}{\partial \chi_i^{+}} \tag{11}$$

[1] Note that in [3] and [4] the notation $\Delta_{k+1} = Tf(D_k)$ is used, which may cause confusion as the distribution $Tf(D_k)$ belongs to the tangent bundle $TM^{+}$, and must be shifted backward to $TM$ to obtain the distribution $\Delta_{k+1}$.

with the second term belonging to the distribution $K$. Its components can be expressed in terms of the base coordinates $x^+$ exclusively iff they do not depend on the fibre coordinates $\chi_i^+$. According to (10), this is equivalent to the property that the Lie derivatives

$$L_{K_i}\{Tf(\Xi)\}_I \equiv 0 \tag{12}$$

for $i = 1,\ldots,m$, $I = 1,\ldots,n$. Due to (8), the components of $Tf(\Xi)$ are actually the scalar products

$$\{Tf(\Xi)\}_I = \left\langle dx_I^+ \circ Tf, \Xi \right\rangle. \tag{13}$$

Replacing $\{Tf(\Xi)\}_I$ in (12) by the right-hand side of (13), we obtain, using the differentiation rule of the scalar product,

$$L_{K_i}\left\langle \mathrm{d}x_I^+ \circ Tf, \Xi \right\rangle = \left\langle \mathrm{d}\left\langle \mathrm{d}x_I^+ \circ Tf, K_i \right\rangle, \Xi \right\rangle + \left\langle \mathrm{d}x_I^+ \circ Tf, [K_i, \Xi] \right\rangle \equiv 0. \tag{14}$$

Note that $\left\langle \mathrm{d}x_I^+ \circ Tf, K_i \right\rangle = \left\langle \mathrm{d}x_I^+, Tf(K_i) \right\rangle \equiv 0$ since $K_i \in K$ and therefore, $Tf(K_i) \equiv 0$ for all $i = 1,\ldots,m$. Consequently, $\left\langle \mathrm{d}x_I^+ \circ Tf, [K_i, \Xi] \right\rangle \equiv 0$ for all $i = 1,\ldots,m$ whenever the Lie brackets $[K_i, \Xi] \in K$ for $i = 1,\ldots,m$. □

Example 1 in Section 7 illustrates the both situations, i.e. when $Tf(\Xi)$ cannot be shifted back and when it can be shifted back; see formulas (37) and (38), respectively.

Below, detailed algorithms are given for computation of the sequences $D_k$ and $\Delta_k$, respectively. Both algorithms require the state equations of the control system to be given. Algorithm 1 gives the sequence $D_k$, or informs that the computation of the sequence cannot be completed, because either $D_{k-1} + K$ is not involutive or $D_{k-1} \cap K$ is not constant dimensional around the equilibrium point. In the latter case the control system is not static state feedback linearizable. Algorithm 2 computes the sequence $\Delta_k$.

**Algorithm 1: Computation of the sequence $D_k$**

**Initialization.** Define the distribution $K$ as the kernel of the tangent map $Tf$ of the state transition map $f$ in (1), $Tf(K) = 0$, and $D_0$ as in (4). Set $D_0 := \mathrm{span}\{\Xi^0\}$, such that the vector fields $\Xi^0$ respect $K$, $k := 1$ and go to step $k$.

**Step $k$.** Check whether $D_{k-1} + K$ is involutive and $D_{k-1} \cap K$ constant dimensional around $(x_0, u_0)$. If not, then stop (the system is not static state feedback linearizable), otherwise continue.

Note that some vector fields, added at the previous step (or for step 1 at initialization) into $D_{k-1}$ may not respect the distribution $K$. If this is so, find the same number of independent linear combinations of $\Xi^0,\ldots,\Xi^{k-2}$ and $[Tf(\Xi^{k-2})]^-$, denoted by $\Xi^{k-1}$ that respect the distribution $K$, and be such that $D_{k-1} = \mathrm{span}\{\Xi^0, \Xi^1, \ldots, \Xi^{k-1}\}$, $\Xi^{k-1} \notin D_{k-2}$. Finally, define[2]

$$D_k = \mathrm{span}\left\{\Xi^0, \ldots, \Xi^{k-1}, [Tf(\Xi^{k-1})]^-\right\}$$

and check whether $\dim D_k = n + m$. If yes, then set $N := k$ and stop (the sequence $D_k$ has been found), otherwise check whether $\dim D_k > \dim D_{k-1}$. If yes, then set $\Xi^k := [Tf(\Xi^{k-1})]^-$, $k := k+1$ and go to the next step. Otherwise, the system is not static state feedback linearizable.

Note that $N$ is the minimal integer such that $\dim D_N = \dim D_{N+1} = \ldots = n + m$.

**Algorithm 2: Computation of the sequence $\Delta_k$**

Algorithm 1 can be modified accordingly to compute the sequence $\Delta_k$. In that case one starts with $\Delta_1 = \mathrm{span}\{\Xi^1\}$ instead of $D_0$, and at the $k$th ($k = 2,\ldots,N$) step defines $\Delta_k = \mathrm{span}\{\Xi^1,\ldots,\Xi^k\}$.

The linearizability conditions for system (1) are formulated in the following theorem.

[2] Some of the vector fields $Tf(\Xi^{k-1})$ may be zero since the corresponding vector fields $\Xi^{k-1} \in K$.

**Theorem 1.** [3] *System* (1) *is static state feedback linearizable around the equilibrium point* $(x_0, u_0)$ *iff*
1. *the distributions* $D_k + K$ *are involutive and* $D_k \cap K$ *are constant dimensional for all* $k = 0, \ldots, N-1$;
2. $\dim D_N = n + m$.

## 4. LINEARIZABILITY CONDITION IN TERMS OF DIFFERENTIAL ONE-FORMS

In [5] the linearizability conditions for system (1) are given in terms of the decreasing sequence of codistributions $\mathscr{H}_k \subset T^*M$, defined locally around the equilibrium point by[3]

$$\mathscr{H}_1 = \operatorname{span}\{\mathrm{d}x\}, \quad \mathscr{H}_{k+1} = \operatorname{span}\left\{\omega \in \mathscr{H}_k \,|\, \omega^+ \circ Tf \in \mathscr{H}_k\right\}, \quad k \geq 1. \tag{15}$$

There exists an integer $N^* \leq n$ such that for $1 \leq k \leq N^*$, $\mathscr{H}_{k+1} \subset \mathscr{H}_k$ but $\mathscr{H}_{N^*+1} \neq \mathscr{H}_{N^*}$ and $\mathscr{H}_{N^*+1} = \mathscr{H}_{N^*+2} := \mathscr{H}_\infty$. Obviously, $N^*$ is the minimal integer satisfying $\mathscr{H}_{N^*+1} = \mathscr{H}_{N^*+2}$ and $\mathscr{H}_\infty$ is the maximal codistribution, invariant with respect to the forward shift. To compute the one-forms $\omega^+ \circ Tf$ in (15), consider first the forward shift of an arbitrary one-form

$$\omega \;=\; \sum_{I=1}^{n} \omega_I(x)\mathrm{d}x_I \in TM, \tag{16}$$

obtained by replacing all the state coordinates $x_I$ in (16) by their forward shifts $x_I^+$:

$$\omega^+ = \sum_{I=1}^{n} \omega_I(x^+)\mathrm{d}x_I^+ \in TM^+.$$

Then the one-form

$$\omega^+ \circ Tf = \sum_{J=1}^{n} \omega_J(x^+)\left(\sum_{I=1}^{n} \frac{\partial f_J}{\partial x_I}\mathrm{d}x_I + \sum_{i=1}^{m} \frac{\partial f_J}{\partial u_i}\mathrm{d}u_i\right) \in T(M \times U),$$

where, by replacing $x^+$ in $\omega_J(x^+)$ by $f(x,u)$, one obtains

$$\omega^+ \circ Tf = \sum_{J=1}^{n} \omega_J \circ f(x,u)\left(\sum_{I=1}^{n} \frac{\partial f_J}{\partial x_I}\mathrm{d}x_I + \sum_{i=1}^{m} \frac{\partial f_J}{\partial u_i}\mathrm{d}u_i\right) \in T(M \times U).$$

The linearizability conditions for system (1) are formulated in the following theorem.

**Theorem 2.** [5] *System* (1) *is static state feedback linearizable around the equilibrium point* $(x_0, u_0)$ *iff*
1. $\mathscr{H}_k$ *is completely integrable for all* $k = 1, \ldots, N^*$,
2. $\mathscr{H}_\infty := \mathscr{H}_{N^*+1} = \{0\}$.

In [5] the complete integrability has been checked via the Frobenius Theorem.

**Theorem 3** (Frobenius)**.** *Let* $W = \operatorname{span}\{\omega_1, \ldots, \omega_r\}$ *be a vector space of* 1*-forms.* $W$ *is locally completely integrable iff* $\mathrm{d}\omega_k \wedge \omega_1 \wedge \ldots \wedge \omega_r = 0$ *for all* $k = 1, \ldots, r$.

However, Theorem 4 below suggests an alternative rule which does not require the computation of exterior derivatives of the basis vectors of $\mathscr{H}_k$, as well as their wedge products with the basis vectors of $\mathscr{H}_k$. The proof of Theorem 4 is based on the following Lemma.

[3] In [5] the condition is given as $\omega^+ \in \mathscr{H}_k$. However, this may yield confusion because the one-forms $\omega^+$ (as the forward shifts of the one-forms $\omega \in TM$) belong to $TM^+$ and the elements of $\mathscr{H}_k$ belong to $TM$. Therefore, one has to "bring back" the one-forms $\omega^+$ to $TM$ with the help of the tangent map $Tf$.

**Lemma 1.** *If the codistributions $\mathscr{H}_k, \mathscr{H}_{k+1}, \ldots, \mathscr{H}_1$ are integrable, or equivalently, one can define the state coordinates $X$, adapted to $\mathscr{H}_k, \mathscr{H}_{k-1}, \ldots, \mathscr{H}_1$, such that*

$$\begin{aligned} &\bar{X}^k_{r_k} = \bar{\sigma}^k_{r_k}(x), \qquad r_k = 1, \ldots, \dim \mathscr{H}_k, \\ &X^\lambda_{i_\lambda} = \sigma^\lambda_{i_\lambda}(x), \qquad i_\lambda = 1, \ldots, n_\lambda, \\ &\mathscr{H}_\lambda = \operatorname{span}\left\{ \mathrm{d}\bar{X}^k_{r_k}, \mathrm{d}X^{k-1}_{i_{k-1}}, \ldots, \mathrm{d}X^\lambda_{i_\lambda} \right\}, \\ &\forall \lambda = 1, \ldots, k-1, \end{aligned} \tag{17}$$

**if and only if** *the forward shift $\left(\omega^k_{\mu_k}\right)^+ \circ Tf$ of an arbitrary 1-form $\omega \in \mathscr{H}_k$ can be expressed in terms of $\left\{ \bar{X}^k_{r_k}, X^{k-1}_{i_{k-1}} \right\}$ only.*

*Proof.* Each codistribution $\mathscr{H}_\lambda$ for all $\lambda = 1, \ldots, k$ defines a corresponding subspace $\mathscr{B}_\lambda \subset M$ such that $\mathscr{H}_\lambda$ is the cotangent bundle of $\mathscr{B}_\lambda$ and $\dim \mathscr{H}_\lambda = \dim \mathscr{B}_\lambda$ if and only if all $\mathscr{H}_\lambda$ are integrable. Or, equivalently, the integrals $\left\{ \bar{X}^k_{r_k}, X^{k-1}_{i_{k-1}}, \ldots, X^\lambda_{i_\lambda} \right\}$ of $\mathscr{H}_\lambda$ can also be defined as the coordinates of the subspace $\mathscr{B}_\lambda$. This means that if an arbitrary 1-form $\omega \in \mathscr{H}_\lambda$ is defined in the subspace $\mathscr{B}_\lambda$, then its components can be expressed only in terms of variables $\left\{ \bar{X}^k_{r_k}, X^{k-1}_{i_{k-1}}, \ldots, X^\lambda_{i_\lambda} \right\}$ as the integrals of $\mathscr{H}_\lambda$.

Finally, according to the definition of the codistributions $\mathscr{H}_k$ for an arbitrary 1-form $\omega \in \mathscr{H}_k$ its forward shift $\omega^+ \circ Tf$ belongs to $\mathscr{H}_{k-1}$, which is assumed to be integrable. That is, the components of $\omega^+ \circ Tf$ can be expressed only in terms of the integrals $\left\{ \bar{X}^k_{r_k}, X^{k-1}_{i_{k-1}} \right\}$ of $\mathscr{H}_{k-1}$. □

**Theorem 4.** *If the codistribution $\mathscr{H}_k$ is not integrable, then there does not exist the complete cobasis for $\mathscr{H}_k$, where the components of the basis 1-forms can be expressed without using the backward shifts of the state coordinates.*

*Proof.* Suppose $\mathscr{H}_k$ is not integrable, but $\mathscr{H}_{k-1}, \ldots, \mathscr{H}_1$ are; then the number of the independent integrals $\bar{X}^k_{\bar{r}_k}$ of $\mathscr{H}_k$ equals $\dim \bar{\mathscr{H}}_k$, where $\bar{\mathscr{H}}_k \subset \mathscr{H}_k$ is the maximal integrable subspace of $\mathscr{H}_k$. Then one can define at least $\dim \mathscr{H}_k - \dim \bar{\mathscr{H}}_k$ independent 1-forms $\omega^k_{\lambda_k}$, where $\dim \bar{\mathscr{H}}_k + 1 \le \lambda_k \le \dim \mathscr{H}_k$, and whose components cannot be expressed in terms of integrals $\left\{ \bar{X}^k_{\bar{r}_k} \right\}$. Consequently, we need some additional variables $I^k_{\mu_k}$ to express the components of $\omega^k_{\lambda_k}$, since the other adapted coordinates $\left\{ X^{k-1}_{i_{k-1}}, \ldots, X^1_{i_1} \right\}$ cannot be used for this purpose, as will be shown below.

First we will prove that the components of $\omega^k_{\lambda_k} \notin \bar{\mathscr{H}}_k$, but $\omega^k_{\lambda_k} \in \mathscr{H}_k$, can be expressed in terms of $\left\{ \left(\bar{X}^k_{r_k}\right)^+, \left(I^k_{\mu_k}\right)^+ \right\}$, and alternatively also in terms of $\left\{ \bar{X}^k_{r_k}, X^{k-1}_{i_{k-1}} \right\}$. The latter yields that the forward shifts $\left(I^k_{\mu_k}\right)^+$ must also be the integrals of $\mathscr{H}_{k-1}$ and the variables $I^k_{\mu_k}$ are the backward shifts of some integrals $X^{k-1}_{i_{k-1}}$ of $\mathscr{H}_{k-1}$.

Really, for $\omega^k_{\lambda_k} \notin \bar{\mathscr{H}}_k$, but $\omega^k_{\lambda_k} \in \mathscr{H}_k$, its components can be expressed in terms of $\left\{ \bar{X}^k_{r_k}, I^k_{\mu_k} \right\}$, where $1 \le r_k \le \dim \bar{\mathscr{H}}_k$ and $\dim \bar{\mathscr{H}}_k + 1 \le \mu_k \le \dim \mathscr{H}_k$. Consequently, the components of the forward shifts $\left(\omega^k_{\lambda_k}\right)^+ \circ Tf$ of the 1-forms $\omega^k_{\lambda_k}$ can be expressed in terms of $\left\{ \left(\bar{X}^k_{r_k}\right)^+, \left(I^k_{\mu_k}\right)^+ \right\}$. At the same time, according to the definitions of the sequence of codistributions $\mathscr{H}_k$, the 1-forms $\left(\omega^k_{\lambda_k}\right)^+ \circ Tf$ belong to $\mathscr{H}_{k-1}$. Due to the integrability of $\mathscr{H}_{k-1}$, the components of $\left(\omega^k_{\lambda_k}\right)^+ \circ Tf \in \mathscr{H}_{k-1}$ can, according to Lemma 1, be expressed in terms of the integrals $\left\{ \bar{X}^k_{r_k}, X^{k-1}_{i_{k-1}} \right\}$ of $\mathscr{H}_{k-1}$.

Let us prove now that the variables $I^k_{\mu_k}$ cannot be the functions of other variables $\left\{X^{k-1}_{i_{k-1}},\dots,X^1_{i_1}\right\}$. The proof is by contradiction. Suppose that one can express $I^k_{\mu_k}$ in terms of $\left\{X^{k-1}_{i_{k-1}},\dots,X^1_{i_1}\right\}$. That is, one can alternatively express the components of $\omega^k_{\lambda_k}\in\mathscr{H}_k$ as the functions of variables $\left\{\bar{X}^k_{r_k},I^k_{\mu_k}\right\}$ in terms of $\left\{\bar{X}^k_{r_k},X^{k-1}_{i_{k-1}},\dots,X^1_{i_1}\right\}$. Consequently, the components of $\left(\omega^k_{\lambda_k}\right)^+\circ Tf\in\mathscr{H}_{k-1}$ can be expressed in terms of the forward shifts $\left\{\left(\bar{X}^k_{r_k}\right)^+,\left(X^{k-1}_{i_{k-1}}\right)^+,\dots,\left(X^1_{i_1}\right)^+\right\}$. However, according to the definition of codistributions $\mathscr{H}_k$, the forward shifts $\left\{\left(X^{k-1}_{i_{k-1}}\right)^+,\dots,\left(X^1_{i_1}\right)^+\right\}$ are not the integrals of $\mathscr{H}_{k-1}$, which leads to contradiction.

To conclude, if $\mathscr{H}_k$ is not integrable, then the variables $I^k_{\mu_k}$ are the backward shifts of the coordinates $X^{k-1}_{i_{k-1}}$, but one cannot express them in terms of coordinates $X$. Therefore, one cannot express the components of all its basis 1-forms in terms of the adapted coordinates $X$ as in (17); one needs to use additionally the backward shifts of the state coordinates. □

The converse statement of Theorem 5 also holds. The proof is based on the application of the Frobenius Theorem and is trivial.

## 5. RELATIONSHIP BETWEEN THE LINEARIZABILITY CONDITIONS

In order to find the explicit relationship between the two sets of linearizability conditions, formulated by Theorems 1 and 2, respectively, we first reformulate Theorem 1 in terms of the distributions $\Delta_k$. In order to do so, we need the following lemma.

**Lemma 2.** *The distribution $D_k+K$ is involutive and $D_k\cap K$ is constant dimensional iff the distribution $\Delta_{k+1}$ is involutive and constant dimensional for $k=0,\dots,N-2$.*

*Proof.* We prove first that $\dim\Delta_{k+1}=\dim(D_k)-\dim(D_K\cap K)$. Due to (2), $D_K+K=\mathrm{span}\{\Xi_\alpha,\ \ \alpha=1,\dots,\rho_k,\ \ K_i,i=1,\dots,m\}$; there $\Xi_\alpha\in D_K$, $\alpha=1,\dots,\rho_k$ are linearly independent vector fields such that $Tf(\Xi_\alpha)\not\equiv 0$. Since $\rho_k=\dim(D_K+K)-\dim K=\dim D_K+\dim K-\dim(D_K\cap K)-\dim K$,

$$\rho_k=\dim D_k-\dim(D_k\cap K). \tag{18}$$

As $K=\ker Tf$, one can rewrite (5) as

$$\Delta^+_{k+1} \quad = \quad Tf(D_k+K), \tag{19}$$

yielding

$$\Delta^+_{k+1} \quad = \quad \mathrm{span}\left\{Tf(\Xi_\alpha)\right\}. \tag{20}$$

Because the number of vector fields $\Xi_\alpha$ (and so also $Tf(\Xi_\alpha)$) is $\rho_k$, $\dim\Delta^+_{k+1}=\dim D_k-\dim(D_k\cap K)$.

Next we prove that $\Delta_{k+1}$ is involutive iff $D_k+K$ is involutive. Assume first that $D_k+K$ is involutive, then all the possible Lie brackets of the vector fields $\Xi_\alpha$ and $K_i$ must also belong to $D_k+K$. By (3),

$$[K_i,K_j]\in D_K+K. \tag{21}$$

According to Proposition 1, the vector fields $Tf(\Xi_\alpha)$ exist iff $\Xi_\alpha$ *respect the distribution* $K$, meaning

$$[\Xi_\alpha,K_i]\in K,\ \ \alpha=1,\dots,\rho_k,\ \ i=1,\dots,m. \tag{22}$$

Finally, the third involutivity condition for $D_k+K$ reads

$$\left[\Xi_\alpha,\,\Xi_\beta\right] \quad = \quad \sum_{\gamma=1}^{\rho_k}c_{\alpha\beta\gamma}\Xi_\gamma+\sum_{j=1}^{m}c_{\alpha\beta j}K_j. \tag{23}$$

We have to prove next that all the possible Lie brackets of the vector fields $Tf(\Xi_\alpha)$ belong to the distribution $\Delta^+_{k+1}$, defined by (20), since then and only then the distribution $\Delta^+_{k+1}$ (and also $\Delta_{k+1}$) is involutive. Applying the tangent map $Tf$ to (23) and taking into account $Tf(K) \equiv 0$ yields for all $\alpha$ and $\beta$

$$Tf\left(\left[\Xi_\alpha, \Xi_\beta\right]\right) = \sum_{\gamma=1}^{\rho_k} c_{\alpha\beta\gamma} Tf(\Xi_\gamma). \tag{24}$$

According to the property of the Lie brackets (see, e.g., [4], p. 50),

$$Tf\left(\left[\Xi_\alpha, \Xi_\beta\right]\right) \quad = \quad \left[Tf(\Xi_\alpha), Tf(\Xi_\beta)\right] \tag{25}$$

holds and, consequently, because of (24),

$$\left[Tf(\Xi_\alpha), Tf(\Xi_\beta)\right] \quad = \quad \sum_{\gamma=1}^{\rho_k} c_{\alpha\beta\gamma} Tf(\Xi_\gamma), \tag{26}$$

proving the involutivity of $\Delta_{k+1}$.

Conversely, assume that $\Delta_{k+1}$ as the backward shift of $\Delta^+_{k+1}$, defined by (20), is involutive. Then (26) holds and due to (25), also (23) must hold. Since the conditions (21) and (22) are also satisfied, the involutivity of $D_k + K$ follows. □

We are now ready to reformulate Theorem 1.

**Theorem 5.** *System* (1) *is static state feedback linearizable around the equilibrium point* $(x_0, u_0)$, *iff*
1. *the distributions* $\Delta_{k+1}$ *are involutive and constant dimensional for all* $k = 0, \ldots, N-1$;
2. $\dim\Delta_N = n$.

We will prove that all distributions $\Delta_k$, $k = 1, \ldots, N-1$ are involutive and *constant dimensional* iff all codistributions $\mathscr{H}_k$, $k = 1, \ldots, N$ are integrable and *constant dimensional*. For that it is enough to show that $\Delta_k$ is the maximal annihilator of $\mathscr{H}_{k+1}$ for $k = 1, \ldots, N-1$.

**Theorem 6.** *The codistribution* $\mathscr{H}_{k+1}$ *is the maximal annihilator of the distribution* $\Delta_k$, *for* $k = 1, \ldots, N-1$. *That is, on the manifold* $M$, $\langle \mathscr{H}_{k+1}, \Delta_k \rangle \equiv 0$, *and* $\operatorname{codim}\mathscr{H}_{k+1} = \dim\Delta_k$.

*Proof.* The proof is by induction. First we prove that the codistribution $\mathscr{H}_2$ is the maximal annihilator of the distribution $\Delta_1$. Due to definitions (4) and (15), obviously $\langle \mathscr{H}_1, D_0 \rangle \equiv 0$. Due to $\mathscr{H}_2^+ \circ Tf \in \mathscr{H}_1$ also

$$\left\langle \mathscr{H}_2^+ \circ Tf, D_0 \right\rangle \quad \equiv \quad 0. \tag{27}$$

The left-hand side of (27) may be interpreted as the multiplication of three matrices. Matrix $\mathscr{H}_2^+$ is a $\dim\mathscr{H}_2 \times n$-matrix, whose rows are the basis elements of $\mathscr{H}_2^+$. The second one, $Tf$, is the $n \times (n+m)$ Jacobi matrix of the state transition map $f$. The third matrix $D_0$ is the $(n+m) \times m$-matrix, whose columns are the basis vectors of $D_0$. Due to the associativity property of matrix multiplication, one can rewrite (27) as

$$\left\langle \mathscr{H}_2^+, Tf(D_0) \right\rangle \equiv \left\langle \mathscr{H}_2^+, \Delta_1^+ \right\rangle \equiv 0,$$

and as the value of a constant does not change by the backward shift, the latter yields $\langle \mathscr{H}_2, \Delta_1 \rangle \equiv 0$. Therefore, $\mathscr{H}_2$ annihilates $\Delta_1$ on the manifold $M$.

To show that $\mathscr{H}_2$ is also the *maximal* annihilator of $\Delta_1$, suppose conversely that there exists a 1-form $\omega \in TM$ not belonging to $\mathscr{H}_2$ but still annihilating $\Delta_1$: $\langle \omega, \Delta_1 \rangle = 0$ and $\omega \notin \mathscr{H}_2$. The relationship $\langle \omega, \Delta_1 \rangle = 0$ must be valid also after applying to it the forward shift and the associativity property of matrix multiplication

$$\left\langle \omega^+, \Delta_1^+ \right\rangle = \left\langle \omega^+, Tf(D_0) \right\rangle = \left\langle \omega^+ \circ Tf, D_0 \right\rangle = 0. \tag{28}$$

According to (28), $\omega^+ \circ Tf$ must belong to $\mathscr{H}_1$ as the maximal annihilator of $D_0$. From above, we have $\omega^+ \circ Tf \in \mathscr{H}_1$ and $\omega \notin \mathscr{H}_2$, which leads to a contradiction. Therefore, $\mathscr{H}_2$ is the maximal annihilator of $\Delta_1$ on the manifold $M$, or, equivalently, due to (4), the maximal annihilator of $D_1$ on the manifold $M \times U$.

Next suppose that $\mathscr{H}_k$ is the maximal annihilator of $D_{k-1}$ on the manifold $M \times U$. Then $\langle \mathscr{H}_k, D_{k-1} \rangle = 0$. According to definition (15), $\mathscr{H}_{k+1}^+ \circ Tf \in \mathscr{H}_k$, yielding $\left\langle \mathscr{H}_{k+1}^+ \circ Tf, D_{k-1} \right\rangle = 0$.

Again, due to the associativity property of a matrix multiplication, definition (15), and the invariance of a scalar product with respect to the (backward) shift, we may write

$$\left\langle \mathscr{H}_{k+1}^+, Tf(D_{k-1}) \right\rangle = \left\langle \mathscr{H}_{k+1}^+, \Delta_k^+ \right\rangle = 0 \Rightarrow \langle \mathscr{H}_{k+1}, \Delta_k \rangle = 0.$$

So, $\mathscr{H}_{k+1}$ annihilates $\Delta_k$. In order to show that $\mathscr{H}_{k+1}$ is also the *maximal* annihilator of $\Delta_k$ on $M$, suppose contrarily the existence of a one-form $\theta \in TM$ such that $\theta \notin \mathscr{H}_{k+1}$ and $\langle \theta, \Delta_k \rangle = 0$. As before, one can show that this leads to a contradiction. Consequently, $\mathscr{H}_{k+1}$ is the maximal annihilator of $\Delta_k$ on $M$. Moreover, because $\dim \Delta_N = n$ ja $\mathscr{H}_{N^*+1}$ is the first zero codistribution in the sequence, also $N^* = N$. □

**Corollary 1.**

1. *From involutivity and constant dimensionality of distribution $\Delta_k$ integrability and constant dimensionality of codistribution $\mathscr{H}_{k+1}$ follows for all $k = 1, \ldots, N-1$ and vice versa.*
2. $\mathscr{H}_\infty = \{0\}$ *iff* $\dim \Delta_N = n$, *or, equivalently*, $\dim D_N + K = m + n$.

*Proof.* Follows directly from Theorem 6. □

The result of Corollary 1 enables combination of both methods. This will be done in Section 6, where the state coordinate transformation is completed by using the $\mathscr{H}_k$ subspaces even if one checks the linearizability condition via the sequence of distributions $\Delta_k$.

## 6. METHOD FOR FINDING THE NEW STATE COORDINATES

The new state coordinates $z_I$, necessary for the static state feedback linearization of system (1), can be found via integration of the basis vectors of the codistributions $\mathscr{H}_k$, $k = 1, \ldots, N^*$, constructed in a specific way by using the detailed algorithm below; see also [5]. Note that the basis vectors of the codistributions $\mathscr{H}_{k+1}$, found either by (15) or by the method in [5], are not necessarily exact and the state coordinates cannot be found from arbitrary exact basis vectors. When linearizability conditions are satisfied, $\mathscr{H}_{k+1}$ is integrable, and one can always find the exact basis for $\mathscr{H}_{k+1}$, which can be used for finding the new coordinates.

**Algorithm 3: Computation of the state transformation** $z = S(x)$

**Initialization.** Given the sequence $\mathscr{H}_k$, $k = 1, \ldots, N$, start with the last nonzero codistribution $\mathscr{H}_N = \operatorname{span}\left\{ \mathrm{d}z_{i_N}^N,\ i_N = 1, \ldots,\ n_N \right\}$. Integrate its basis one-forms to get the first $n_N (= \dim \mathscr{H}_N)$ state coordinates $\left\{ z_1^N, \ldots, z_{n_N}^N \right\}$. Set $k := 1$.

**Step** $k$**.** Start with the codistribution $\mathscr{H}_{N-k+1} = \operatorname{span}\left\{ \mathrm{d}z_{i_N}^N, \ldots,\ \mathrm{d}z_{i_{N-k+1}}^{N-k+1} \right\}$, $i_{N-k+1} = 1, \ldots, n_{N-k+1}$, $n_{N-k+1} = \dim \mathscr{H}_{N-k+1} - \dim \mathscr{H}_{N-k+2}$. The shifts of $z_{i_{N-k+1}}^{N-k+1}$ will be defined as the next subset of the new state coordinates $z_{i_{N-k+1}}^{N-k} := \left( z_{i_{N-k+1}}^{N-k+1} \right)^+$, satisfying for all $i_{N-k+1}$ the conditions $\mathrm{d}z_{i_{N-k+1}}^{N-k} \in \mathscr{H}_{N-k}$, $\mathrm{d}z_{i_{N-k+1}}^{N-k} \notin \mathscr{H}_{N-k+1}$.

Add $n_{N-k} - n_{N-k+1}$ more independent integrals $\left\{ z_{i_{N-k+1}+1}^{N-k}, \ldots,\ z_{i_{N-k}}^{N-k} \right\}$ so that their differentials complete the basis of $\mathscr{H}_{N-k} = \operatorname{span}\left\{ \mathrm{d}z_{i_N}^N, \ldots, \mathrm{d}z_{i_{N-k}}^{N-k} \right\}$, $i_{N-k} = 1, \ldots, n_{N-k}$. Check whether $k = N-1$. If yes, then stop. Otherwise set $k := k+1$ and go to the next step.

Alternatively, one may compute, according to Algorithm 2, the sequence of distributions $\Delta_k = \operatorname{span}\{\Xi^1, \ldots, \Xi^k\}$, $k = 1, \ldots, N$. Again, as all basis vectors of $\mathscr{H}_{k+1}$ are not necessarily exact, the vector

fields $\Xi^1,\dots,\Xi^k$ do not necessarily commute. Therefore, one has to replace them by their commuting linear combinations $\hat{\Xi}^1,\dots,\hat{\Xi}^k$ such that $\Delta_k = \mathrm{span}\{\hat{\Xi}^1,\dots,\hat{\Xi}^k\}$. Next, define the canonical parameters of the vector fields $\hat{\Xi}^l$ that satisfy the condition $\langle \mathrm{d}X^k_{i_k}, \hat{\Xi}^l_{i_l}\rangle = \delta_{kl}\delta_{i_k i_l}$. By Theorem 6, the canonical parameters $X^k$ of the distribution $\Delta_k$ provide an integrable basis for the codistribution $\mathscr{H}_{k+1}$, $k = 1,\dots,N-1$. However, this set of state coordinates $X^k$ is in general not yet suitable for linearization. Therefore, one may continue with Algorithm 3 to find the new state coordinates.

## 7. DISCUSSION AND EXAMPLES

We have implemented both methods for checking feedback linearizability and the method for finding the new state coordinates in the package NLControl built within the computer algebra system Mathematica. Since the functions from the package NLControl cannot be used outside the Mathematica environment, we have developed a webMathematica-based application that allows the functions from NLControl to be used on the world-wide web, in such a way that no other software except for an internet browser is necessary. The developed webpage is available at http://webmathematica.cc.ioc.ee/webmathematica/NLControl/ .

As the first step of the computations, the backward shift operator, defined by the system equations, has to be found, as described in [5]. For that one has to solve a system of $n$ nonlinear equations with respect to $n$ variables. The solution of such a system may exhibit a much more complex expression than the state transition map of the original control system and it may also happen that Mathematica is not able to find the solution. In a few cases the solution cannot, in principle, be expressed in terms of the elementary functions. Even in the latter case it is still possible, though extremely rare, that the system is feedback linearizable.

In principle, if the system is linearizable, then the linearization procedure can be carried out without knowing the backward shift operator. However, in practice it may be complicated, because in order to compute the backward shift of the one-form (or, alternatively, that of the vector field), the variables on negative time-instances may occur, but can be eliminated by using certain replacement rules defined by the control system. The backward shift operator gives explicit rules for replacing the mentioned variables; otherwise one has to try to eliminate them just by using the relations imposed by the original control system equations. The latter method is much less reliable and works successfully only with small systems. For instance, to find the backward shift operator for the system

$$\begin{aligned} x_1^+ &= u x_2 \sin x_2, \\ x_2^+ &= u, \end{aligned} \tag{29}$$

one has to express $x_2$ from $x_1^+ = u x_2 \sin x_2$, which is beyond the abilities of Mathematica 7.0. However, if the method from [5] (see the proof of Lemma 4.3) is applied to find $\mathscr{H}_2 = \mathrm{span}\{-\mathrm{d}x_1 + (x_2 \sin x_2)^- \mathrm{d}x_2\}$ for system (29), then we actually need only the backward shift of the product $x_2 \sin x_2$, that is

$$(x_2 \sin x_2)^- = x_1/x_2.$$

The system is feedback linearizable by the state transformation $z = (x_1/x_2, x_2 \sin x_2)$. But it is not possible to express the static state feedback in the form $u = \alpha(x,v)$, because $v = u \sin u$ and thus $u$ cannot be expressed in terms of elementary functions. However, replacing $\sin x_2$ by $e^{x_2}$, for example, in (29), yields a system, for which the backward shift operator can be found easily, since Mathematica is able to find the solution of the equation $x_1^+ = u x_2 \ln x_2$ as $x_2 =$ `ProductLog[`$x_1^+/u$`]`[4].

To compute the sequence of codistributions of differential one-forms, three alternative methods are available. The first of them uses the method from [5], the second is based on the formula $\mathscr{H}_{k+1} = (\mathscr{H}_k \bigcap \mathscr{H}_k^+)^-$, for $k \geq 1$, and the third method computes the sequence of distributions $\Delta_k$ and then finds $\mathscr{H}_{k+1}$'s as their maximal annihilators.

[4] `ProductLog[`$z$`]` represents the solution for $w$ in $z = we^w$.

In the majority of cases the fastest method is the one based on the method from [5]. Even if our goal is only to check the feedback linearizability property of the system, which means that using the third method, we may only compute the distributions $\Delta_k$ and check their involutivity, the first method in most cases still works faster. Of course, there exist exceptions, often characterized by the occurrence of exponential and logarithmic functions, for which the distributions $\Delta_k$ give the result faster. Unfortunately, it is not possible to predict the fastest linearization method only by visual inspection of the function $f$.

The only difficulty known to us, which may occur in using the first method (and occurs extremely rarely), is that in case the expressions, found on the previous steps, have not been enough simplified, there is a chance that Mathematica may be unable to solve the system of equations and the computation fails.

The second method turns out to be slower than the first method, but in most cases it is faster than method 3. If the basis vectors of $\mathscr{H}_k$ are not simplified enough, some basis vectors can be lost from the intersection $\mathscr{H}_k \bigcap \mathscr{H}_k^+$. Addition of too many simplification commands into the program makes it work slowly.

If we need only to check the feedback linearizability property, an additional method is possible: instead of the subspaces $\mathscr{H}_k$, one may use the subspaces $\mathscr{I}_k$, introduced in [10] and defined by $\mathscr{I}_1 = \mathrm{span}_{\mathscr{K}}\{\mathrm{d}x(0)\}$, $\mathscr{I}_{k+1} = \mathscr{I}_k \bigcap \mathscr{I}_k^+$ for $k \geq 1$. Between the sequences $\mathscr{H}_k$ and $\mathscr{I}_k$ the relations $\mathscr{I}_\infty = \mathscr{H}_\infty$ and $\mathscr{I}_k = \delta^{k-1}\mathscr{H}_k$ hold, where $\delta$ is the forward-shift operator. Moreover, the subspace $\mathscr{I}_k$ is completely integrable iff $\mathscr{H}_k$ is completely integrable. This method is especially useful when it is complicated to find the backward shift operator or it cannot be found at all. Then $\mathscr{I}_k$ provides us with the only practical way for checking feedback linearizability. On the other hand, if the function $f$ is complex and the backward shift operator is defined by a simple expression, then the computation of $\mathscr{I}_k$'s usually takes more time than that of $\mathscr{H}_k$'s and $\Delta_k$'s. In case the subspaces are not integrable, the expression of $I_k$ is usually more complex than the corresponding $\mathscr{H}_k$.

Note that the algorithm for finding the state coordinates actually requires integration (solution) of differential equations and is thus only constructive if the latter subproblem is solvable. Though over the years the capabilities of Mathematica to integrate the one-forms for medium-size medium-complexity problems have improved, it has still not good enough facilities for this task. To improve the capabilities of Mathematica in this respect, we have implemented an additional function `IntegrateOneForms`. This function replaces the integration of the set of one-forms by solution of the sequence of linear homogeneous PDEs and is based on the algorithm described in [11]; see also [12]. Despite this extension, the solution of the set of the partial differential equations is still often unsuccessful, especially for high-complexity examples.

The final change of variables, which represents the system in the new state coordinates, requires again the inverse state transformation to be found and thus is a potential point of failure. The latter transformation is sometimes a quite time-demanding operation and may take more time than all the previous computations.

Note that none of the methods discussed in this paper is adapted for approximate calculations, therefore all decimal fractions are transformed to rational numbers.

Finally, note that the results on state feedback linearization with Mathematica for continuous-time nonlinear systems have been reported in [13] and with Maple in [14].

To illustrate the results of this paper, we provide two examples. One of them is an academic example, the other is the model of a truck trailer.

**Example 1.** Consider the system

$$\begin{aligned} x_1^+ &= \ln\left(e^{x_1} + x_3 + u_1\right), \\ x_2^+ &= x_2 u_2 \left(1 + e^{-x_1}\left(u_1 + x_3\right)\right), \\ x_3^+ &= x_2 e^{-x_1} + x_3. \end{aligned} \tag{30}$$

One may define the new state coordinates $z$ from the sequence of codistributions $\mathscr{H}_1 = \mathrm{span}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3\}$, $\mathscr{H}_2 = \mathrm{span}\{\mathrm{d}x_3\}$, computed for equations (30). According to the results of Section 6, $\mathscr{H}_2$ defines the first new state coordinate $z_1^2 = x_3$. The next new state coordinate is its forward shift

$z_2^1 = z_1^{2+} = x_3^+ = x_2 e^{-x_1} + x_3$. Finally, to complete the span of the 1-forms $\mathrm{d}z_1^2$ and $\mathrm{d}z_2^1$ to the cobasis $\mathscr{H}_1$ of the state space, we add the 1-form $\mathrm{d}z_1^1 = \mathrm{d}x_1$. Consequently, the last new state coordinate is $z_1^1 = x_1$. The state equations in the new coordinates are

$$\begin{aligned} z_1^{1+} &= \ln\left(e^{z_1^1} + z_1^2 + u_1\right), \\ z_2^{1+} &= z_2^1 + u_2\left(z_2^1 - z_1^2\right), \\ z_1^{2+} &= z_2^1, \end{aligned} \tag{31}$$

yielding the linear equations

$$z_1^{1+} = v_1, \;\; z_2^{1+} = z_2^1 + v_2, \;\; z_1^{2+} = z_2^1 \tag{32}$$

after applying the static state feedback, defined by $v_1 = \ln\left(e^{z_1^1} + z_1^2 + u_1\right)$, $v_2 = u_2\left(z_2^1 - z_2^1\right)$.

Alternatively, one may find the new state coordinates from the sequence of distributions $\Delta_k$. Compute the tangent map of the state transition map as follows:

$$Tf = \begin{pmatrix} \frac{e^{x_1}}{e^{x_1}+u_1+x_3} & 0 & \frac{1}{e^{x_1}+u_1+x_3} & \frac{1}{e^{x_1}+u_1+x_3} & 0 \\ -e^{-x_1}u_2x_2(u_1+x_3) & u_2(1+e^{-x_1}(u_1+x_3)) & e^{-x_1}u_2x_2 & e^{-x_1}u_2x_2 & x_2(1+e^{-x_1}(u_1+x_3)) \\ -e^{-x_1}x_2 & e^{-x_1} & 1 & 0 & 0 \end{pmatrix} \tag{33}$$

and its kernel $K = \mathrm{Ker}Tf = \mathrm{span}\{K_1, K_2\}$, where

$$\begin{aligned} K_1 &= e^{-2x_1}x_2\tfrac{\partial}{\partial x_1} + e^{-2x_1}x_2\left(e^{2x_1} + x_2\right)\tfrac{\partial}{\partial x_2} - e^{-x_1}x_2\tfrac{\partial}{\partial x_3} - u_2\tfrac{\partial}{\partial u_2}, \\ K_2 &= -e^{-x_1}\tfrac{\partial}{\partial x_1} - e^{-x_1}x_2\tfrac{\partial}{\partial x_2} + \tfrac{\partial}{\partial u_1}. \end{aligned} \tag{34}$$

Next, compute, according to (5), the distribution

$$\Delta_1 = \{Tf(D_0)\}^-, \tag{35}$$

where $D_0 = \mathrm{span}\{\partial/\partial u_1, \partial/\partial u_2\}$. The distribution $D_0 + K$ is involutive as both $K$ and $D_0$ are involutive. Next, observe that the vector field $\partial/\partial u_1$ commutes with the vector fields $K_1$ and $K_2$ but the vector field $\partial/\partial u_2$ does not respect the distribution $K$ since

$$\left[\frac{\partial}{\partial u_2}, K_1\right] = \frac{\partial}{\partial u_2} \notin K. \tag{36}$$

According to Proposition 1, one cannot, due to (36), express the components of the vector field $Tf(\partial/\partial u_2)$ in terms of $x^+$ exclusively. Really (see 30),

$$Tf\left(\frac{\partial}{\partial u_2}\right) = x_2\left(1 + e^{-x_1}(u_1 + x_3)\right)\frac{\partial}{\partial x_2^+} = \frac{x_2^+}{u_2}\frac{\partial}{\partial x_2^+}. \tag{37}$$

Choose another basis for $D_0$ that respects $K$, for example,

$$D_0 = \mathrm{span}\{\partial/\partial u_1, u_2\partial/\partial u_2\}.$$

Then we have

$$\begin{aligned} Tf\left(\tfrac{\partial}{\partial u_1}\right) &= e^{-x_1^+}\tfrac{\partial}{\partial x_1^+} + x_2^+ e^{-x_1^+}\tfrac{\partial}{\partial x_2^+}, \\ Tf\left(u_2\tfrac{\partial}{\partial u_2}\right) &= x_2^+\tfrac{\partial}{\partial x_2^+}, \end{aligned} \tag{38}$$

and

$$\Delta_1 = \operatorname{span}\left\{ e^{-x_1}\frac{\partial}{\partial x_1} + x_2 e^{-x_1}\frac{\partial}{\partial x_2}, x_2\frac{\partial}{\partial x_2} \right\}. \tag{39}$$

Since $\Delta_2 = \Delta_N$, its basis vector fields span the complete tangent bundle $T(M \times U)$, and we have to find only one vector field $\Xi_1^2$, which does not belong to $\Delta_1$ and commutes with $\Xi_1^1 = e^{-x_1}\frac{\partial}{\partial x_1} + x_2 e^{-x_1}\frac{\partial}{\partial x_2}$ and $\Xi_2^1 = x_2\frac{\partial}{\partial x_2}$. We may take, for example, $\Xi_1^2 = \partial/\partial x_3$. Consequently, we get the new commutative basis of $\Delta_2$ as $\Xi_1^1$, $\Xi_2^1$ and $\Xi_1^2$; those canonical parameters are

$$X_1^1 = e^{x_1}, \quad X_2^1 = -x_1 + \ln x_2, \quad X_1^2 = x_3 \tag{40}$$

yielding the new state equations:

$$X_1^{1+} = X_1^1 + X_1^2 + u_1, \;\; X_2^{1+} = \ln\left(u_2 e^{X_2^1}\right), \;\; X_1^{2+} = e^{X_2^1} + X_1^1. \tag{41}$$

However, according to the results of Section 6, equations (41) are not yet linearizable via static state feedback, and one needs an additional coordinate transformation.

The additional coordinate transformation is defined below according to the method suggested in [3]. Starting with the coordinates $\{X_1^1, X_2^1, X_1^2\}$, define the first set $\tilde{z}^2$ of the new coordinates $\tilde{z}$ as $\tilde{z}_1^2 = X_1^2 = x_3$. The second set $\tilde{z}^1$ of the new coordinates is defined in the following way. First we take $\tilde{z}_2^1 = \left(\tilde{z}_2^1\right)^+$, and complete the set of new coordinates by $\tilde{z}_1^1 = X_1^1$, leading to the transformation

$$\tilde{z}_1^1 = X_1^1, \tilde{z}_2^1 = \left(X_1^1\right)^+, \quad \tilde{z}_1^2 = X_1^2. \tag{42}$$

The new state equations are

$$\begin{aligned} \left(\tilde{z}_1^1\right)^+ &= (\tilde{z}_1^1 + \tilde{z}_1^2 + u_1), \\ \left(\tilde{z}_2^1\right)^+ &= \tilde{z}_2^1 + \left(\tilde{z}_2^1 - \tilde{z}_1^2\right) u_2, \\ \left(\tilde{z}_1^2\right)^+ &= \tilde{z}_2^1. \end{aligned} \tag{43}$$

**Example 2.** Consider the model of a truck trailer [15]:

$$\begin{aligned} x_1^+ &= x_1 + \frac{vt}{l}u - \frac{vt}{L}x_1, \\ x_2^+ &= x_2 + \frac{vt}{L}x_1, \\ x_3^+ &= x_3 + vt\sin\left(x_2 + \frac{vt}{2L}x_1\right), \\ x_4^+ &= x_4 + vt\cos\left(x_2 + \frac{vt}{2L}x_1\right), \end{aligned} \tag{44}$$

where $x_1$ is the angle between the truck and the trailer, $x_2$ is the angle between the trailer and the road, $x_3$ and $x_4$ are the vertical and horizontal positions of the rear end of the trailer, respectively, $L$ is the length of the trailer, $l$ is the length of the truck, $t$ is the sampling interval, and $v$ is the velocity of the truck. The input parameter $u$ is the steering angle. We are going to check, using Theorems 5 and 2, respectively, whether system (44) is static state feedback linearizable. Compute first, for (44),

$$Tf = \begin{pmatrix} 1 - \frac{tv}{L} & 0 & 0 & 0 & \frac{tv}{l} \\ \frac{tv}{L} & 1 & 0 & 0 & 0 \\ \frac{t^2v^2}{2L}\cos\left(x_2 + \frac{vtx_1}{2L}\right) & tv\cos\left(x_2 + \frac{vtx_1}{2L}\right) & 1 & 0 & 0 \\ -\frac{t^2v^2}{2L}\sin\left(x_2 + \frac{vtx_1}{2L}\right) & -tv\sin\left(x_2 + \frac{vtx_1}{2L}\right) & 0 & 1 & 0 \end{pmatrix}$$

and find $K = \ker Tf$, spanned by the vector field

$$-\frac{Ltv}{l(L-tv)}\frac{\partial}{\partial x_1}+\frac{t^2v^2}{l(L-tv)}\frac{\partial}{\partial x_2}-\frac{t^3v^3}{2l(L-tv)}\cos\left(x_2+\frac{tvx_1}{2L}\right)\frac{\partial}{\partial x_3}+\frac{t^3v^3}{2l(L-tv)}\sin\left(x_2+\frac{tvx_1}{2L}\right)\frac{\partial}{\partial x_4}+\frac{\partial}{\partial u}.$$

Next compute the distribution $\Delta_1$, spanned by the vector field

$$Tf\left(\frac{\partial}{\partial u}\right) = \frac{tv}{l}\frac{\partial}{\partial x_1^+}. \tag{45}$$

In order to simplify the computations, we multiply the vector field (45) by the constant $l/tv$, obtaining $\Delta_1^+ = \operatorname{span}\left\{\Xi_1^+\right\}$, where $\Xi_1 = \partial/\partial x_1$. Obviously, the distribution $\Delta_1$ is involutive. Applying the tangent map $Tf$ to the distribution $\Delta_1 = \operatorname{span}\{\partial/\partial x_1\}$, we obtain

$$\begin{aligned}\Delta_2^+ &= \operatorname{span}\left\{\frac{\partial}{\partial x_1^+}, 2\frac{\partial}{\partial x_2^+}+tv\cos\left(\frac{tv(tvu-lx_1^+)}{2l(L-2tv)}+x_2^+\right)\frac{\partial}{\partial x_3^+}\right.\\ &\left.-tv\sin\left(\frac{tv(tvu-lx_1^+)}{2l(L-2tv)}+x_2^+\right)\frac{\partial}{\partial x_4^+}\right\}.\end{aligned} \tag{46}$$

The distribution $\Delta_2^+$ (and therefore, also $\Delta_2$) is not involutive and therefore, according to Theorem 5, system (44) is not static state feedback linearizable.

Alternatively, one may compute the codistributions: $\mathscr{H}_1 = \operatorname{span}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4\}$, $\mathscr{H}_2 = \operatorname{span}\{\mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4\}$, and

$$\mathscr{H}_3 = \operatorname{span}\left\{tv\mathrm{d}x_2+2\operatorname{cosec}\left(x_2+\frac{tv(tvu^- - lx_1)}{2l(L-tv)}\right)\mathrm{d}x_4,\ \mathrm{d}x_3+\cot\left(x_2+\frac{tv(tvu^- - lx_1)}{2l(L-tv)}\right)\mathrm{d}x_4\right\}.$$

Again, it is easy to check, for example via the Frobenius Theorem, that the codistribution $\mathscr{H}_3$ is not integrable, and therefore according to Theorem 2, system (44) is not static state feedback linearizable.

## 8. CONCLUSION

In this paper the problem of static state feedback linearizability of a discrete-time nonlinear control system has been addressed. The paper focuses on establishing the explicit relationship between the two sets of necessary and sufficient linearizability conditions. The first set of conditions is formulated in terms of involutivity of an increasing sequence of certain distributions of vector fields. The second set of conditions is formulated in terms of integrability of the decreasing sequence of the codistributions of differential one-forms. We have demonstrated that the distributions used in the first set of conditions are the maximal annihilators of the corresponding codistributions used in the second set of conditions. Moreover, two methods have been compared from the point of view of computational complexity. Note that the new state coordinates are the canonical parameters of the commutative basis vector fields (with specific properties) of the distribution $\Delta_N$ and also the integrals of the corresponding codistributions $\mathscr{H}_1$. The method based on one-forms is easier than that based on vector fields, since in the latter case the new state coordinates are computed in two steps. At the first step one has to find canonical parameters of an arbitrary commutative basis vector field of the distribution $\Delta_N$, which actually corresponds to finding the integrals of the corresponding codistribution $\mathscr{H}_1$. At the second step one modifies the integrals of $\mathscr{H}_1$ to get the integrals with specific properties.

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## Lineariseeritavus staatilise olekutagasisidega: kahe meetodi vahelised seosed

Tanel Mullari, Ülle Kotta ja Maris Tõnso

On käsitletud diskreetse ajaga mittelineaarsete juhtimissüsteemide lineariseerimise võimalikkust staatilise olekutagasisidega. Põhitähelepanu on pööratud kahe tarvilike ja piisavate tingimuste kogumi omavaheliste seoste uurimisele. Esimene neist on esitatud juhtimissüsteemi defineeritud vektorväljadega genereeritud jaotuste jada involutiivsuse kaudu. Teine kogum esitub juhtimissüsteemi defineeritud diferentsiaalsete 1-vormidega genereeritud kaasjaotuste jada integreeruvuse kaudu. Artiklis on tõestatud, et esimese tingimuste kogumi puhul esinevad jaotused on teise tingimuste kogumi vastavate kaasjaotuste maksimaalsed annihilaatorid. On võrreldud mõlemat liiki tingimuste kontrolli ja vaadeldud uute olekukoordinaatide leidmist nende rakendusliku lihtsuse seisukohast arvutialgebra süsteemis Mathematica. On esitatud algoritmid lineariseerimiseks vajaliku koordinaatteisenduse leidmiseks. Teoreetilisi tulemusi illustreerivad kaks näidet.