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MATHEMATICS

Convergence of the p-Bieberbach polynomials in regions with zero angles

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Abstract. Uniform convergence of the p-Bieberbach polynomials is proved in the case of a simply connected region bounded by a piecewise quasiconformal curve with certain interior zero angles on the corner where two arcs meet.

Key words: uniform approximation, extremal polynomials, Bieberbach polynomials, conformal mappings.

1. INTRODUCTION

Finding the Riemann mapping function for a given region is a very famous and important problem for researchers. The reason is that this function has many applications in some branches of mathematics. There are some construction methods of this function for simple regions differing from those for general regions. So, the best way is to approximate this function by using some extremal polynomials.

Let G be a finite region with $0 \in G$ bounded by Jordan curve $L := \partial G$ and let $w = \varphi(z)$ be a conformal mapping of G onto the disk $\{w : |w| < r_0\}$ with $\varphi(0) = 0$, $\varphi'(0) = 1$, where r_0 is called the conformal radius of $\overline{G}$ with respect to 0. Denote by $A_p^1(G)$, $p > 0$ the set of functions $f(z)$ analytic in G satisfying $f(0) = 0$, $f'(0) = 1$ such that

$$\|f\|_{A_p^1(G)} := \|f'\|_{A_p(G)} := \left(\iint\limits_G |f'(z)|^p \, d\sigma_z \right)^{\frac{1}{p}} < \infty,$$

where $d\sigma_z$ is a two-dimensional Lebesque measure on G, and denote by $\wp_n$ the set of all algebraic polynomials $P_n(z)$ of degree at most n, satisfying $P_n(0) = 0$, $P_n'(0) = 1$.

Consider the following extremal problem:

$$\left\{ \|\varphi - P_n\|_{A_p^1(G)} : P_n \in \wp_n \right\} \longrightarrow \min, \qquad p > 0. \tag{1.1}$$

Using a method similar to the one given in [10, p. 137], it is seen that there exists an extremal polynomial $P_n^*(z)$ furnishing to the problem (1.1). These polynomials $P_n^*(z)$ are determined uniquely in case $p > 1$ [10, p. 142]. In [14] the solution of (1.1) was called p-Bieberbach polynomials and denoted by $B_{n,p}(z)$, and its approximation properties in uniform norm were investigated, i.e.

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$$\|\varphi - B_{n,p}\|_{C(\overline{G})} := \max_{z\in\overline{G}} \{|\varphi(z) - B_{n,p}(z)|\} \leq \frac{c}{n^{\gamma}}, \tag{1.2}$$

where γ depends on the geometric properties of Γ and c is a constant independent from n.

In case $p = 2$ the p-Bieberbach polynomials coincide with the well-known n-th Bieberbach polynomial $B_n(z) \equiv B_{n,2}(z)$ (see, for example, [13,20]). Approximation properties in the uniform norm of $B_n(z)$ on $\overline{G}$ were first observed by Keldysh in [13] for the domains with sufficiently smooth boundaries. A considerable progress in this area was achieved in [2,4,6,7,11,12,16,18,20] and others.

In [14], approximation properties of p-Bieberbach polynomials were investigated in case the region was bounded by a quasiconformal curve. It is well known that these curves do not allow interior zero angles. The effect of zero angles for these extremal polynomials has not yet been studied but our results show this effect. First, we will investigate the approximation rate of $B_{n,p}(z)$ to the function φ in A^1_p-norm (Theorem 1), and by using the well-known Simonenko and Andrievskii method (see, for example, [6] and [18]), the approximation rate of $B_{n,p}(z)$ to the function φ in the uniform norm will be obtained (Theorems 2–6).

2. MATERIAL AND METHODS

Definition 1 [15, p. 97]**.** *The Jordan arc (or a curve) L is called a K-quasiconformal $(K \geq 1)$ arc (or curve) if there is a K-quasiconformal mapping f of a region D containing L such that $f(L)$ is a line segment (or a circle).*

Let $F(L)$ denote the set of all sense-preserving plane homeomorphisms f of regions $D \supset L$ such that $f(L)$ is a line segment or circle and let

$$K_L = \inf\{K(f) : f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of such a mapping f. Then L is K-quasiconformal if and only if $K_L < \infty$. If L is a K-quasiconformal, then $K_L \leq K$.

$D \equiv \mathbb{C}$ gives the global definition of a K-quasiconformal arc or curve consequently. This definition is common in the literature. Through this work, the global definition will be considered.

Definition 2 [2]**.** *For given $K \geqslant 1$, $\alpha \geqslant 0$, we say that $G \in PQ(K,\alpha)$ if $L := \partial G$ is expressed as a union of a finite number K_j-quasiconformal arcs connected at the points $z_0,\ z_1,\ z_{2},...,z_{m-1}$, $K = \max\limits_{1\leq j\leq m}\{K_j\}$, L is locally K-quasiconformal at z_0, and in the local co-ordinate system (x,y) with origin at z_j, $1 \leq j \leq m-1$, the following conditions are satisfied*:

$$\text{(i)}\ \left\{z = x+iy : a_1x^{1+\alpha} \leq y \leq a_2x^{1+\alpha}, 0 \leq x \leq \varepsilon_1\right\} \subset \overline{G},$$

$$\text{(ii)}\ \{z = x+iy : |y| \geq \varepsilon_2 x,\ 0 \leq x \leq \varepsilon_1\} \subset CG$$

for some constants $-\infty < a_1 < a_2 < \infty$, $\varepsilon_i > 0$, $i = 1,2$.

It is clear from Definition 2 that each domain $G \in PQ(K,\alpha)$ may have $m-1$ interior zero angles. If G does not have those interior zero angles (i.e $\alpha = 0$), then G is bounded by a K-quasiconformal curve and we say that $G \in Q(K)$.

Throughout this paper, $c, c_1, c_2, ...$ are positive constants and $\varepsilon, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive numbers which in general depend on G. By the notation "$a \preccurlyeq b$" we mean that $a \leq c_1 b$ for a constant c_1, which does not depend on a and b. The relation "$a \asymp b$" indicates that $c_2 b \leq a \leq c_3 b$, where c_2, c_3 are independent of a and b.

Let $G \subset \mathbb{C}$ be a finite region bounded by Jordan curve L and let $w = \Phi(z)$ $(w = \widehat{\varphi}(z))$ be the conformal mapping of $\Omega := ext\overline{G}$ (G) onto

$$\widehat{\Delta} = \{w : |w| > 1\} \quad (\{w : |w| < 1\}),$$

normalized by

$$\Phi(\infty) = \infty,\ \ \Phi'(\infty) > 0 \quad \left(\widehat{\varphi}(0) = 0,\ \ \widehat{\varphi}'(0) > 0\right).$$

The level curve (exterior or interior) can be defined for $t > 0$ as

$$L_t := \{z : |\widehat{\varphi}(z)| = t,\ \text{if}\ t < 1\ ;\ |\Phi(z)| = t,\ \text{if}\ t > 1\}, L_1 \equiv L.$$

Let us denote

$$G_t := intL_t, \Omega_t := extL_t, \quad \text{and} \quad d(z,L) := \inf\{|\varsigma - z| : \varsigma \in L\}.$$

Let L be a K-quasiconformal curve. By using the facts in ([15, p. 97]; [5, p. 76]; and [8, p. 26]) we can find a $C(K)$-quasiconformal reflection $\alpha(.)$ across L such that it satisfies the following

$$\begin{gathered} |z_1 - \alpha(z)| \asymp |z_1 - z|,\ z_1 \in L,\ \varepsilon < |z| < \tfrac{1}{\varepsilon}, \\ |\alpha_{\bar z}| \asymp |\alpha_z| \asymp 1, \varepsilon < |z| < \tfrac{1}{\varepsilon}, \\ |\alpha_{\bar z}| \asymp |\alpha_z|^2,\ |z| < \varepsilon,\ |\alpha_{\bar z}| \asymp |z|^{-2},\ |z| > \tfrac{1}{\varepsilon} \end{gathered} \tag{2.1}$$

and Jacobian $J_\alpha = |\alpha_z|^2 - |\alpha_{\bar z}|^2$ of $\alpha(.)$ satisfied $J_\alpha \asymp 1$.

Lemma 1 [8, p. 97]**.** *Suppose that the function $w = F(\zeta)$ is K-quasiconformal mapping of the plane onto itself and $F(\infty) = \infty$. Assume also that $\zeta_i \in \mathbb{C}$, $w_i = F(\zeta_i)$, $i = 1,2,3$. Then,*
(a) *The statements $|\zeta_1 - \zeta_2| \preccurlyeq |\zeta_1 - \zeta_3|$ and $|w_1 - w_2| \preccurlyeq |w_1 - w_3|$ are equivalent.*
(b) *If $|\zeta_1 - \zeta_2| \preccurlyeq |\zeta_1 - \zeta_3|$, then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\frac{1}{K}} \preccurlyeq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preccurlyeq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K}.$$

Lemma 2 [1]**.** *Let $L := \partial G$ be a quasiconformal curve. Then for every $z \in L$ there exists an arc $\beta(z_0,z)$ in G joining z_0 to z with the following properties*:
(a) *$d(\xi,L) \asymp |\xi - z|$ for every $\xi \in \beta(z_0,z)$;*
(b) *if $\widetilde{\beta}(\xi_1,\xi_2)$ is the subarc of $\beta(z_0,z)$ joining ξ_1 to ξ_2, then*

$$mes\ \widetilde{\beta}(\xi_1,\xi_2) \preccurlyeq |\xi_1 - \xi_2|$$

for every pair $\xi_1,\xi_2 \in \beta(z_0,z)$.

Lemma 3 [6]**.** *Let $L := \partial G$ be a quasiconformal curve. Then*

$$mes\ \ell \asymp mes\ \alpha(\ell)$$

for every rectifiable arc $\ell \subset G$.

3. APPROXIMATION IN THE $A_p^1(G)$-NORM

Suppose that $G \in PQ(K,\alpha)$ for some $K \geqslant 1$ and $\alpha \geq 0$ is given. For the sake of simplicity, but without loss of generality, we assume that $m = 2$, $z_1 = -1$, $z_2 = 1$; $(-1,1) \subset G$ and let the local co-ordinate axes be parallel to Ox and Oy in the co-ordinate system

$$L^1 := \{z : z \in L,\ \mathrm{Im}z \geq 0\},\ L^2 := \{z : z \in L,\ \mathrm{Im}z \leq 0\}.$$

Then z_0 is taken as an arbitrary point on L^2 (or on L^1 subject to the chosen direction).

We recall that the region $G \in PQ(K,\alpha)$ has interior zero angles in the nearest neighbourhood of each point $z_1 = -1$ and $z_2 = 1$, respectively.

We can say that the function $w = \widehat{\varphi}(z)$ for the domain $G \in PQ(K,\alpha)$ satisfies the conditions described in Lemma 1 in the neighbourhood of point $z_{1,2} = \mp 1$. So, we can easily get

$$\begin{array}{rcl} d(z,L) & \preccurlyeq & (|\widehat{\varphi}(z)|-1)^{K^{-1}}; \quad |z\pm 1| \preccurlyeq |\widehat{\varphi}(z)-\widehat{\varphi}(\pm 1)|^{K^{-1}}, \\ \forall z & \in & M := \{z \in G : |z\pm 1| > \varepsilon_1\}. \end{array} \tag{3.1}$$

On the other hand, using the properties of the function $w = \widehat{\varphi}(z)$ in the neighbourhood of point $z_{1,2} = \mp 1$ (see [6,9]), we obtain

$$|z\pm 1| \preccurlyeq [-\ln|\widehat{\varphi}(z)-\widehat{\varphi}(\pm 1)|]^{-\alpha^{-1}}. \tag{3.2}$$

Because each $L^j, j = 1,2$ is a K_j-quasiconformal arc, $\alpha^j(.)$ must be the quasiconformal reflection across L^j.

Let us also denote:

$$\gamma_1^1 := \alpha^1\left(\left\{z = x+iy : y = \frac{2a_1+a_2}{3}(x+1)^{1+\alpha}\right\}\right),$$

$$\gamma_1^2 := \alpha^2\left(\left\{z = x+iy : y = \frac{a_1+2a_2}{3}(x+1)^{1+\alpha}\right\}\right),$$

and

$$\gamma_2^1 := \alpha^1\left(\left\{z = x+iy : y = \frac{2a_1+a_2}{3}(1-x)^{1+\alpha}\right\}\right),$$

$$\gamma_2^2 := \alpha^2\left(\left\{z = x+iy : y = \frac{a_1+2a_2}{3}(1-x)^{1+\alpha}\right\}\right),$$

where the constants a_i, $(i = 1,2)$ are taken from Definition 2.

It is easy to check from Lemma 3 that

$$mes\ \gamma_i^j(\xi_1,\xi_2) \preccurlyeq |\xi_1-\xi_2|$$

for all $\xi_1,\xi_2 \in \gamma_i^j$, $i,j = 1,2$.

Let $N = N(R_0)$ be a sufficiently large natural number. For $n > N$ and arbitrary $0 < \varepsilon < 1$, let us choose $R = r_0 + cn^{\varepsilon-1}$ such that $r_0 < R < R_0$ and points z_i^j, $i,j = 1,2$ such that they are in the intersection of L_R and γ_i^j.

According to the positive direction on L_R, these points divide L_R into four parts as follows

$$L_R^1 := L_R(z_3^1,z_2^1),\ \ L_R^2 := L_R\left(z_2^2,z_3^2\right),\ L_R^3 := L_R\left(z_2^1,z_2^2\right),\ \ L_R^4 := L_R\left(z_3^2,z_3^1\right)$$

and $\gamma_i^j(R)$ is a subarc of γ_i^j joining points ∓ 1 with z_i^j. Denote

$$\Gamma_R^j := \gamma_1^j(R)\cup L_R^j \cup \gamma_2^j(R),\ \ U := int(\Gamma_R^1\cup\Gamma_R^2)\ \text{ and }\ U_R := U\setminus\overline{G}.$$

We can extend the function $\varphi(z)$ to U in the following way

$$\widetilde{\varphi}(z) := \begin{cases} \varphi(z) & ; z\in\overline{G}, \\ \frac{r_0^2}{\varphi(\alpha^j(z))} & ; z\in U_R,\ j = 1,2. \end{cases}$$

Then,

$$\widetilde{\varphi}_{\overline{z}}(z)=\begin{cases}0 & ;z\in G,\\ \varphi'(\alpha^j(z))\alpha^j_{\overline{z}}(z) & ;z\in U_R,\quad j=1,2.\end{cases}$$

From the Cauchy–Pompeiu formula [15, p. 148] we get

$$\varphi(z)=\frac{1}{2\pi i}\int\limits_{\Gamma_R}\frac{\widetilde{\varphi}(\xi)}{\xi-z}d\xi-\frac{1}{\pi}\iint\limits_{U_R}\frac{\widetilde{\varphi}_{\overline{\xi}}(\xi)}{\xi-z}d\sigma_\xi,\quad z\in G.$$

Then, using the above notations we obtain

$$\varphi(z)=\frac{1}{2\pi i}\int\limits_{L_R}\frac{f(\xi)}{\xi-z}d\xi+\frac{1}{2\pi i}\sum_{i=1}^{2}\sum_{j=1}^{2}\int\limits_{\gamma_i^j(R)}\frac{\widetilde{\varphi}(\xi)-\varphi((-1)^j)}{\xi-z}d\xi-\frac{1}{\pi}\iint\limits_{U_R}\frac{\widetilde{\varphi}_{\overline{\xi}}(\xi)}{\xi-z}d\sigma_\xi,\tag{3.3}$$

where

$$f(\xi)=\begin{cases}\widetilde{\varphi}(\xi) & ;\xi\in L_R^1\cup L_R^2\\ \varphi(-1) & ;\xi\in L_R^3\\ \varphi(1) & ;\xi\in L_R^4\end{cases}.$$

Since the first part of (3.3) is analytic in $\overline{G}$, there exists a polynomial $P_{n-1}(z)$ where $\deg P_{n-1}(z)\le n-1$ [19, p. 142] such that

$$\left|\frac{1}{2\pi i}\int\limits_{L_R}\frac{f(\xi)}{(\xi-z)^2}d\xi-P_{n-1}(z)\right|<\frac{c}{n}.\tag{3.4}$$

Let $Q_n(z):=\int_0^z P_{n-1}(t)dt$. Then $Q_n(0)=0$ and from (3.3) and (3.4) we have

$$\left|\varphi'(z)-Q_n'(z)\right|\le\frac{c}{n}+\frac{1}{2\pi}\left|\sum_{i=1}^{2}\sum_{j=1}^{2}\int\limits_{\gamma_i^j(R)}\frac{\widetilde{\varphi}(\xi)-\varphi((-1)^j)}{(\xi-z)^2}d\xi\right|+\left|-\frac{1}{\pi}\iint\limits_{U_R}\frac{\widetilde{\varphi}_{\overline{\xi}}(\xi)}{(\xi-z)^2}d\sigma_\xi\right|,$$

and let us take integrals over G of the p-th power of both sides

$$\begin{aligned}\iint\limits_G\left|\varphi'(z)-Q_n'(z)\right|^p d\sigma_z\preccurlyeq\frac{1}{n^p}\quad &+\quad\iint\limits_G\left|\sum_{i=1}^{2}\sum_{j=1}^{2}\int\limits_{\gamma_i^j(R)}\frac{\widetilde{\varphi}(\xi)-\varphi((-1)^j)}{(\xi-z)^2}d\xi\right|^p d\sigma_z\\ &+\quad\iint\limits_G\left|\iint\limits_{U_R}\frac{\widetilde{\varphi}_{\overline{\xi}}(\xi)}{(\xi-z)^2}d\sigma_\xi\right|^p d\sigma_z.\end{aligned}\tag{3.5}$$

From the Calderon–Zygmund inequality [5, p. 98], we obtain

$$\iint\limits_G\left|\iint\limits_{U_R}\frac{\widetilde{\varphi}_{\overline{\xi}}(\xi)}{(\xi-z)^2}d\sigma_\xi\right|^p d\sigma_z\preccurlyeq\iint\limits_{U_R}\left|\varphi'(\alpha^j(\xi))\right|^p d\sigma_\xi,\quad j=1,2.\tag{3.6}$$

So, (3.5) and (3.6) give us

$$
\begin{aligned}
\|\varphi' - Q_n'\|^p_{A_p(G)} \preccurlyeq \frac{1}{n^p} &+ \sum_{i=1}^{2}\sum_{j=1}^{2}\left\| \int\limits_{\gamma_i^j(R)} \frac{\widetilde{\varphi}(\xi)-\varphi((-1)^j)}{(\xi-z)} d\xi \right\|^p_{A_p(G)} \\
&+ \iint\limits_{U_R} \left|\varphi'(\alpha^j(\xi))\right|^p d\sigma_\xi . \qquad (3.7)
\end{aligned}
$$

Let us consider two cases of p in the last double integral in (3.7) as: $1 < p < 2$ and $p \geq 2$. If $1 < p < 2$, then using the Hölder inequality [21, p. 105] we obtain

$$
\begin{aligned}
\iint\limits_{U_R} \left|\varphi'(\alpha^j(\xi))\right|^p d\sigma_\xi &\preccurlyeq \left(\iint\limits_{U_R} \left|\varphi'(\alpha^j(\xi))\right|^2 d\sigma_\xi \right)^{\frac{p}{2}} \left(\iint\limits_{U_R} d\sigma_\xi \right)^{1-\frac{p}{2}} \\
&\preccurlyeq \left(\int\int\limits_{\alpha^j(U_R)} \left|\varphi'(\xi)\right|^2 d\sigma_\xi \right)^{\frac{p}{2}} \left(\int\int\limits_{\alpha^j(U_R)} d\sigma_\xi \right)^{1-\frac{p}{2}} \\
&= \left[mes\ \varphi(\alpha^j(U_R))\right]^{\frac{p}{2}} \left[mes\ \alpha^j(U_R)\right]^{1-\frac{p}{2}} . \qquad (3.8)
\end{aligned}
$$

Thus, (3.7) and (3.8) give

$$
\begin{aligned}
\|\varphi' - Q_n'\|^p_{A_p(G)} \preccurlyeq \frac{1}{n^p} &+ \sum_{i=1}^{2}\sum_{j=1}^{1}\left\| \int\limits_{\gamma_i^j(R)} \frac{\widetilde{\varphi}(\xi)-\varphi((-1)^j)}{(\xi-z)^2} d\xi \right\|^p_{A_p(G)} \\
&+ \begin{cases} [mes\ \varphi(\alpha^j(U_R))]^{\frac{p}{2}}[mes\ \alpha^j(U_R)]^{1-\frac{p}{2}} & ;1<p<2,\ j=1,2 \\ \iint\limits_{U_R} \left|\varphi'(\alpha^j(\xi))\right|^p d\sigma_\xi & ;p\geq 2,\ j=1,2. \end{cases} \qquad (3.9)
\end{aligned}
$$

We introduce the following notations:

$$
\begin{aligned}
a \ &: = \min\left\{ \frac{2-(p+2)\alpha}{2(1+\alpha)(K^2+1)}, \frac{p}{2K^2} + \frac{(2-p)}{K^2(K^2+1)} \right\}, \\
b \ &: = \min\left\{ \frac{2-(p+2)\alpha}{2(1+\alpha)(K^2+1)}, \frac{1}{K^2} - \frac{(p-2)\left(K^2-1\right)}{K^2(K^2+1)} \right\}.
\end{aligned}
$$

Theorem 1. *Let $G \in PQ(K,\alpha)$ for some K, $K \geqslant 1$ and α, $0 \leqslant \alpha < \min\left\{\frac{2(p-1)}{p+2}, \frac{2}{p+2}\right\}$ for $p > 1$. Then, for any number $n = 2,3,...$ the p-Bieberbach polynomials $B_{n,p}(z)$ satisfy*

$$\|\varphi - B_{n,p}\|_{A_p^1(G)} \preccurlyeq n^{-\vartheta},$$

where

$$\vartheta \in \begin{cases} \left(0, \frac{1}{p}a\right) & ;1<p<2, \\ \left(0, \frac{1}{p}b\right) & ;2 \leqslant p < 2 + \frac{K^2+1}{K^2-1}. \end{cases}$$

Proof. We shall obtain the proof by using (3.9). Firstly, assume that $G \in PQ(K,\alpha)$ for some $K \geqslant 1$ and $0 \le \alpha < \min\left\{\frac{2(p-1)}{p+2}, \frac{2}{p+2}\right\}$.

Let us choose $g_j(\xi) := \widetilde{\varphi}(\xi) - \varphi((-1)^j)$, $j = 1,2$, and $\vartheta(t) = t^{1-\frac{1+\alpha}{p}}$ by using Lemma 2.4 [3], we obtain

$$\left\| \int\limits_{\gamma_i^j(R)} \frac{\widetilde{\varphi}(\xi) - \varphi\left((-1)^j\right)}{(\xi - z)^2} d\xi \right\|_{A_p} \preccurlyeq \begin{cases} \left|\ln \ell_{i,j}\right|^{\frac{1}{p}} \ell_{i,j}^{\frac{2-(p+2)\alpha}{2p}} & ;\alpha < \frac{2(p-1)}{p+2}, 1 < p \le 2, \\ \ell_{i,j}^{\frac{2-(p+2)\alpha}{2p}} & ;\alpha < \frac{2}{p+2}, \quad p > 2, \end{cases} \tag{3.10}$$

where $\ell_{i,j} := mes\ \gamma_i^j(R)$. On the other hand, we have $d(z_i^j, L^j) \preccurlyeq n^{-\frac{2}{K^2+1}}$ according to [17, Lemma 9.9]. Then, from (2.1), (3.1), and (3.2) we get

$$\ell_{i,j} \preccurlyeq \left|z_i^j - (-1)^j\right| \preccurlyeq d^{\frac{1}{1+\alpha}}(z_i^j, L^j) \preccurlyeq \left(\frac{1}{n}\right)^{\frac{2-\varepsilon}{(1+\alpha)(K^2+1)}} \tag{3.11}$$

for all $\varepsilon > 0$. Combining (3.10) and (3.11), we have

$$\left\| \int\limits_{\gamma_i^j(R)} \frac{\widetilde{\varphi}(\xi) - \varphi\left((-1)^j\right)}{(\xi - z)^2} d\xi \right\|_{A_p(G)} \preccurlyeq \left(\frac{1}{n}\right)^{\frac{2-(p+2)\alpha-\varepsilon}{2p(1+\alpha)(K^2+1)}}, \quad p > 1,$$

where i, $j = 1,2$.

For sufficiently small $0 < \varepsilon_0 < \varepsilon_1$, we shall use the following notations:

$$U_R := V_1^1 \cup V_1^2 \cup V_2^1 \cup V_2^2 \cup V_3,$$

where

$$\begin{aligned} V_i^1 \quad &: \quad = U_R \cap D\left((-1)^i, \varepsilon_0\right) \cap \{z : \operatorname{Im} z > 0\}, i = 1,2; \\ V_i^2 \quad &: \quad = U_R \cap D\left((-1)^i, \varepsilon_0\right) \cap \{z : \operatorname{Im} z < 0\}, i = 1,2; \\ V_3 \quad &: \quad = U_R \setminus [D(-1,\varepsilon_0) \cup D(1,\varepsilon_0)]. \end{aligned}$$

Secondly, assume that $G \in PQ(K;\alpha)$ for some $K \geqslant 1, \alpha \ge 0$. Then for all $\varepsilon > 0$

$$mes\ \varphi\left(\alpha^j\left(V_i^j\right)\right) \preccurlyeq (n)^{\frac{\varepsilon-1}{K^2}}, mes\ \varphi\left(\alpha^j(V_3)\right) \preccurlyeq (n)^{\frac{\varepsilon-1}{K^2}}, i,j = 1,2$$

in [3, p. 658]. Then, from Lemma 1 it is easy to obtain

$$mes\ \alpha^j(U_R) \preccurlyeq (n)^{\frac{\varepsilon-2}{K^2(K^2+1)}}, \quad j = 1,2$$

for all $\varepsilon > 0$. Finally, for all $\varepsilon > 0$

$$\iint\limits_{U_R} \left|\varphi'\left(\alpha^j(\xi)\right)\right|^p d\sigma_\xi \preccurlyeq \left(\frac{1}{n}\right)^{\frac{1-\varkappa}{K^2}-\varepsilon}, \quad 2 \leqslant p < 2 + \frac{K^2+1}{K^2-1},$$

where $\varkappa := (p-2)\frac{K^2-1}{K^2+1}$ and $j = 1,2$ in [14, Lemma 3.2].

We get the following result by using these estimates and (3.9): let $G \in PQ(K;\alpha)$ for some $K \geqslant 1$, $0 \leqslant \alpha < \min\left\{\frac{2(p-1)}{p+2}, \frac{2}{p+2}\right\}$. Then there exists a polynomial $Q_n(z)$ with $Q_n(0) = 0$ for any number $n = 2,3,4,\ldots$ such that

$$\|\varphi - Q_n\|_{A_p^1(G)} \preccurlyeq \left(\frac{1}{n}\right)^{\vartheta}, \tag{3.12}$$

where

$$\vartheta \in \begin{cases} \left(0, \min\left\{\frac{2-(p+2)\alpha}{2p(1+\alpha)(K^2+1)}, \frac{p}{2K^2} + \frac{(2-p)}{pK^2(K^2+1)}\right\}\right) & ;1 < p < 2, \\ \left(0, \min\left\{\frac{2-(p+2)\alpha}{2p(1+\alpha)(K^2+1)}, \frac{1}{pK^2} - \frac{(p-2)(K^2-1)}{pK^2(K^2+1)}\right\}\right) & ;2 \leqslant p < 2 + \frac{K^2+1}{K^2-1} \end{cases}.$$

Now let us consider the polynomial

$$\widetilde{Q}_n(z) := Q_n(z) + \left[1 - Q_n'(0)\right] z.$$

It is clear that $\widetilde{Q}_n \in \wp$ satisfies normalization conditions $\widetilde{Q}_n(0) = 0$, $\widetilde{Q}_n'(0) = 1$. From the Mean Value Theorem we have

$$\left|1 - Q_n'(0)\right| \preccurlyeq \frac{1}{\pi d^{\frac{2}{p}}(0,L)} \left\|\varphi' - Q_n'\right\|_{A_p(G)}$$

and by means of (3.12) we obtain

$$\left\|\varphi - \widetilde{Q}_n\right\|_{A_p^1(G)} \preccurlyeq \left(\frac{1}{n}\right)^{\vartheta}.$$

So, if we consider the extremal property of the polynomials $B_{n,p}(z)$, then we have

$$\|\varphi - B_{n,p}\|_{A_p^1(G)} \preccurlyeq \left(\frac{1}{n}\right)^{\vartheta}. \tag{3.13}$$

This gives the proof of Theorem 1. □

4. RESULTS

For the given real numbers $K \geqslant 1$, $p > 1$, and $0 \leqslant \alpha < 1$ let us have the following notations:

$$\alpha^* \in \left[0, \sqrt{2}-1\right), \; \Delta(p,K) := \left(\frac{4-p}{2} + \frac{p+2}{8(K^2+1)}\right)^2 - 4\left(\frac{2-p}{2} - \frac{1}{4(K^2+1)}\right)$$

and

$$\beta := \beta(p,K) := \frac{1}{2}\sqrt{\Delta(p,K)} - \left(\frac{4-p}{4} + \frac{p+2}{16(K^2+1)}\right).$$

Theorem 2. *Let $G \in PQ(K,\alpha)$ for some K, $K \geqslant 1$ and α, $0 \leqslant \alpha < \min\{\frac{2(p-1)}{p+2}, \beta\}$ for $2 - \frac{1}{2(K^2+1)} < p < 2$. Then the p-Bieberbach polynomials $B_{n,p}(z)$ $(n \geqslant 2)$ satisfy*

$$\|\varphi - B_{n,p}\|_{C(\overline{G})} \preccurlyeq n^{-\gamma}$$

for each γ with $0 < \gamma < \frac{2-(p+2)\alpha}{2p(1+\alpha)(K^2+1)} - \frac{2}{p}(2 + 2\alpha - p)$.

Corollary 1. *Let $G \in PQ(K,\alpha)$ for some K, $1 \leqslant K \leqslant \sqrt{\frac{1}{2(2-p)}-1}$ and α, $0 \leqslant \alpha < \min\left\{\frac{2(p-1)}{p+2}, \beta(p,K)\right\}$ for $\frac{7}{4} \leqslant p < 2$. Then, the p-Bieberbach polynomials $B_{n,p}(z)$ $(n \geqslant 2)$ satisfy*

$$\|\varphi - B_{n,p}\|_{C(\overline{G})} \preccurlyeq n^{-\gamma}$$

for each γ with $0 < \gamma < \frac{2-(p+2)\alpha}{2p(1+\alpha)(K^2+1)} - \frac{2}{p}(2+2\alpha-p)$.

Although the approximation rate in Theorem 2 and Corollary 1 is the same, p has a lower bound depending on arbitrary $K \geqslant 1$ in Theorem 2 and K has an upper bound depending on arbitrary p, $p \in \left[\frac{7}{2}, 2\right)$ in Corollary 1.

Theorem 3. *Let $G \in PQ(K,\alpha)$ for some K, $1 \leqslant K < \sqrt{1+\frac{1}{p-2}}$ and α, $\sqrt{2}-1 \leqslant \alpha < \frac{2}{p+2}$ for $2 \leqslant p < 2\sqrt{2}$. Then the p-Bieberbach polynomials $B_{n,p}(z)$ $(n \geqslant 2)$ satisfy*

$$\|\varphi - B_{n,p}\|_{C(\overline{G})} \preccurlyeq n^{-\gamma}$$

for each γ with $0 < \gamma < \frac{2-(p+2)\alpha}{2p(1+\alpha)(K^2+1)} - \frac{2}{p}(2+2\alpha-p)$.

Theorem 4. *Let $G \in PQ(K,\alpha)$ for some K, $\sqrt{1+\frac{1}{p-2}} \leqslant K$ and for some α,*

$$\min\left\{\sqrt{2}-1, \frac{2(p-2)K^2-2(p-1)}{(8-p)K^2+2(p-1)}\right\} \leqslant \alpha < \frac{2}{p+2}$$

for $2 < p < 2\sqrt{2}$. Then the p-Bieberbach polynomials $B_{n,p}(z)$ $(n \geqslant 2)$ satisfy

$$\|\varphi - B_{n,p}\|_{C(\overline{G})} \preccurlyeq n^{-\gamma}$$

for each γ with $0 < \gamma < \frac{2-(p+2)\alpha}{2p(1+\alpha)(K^2+1)} - \frac{2}{p}(2+2\alpha-p)$.

Theorem 5. *Let $G \in PQ(K,\alpha)$ for some K, $\sqrt{1+\frac{1}{p-2}} \leqslant K$ and α, $\sqrt{2}-1 \leqslant \alpha < \min\left\{\frac{2(p-2)K^2-2(p-1)}{(8-p)K^2+2(p-1)}, \frac{(3-p)K^2+p-1}{4K^2(K^2+1)} + \frac{p-2}{2}\right\}$ for $2 < p < 2\sqrt{2}$. Then the p-Bieberbach polynomials $B_{n,p}(z)$ $(n \geqslant 2)$ satisfy*

$$\|\varphi - B_{n,p}\|_{C(\overline{G})} \preccurlyeq n^{-\gamma}$$

for each γ with $0 < \gamma < \left(\frac{1}{pK^2} - \frac{(p-2)\left(K^2-1\right)}{pK^2(K^2+1)}\right) - \frac{2}{p}(2+2\alpha-p)$.

Theorem 6. *Let $G \in PQ(K,\alpha)$ for some K, $K \geqslant 1$ and α, $0 \leqslant \alpha < \min\left\{\frac{2}{p+2}, \alpha^*\right\}$ for $2(1+\alpha^*) \leqslant p < 2 + \frac{K^2+1}{K^2-1}$. Then the p-Bieberbach polynomials $B_{n,p}(z)$ $(n \geqslant 2)$ satisfy*

$$\|\varphi - B_{n,p}\|_{C(\overline{G})} \preccurlyeq n^{-\gamma}$$

for each γ with $0 < \gamma < \frac{1}{p}\min\left\{\frac{2-(p+2)\alpha}{2(1+\alpha)(K^2+1)}, \frac{1}{K^2} - \frac{(p-2)\left(K^2-1\right)}{K^2(K^2+1)}\right\}$.

5. THE PROOF OF THEOREMS 2–6

To prove Theorems 2–6 we shall use a similar method to the one of Andrievskii and Simonenko employed in the proofs of analogous theorems for $p = 2$ (see [7,11] and [18]).

Lemma 4 [14]**.** *Let $G \subset \mathbb{C}$ be a simply connected domain so that*

$$\|\varphi - B_{n,p}\|_{A^1_p(G)} \preccurlyeq n^{-\mu}$$

for each $\mu \in (0,1)$, $n = 2,3,...$, and

$$\|P_n\|_{C(\overline{G})} \preccurlyeq \|P_n\|_{A^1_p(G)} \begin{cases} 1 & ,p > 2, \\ \sqrt{\log n} & ,p = 2, \\ n^{\eta} & ,0 < p < 2,\ \eta > 0, \end{cases} \tag{5.1}$$

for all polynomials $P_n(z)$ of degree $\leq n$ and $P_n(0) = 0$. Then

$$\|\varphi - B_{n,p}\|_{C(\overline{G})} \preccurlyeq n^{\eta-\mu},$$

where $0 < p < 2$.

Therefore, replacing μ by ϑ in (3.12) and taking $\eta = \frac{2}{p}(2\alpha + 2 - p)$ from Corollary 4.1 in [3], we obtain the proof of Theorems 2–6.

6. CONCLUSION

The main goal of this paper was to find the approximation rate of $B_{n,p}$ to φ in the uniform norm when the region has some certain singularity on the boundary.

It is seen from the theorems (Theorems 2–6) that the approximation rate depends not only on analytic properties (the quasiconformality coefficient K) but also on the geometric properties (the boundary has x^{α} type zero angles) of the region.

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p-Bieberbachi polünoomide ühtlane koonduvus nulliliste nurkadega piirkondades

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On tõestatud p-Bieberbachi polünoomide ühtlane koonduvus ühelisidusatel piirkondadel, mis on piiratud tükiti kvaasikonformse kõveraga, millel on teatud nullilised sisenurgad kahe kaare kokkupuute punktis.