Proceedings of the Estonian Academy of Sciences,
2011, **60**, 1, 12–24
doi: 10.3176/proc.2011.1.02
Available online at www.eap.ee/proceedings

MATHEMATICS

A strong convergence theorem on generalized equilibrium problems and strictly pseudocontractive mappings

Yan Hao

School of Mathematics, Physics and Information Science, Zhejiang Ocean University, Zhoushan 316004, China; zjhaoyan@yahoo.cn

Received 12 October 2009, revised 7 December 2009, accepted 13 August 2010

Abstract. In this paper, we consider a general iterative process for a generalized equilibrium problem and a strictly pseudocontractive mapping. A strong convergence theorem of common elements of the fixed point sets of the strictly pseudocontractive mapping and of the solution sets of the generalized equilibrium problem is established in the framework of Hilbert spaces.

Key words: nonexpansive mapping, inverse-strongly monotone mapping, strictly pseudocontractive mapping, equilibrium problem.

1. INTRODUCTION AND PRELIMINARIES

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H, and $A : C \to H$ a nonlinear mapping. Recall that the mapping A is said to be monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Let $S : C \to C$ be a nonlinear mapping. In this paper, we use $F(S)$ to denote the set of fixed points of S. Recall that the mapping S is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

S is said to be strictly pseudocontractive with the coefficient $k \in [0, 1)$ if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in C.$$

The class of strictly pseudocontractive mappings was introduced by Browder and Petryshyn [1] in 1967.

Let $A : C \to H$ be an inverse-strongly monotone mapping, F a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. In this paper, we consider the following generalized equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F,A)$, i.e.,

$$EP(F,A) = \{x \in C : F(x,y) + \langle Ax, y-x\rangle \geq 0, \quad \forall y \in C\}.$$

To study the generalized equilibrium problems (1.1), we may assume that F satisfies the following conditions:
(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x,y \in C$;
(A3) for each $x,y,z \in C$,

$$\limsup_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.
Next, we give some special cases of the problem (1.1).
(i) If $A \equiv 0$, then the generalized equilibrium problem (1.1) is reduced to the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F)$, i.e.,

$$EP(F) = \{x \in C : F(x,y) \geq 0, \quad \forall y \in C\}.$$

(ii) If $F \equiv 0$, then the problem (1.1) is reduced to the following classical variational inequality.

$$\text{Find } x \in C \text{ such that } \langle Ax, y-x\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

It is known that $x \in C$ is a solution to (1.3) if and only if x is a fixed point of the mapping $P_C(I - \rho A)$, where $\rho > 0$ is a constant and I is the identity mapping.

Recently, many authors considered iterative methods for the problems (1.1) and (1.2); see, for example, [2–13]. In 2007, Takahashi and Takahashi [12] considered the equilibrium problem (1.2) and a nonexpansive mapping by an iterative method. To be more precise, they proved the following Theorem.

Theorem TT1. *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(y_n, u) + \dfrac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S y_n, \quad n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\} \in [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$$\liminf_{n\to\infty} r_n > 0, \ \textit{and} \ \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)} f(z)$.

Recently, Takahashi and Takahashi [13] further considered the generalized equilibrium problem (1.1). They obtained the following result in a real Hilbert space.

Theorem TT2. *Let C be a closed convex subset of a real Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bi-function satisfying (A1), (A2), (A3), and (A4). Let A be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap EP(F,A) \neq \emptyset$. Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n \rangle + \dfrac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S[\alpha_n u + (1-\alpha_n) z_n], \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [0,1]$, and $\{r_n\} \subset [0, 2\alpha]$, satisfy

$$0 < c \leq \beta_n \leq d < 1, \quad 0 < a \leq \lambda_n \leq b < 2\alpha,$$

$$\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = 0, \ \lim_{n\to\infty} \alpha_n = 0, \ \text{and} \ \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to $z = P_{F(S)\cap EP(F,A)} u$.

Very recently, Qin, Kang, and Cho [11] considered the generalized equilibrium problem (1.1) and a strictly pseudocontractive mapping based on an iterative method. To be more precise, they proved the following results.

Theorem QKC. *Let C be a nonempty, closed convex subset of a Hilbert space H. Let F_1 and F_2 be two bi-functions from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping and $B : C \to H$ a β-inverse-strongly monotone mapping. Let $T : C \to C$ be a k-strict pseudo-contraction with a fixed point. Define a mapping $S : C \to C$ by $Sx = kx + (1-k)Tx$, $\forall x \in C$. Assume that $F = EP(F_1, A) \cap EP(F_2, B) \cap F(T) \neq \emptyset$. Let $u \in C$, $x_1 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} F_1(u_n, u) + \langle Ax_n, u - u_n \rangle + \dfrac{1}{r} \langle u - u_n, u_n - x_n \rangle \geq 0, \quad \forall u \in C, \\ F_2(v_n, v) + \langle Bx_n, v - v_n \rangle + \dfrac{1}{s} \langle v - v_n, v_n - x_n \rangle \geq 0, \quad \forall v \in C, \\ y_n = \gamma_n u_n + (1-\gamma_n) v_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S[\alpha_n u + (1-\alpha_n) y_n], \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$, $r \in (0, 2\alpha)$, and $s \in (0, 2\beta)$. If the above control sequences satisfy the following restrictions
(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(c) $\lim_{n\to\infty} \gamma_n = \gamma \in (0,1)$,
then the sequence $\{x_n\}$ defined by the iterative algorithm (1.6) converges strongly to $z \in F$, where $z = P_F u$.

In this paper, we consider a general iterative process for the generalized equilibrium problem (1.1) and a strictly pseudocontractive mapping. A strong convergence of common elements of the fixed point sets of the strictly pseudocontractive mapping and of the solution sets of the generalized equilibrium problem is established in the framework of Hilbert spaces. The results presented in this paper improve and extend the corresponding results announced by Qin, Kang, and Cho [11], Takahashi and Takahashi [12], and Takahashi and Takahashi [13].

In order to prove our main results, we also need the following lemmas.

Lemma 1.1 ([14]). *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a strictly pseudocontractive mapping. Then $I - T$ is demi-closed, that is, if $\{x_n\}$ is a sequence in C with $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x \in F(T)$.*

The following lemma can be found in [2], [3], and [13].

Lemma 1.2. *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then, the following hold:
(a) *T_r is single-valued;*
(b) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(c) $F(T_r) = EP(F)$;
(d) $\|T_s x - T_r x\| \leq \frac{s-r}{s}\|T_s x - x\|$;
(e) *$EP(F)$ is closed and convex.*

Lemma 1.3 ([15]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for each $n \geq 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 1.4 ([16]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ a k-strict pseudo-contraction with a fixed point. Define $S : C \to C$ by $S_a x = ax + (1-a)Sx$ for each $x \in C$. If $a \in [k,1)$, then S_a is nonexpansive with $F(S_a) = F(S)$.*

Lemma 1.5 ([17]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(a) $\sum_{n=1}^{\infty}\gamma_n = \infty$;
(b) $\limsup_{n\to\infty}\delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$. *Then* $\lim_{n\to\infty}\alpha_n = 0$.

2. MAIN RESULTS

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $C \times C$ to $\mathbb{R}$ which satisfy (A1)–(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping, $B : C \to H$ a β-inverse-strongly monotone mapping, and $S : C \to C$ a strictly pseudocontractive mapping with the coefficient $k \in [0,1)$. Assume that $\mathscr{F} := EP(F_1, A) \cap EP(F_2, B) \cap F(S)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner: $x_1 \in C$ and*

$$\begin{cases} F_1(u_n, u) + \langle Ax_n, u - u_n\rangle + \dfrac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ F_2(v_n, v) + \langle Bx_n, v - v_n\rangle + \dfrac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in C, \\ y_n = \alpha_n x + (1-\alpha_n)(\lambda_n u_n + (1-\lambda_n)v_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\gamma_n y_n + (1-\gamma_n)Sy_n), \quad \forall n \geq 1, \end{cases} \tag{2.1}$$

where $x \in C$ is a fixed element, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ are sequences in $(0,1)$ and $\{r_n\}$, $\{s_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following conditions
(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a \le \beta_n \le b < 1$;
(C3) $\lim_{n\to\infty}(\gamma_{n+1} - \gamma_n) = 0$ *and* $0 \le k \le \gamma_n < c < 1$;
(C4) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$ *and* $0 < d \le \lambda_n \le e < 1$;
(C5) $\lim_{n\to\infty}(r_{n+1} - r_n) = \lim_{n\to\infty}(s_{n+1} - s_n) = 0$;
(C6) $0 < f \le r_n \le g < 2\alpha$ *and* $0 < f' \le s_n \le g' < 2\beta$.
Then the sequence $\{x_n\}$ generated in (2.1) converges strongly to some point $\bar{x}$, where $\bar{x} = P_\Omega u$.

Proof. Note that u_n can be rewritten as

$$u_n = T_{r_n}(x_n - r_n A x_n), \quad \forall n \ge 1$$

and v_n can be rewritten as

$$v_n = T_{s_n}(x_n - s_n B x_n), \quad \forall n \ge 1.$$

Fix $p \in \mathscr{F}$. It follows that

$$p = Sp = T_{r_n}(p - r_n A p) = T_{s_n}(p - s_n B p), \quad \forall n \ge 1.$$

Note that $I - r_n A$ is nonexpansive for each $n \ge 1$. Indeed, for any $x, y \in C$, we see from the condition (C6) that

$$\begin{aligned} \|(I - r_n A)x - (I - r_n A)y\|^2 &= \|(x-y) - r_n(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2r_n\langle x-y, Ax-Ay\rangle + r_n^2\|Ax-Ay\|^2 \\ &\le \|x-y\|^2 - r_n(2\alpha - r_n)\|Ax-Ay\|^2 \\ &\le \|x-y\|^2. \end{aligned} \tag{2.2}$$

This shows that $I - r_n A$ is nonexpansive for each $n \ge 1$. In a similar way, we can obtain that $I - s_n B$ is nonexpansive for each $n \ge 1$. It follows that

$$\|u_n - p\| \le \|x_n - p\| \text{ and } \|v_n - p\| \le \|x_n - p\|. \tag{2.3}$$

Putting $w_n = \lambda_n u_n + (1-\lambda_n)v_n$ for each $n \ge 1$, we see that

$$\|w_n - p\| \le \lambda_n\|u_n - p\| + (1-\lambda_n)\|v_n - p\| \le \|x_n - p\|.$$

Put $S_n = \gamma_n I + (1-\gamma_n)S$ for each $n \ge 1$. It follows from Lemma 1.4 that S_n is nonexpansive with $F(S_n) = F(S)$ for each $n \ge 1$. Note that

$$\begin{aligned} \|x_{n+1} - p\| &\le \beta_n\|x_n - p\| + (1-\beta_n)\|S_n y_n - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\big(\alpha_n\|x - p\| + (1-\alpha_n)\|w_n - p\|\big) \\ &\le \beta_n\|x_n - p\| + \alpha_n(1-\beta_n)\|x - p\| + (1-\alpha_n)(1-\beta_n)\|x_n - p\| \\ &= \big(1 - \alpha_n(1-\beta_n)\big)\|x_n - p\| + \alpha_n(1-\beta_n)\|x - p\|. \end{aligned}$$

By mathematical inductions, we can easily see that the sequence $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$. In view of Lemma 1.2, we obtain that

$$\begin{aligned}
\|u_{n+1}-u_n\| &\le \|T_{r_{n+1}}(I-r_{n+1}A)x_{n+1}-T_{r_{n+1}}(I-r_nA)x_n\|+\|T_{r_{n+1}}(I-r_nA)x_n-T_{r_n}(I-r_nA)x_n\|\\
&\le \|(I-r_{n+1}A)x_{n+1}-(I-r_nA)x_n\|+\|T_{r_{n+1}}(I-r_nA)x_n-T_{r_n}(I-r_nA)x_n\|\\
&\le \|x_{n+1}-x_n\|+|r_{n+1}-r_n|\|Ax_n\|+\frac{r_{n+1}-r_n}{r_{n+1}}\|T_{r_{n+1}}(I-r_nA)x_n-(I-r_nA)x_n\|\\
&\le \|x_{n+1}-x_n\|+2M_1|r_{n+1}-r_n|, \qquad (2.4)
\end{aligned}$$

where M_1 is an appropriate constant such that

$$M_1=\max\left\{\sup_{n\ge1}\left\{\frac{\|T_{r_{n+1}}(I-r_nA)x_n-(I-r_nA)x_n\|}{r_{n+1}},\sup_{n\ge1}\{\|Ax_n\|\}\right\}\right\}.$$

In a similar way, we can obtain that

$$\|v_{n+1}-v_n\|\le\|x_{n+1}-x_n\|+2M_2|s_{n+1}-s_n|, \qquad (2.5)$$

where M_2 is an appropriate constant such that

$$M_2=\max\left\{\sup_{n\ge1}\left\{\frac{\|T_{s_{n+1}}(I-s_nB)x_n-(I-s_nB)x_n\|}{s_{n+1}},\sup_{n\ge1}\{\|Bx_n\|\}\right\}\right\}.$$

Note that

$$\|w_{n+1}-w_n\|\le\lambda_{n+1}|u_{n+1}-u_n|+(1-\lambda_{n+1})|v_{n+1}-v_n|+(\|v_n\|+\|u_n\|)|\lambda_{n+1}-\lambda_n|. \qquad (2.6)$$

Substituting (2.4) and (2.5) into (2.6), we arrive at

$$\|w_{n+1}-w_n\|\le\|x_{n+1}-x_n\|+M_3(|r_{n+1}-r_n|+|s_{n+1}-s_n|+|\lambda_{n+1}-\lambda_n|), \qquad (2.7)$$

where M_3 is an appropriate constant such that

$$M_3=\max\{2M_1,2M_2,\sup_{n\ge1}\{\|v_n\|+\|u_n\|\}\}.$$

It follows from (2.7) that

$$\begin{aligned}
\|y_{n+1}-y_n\| &\le (\|x\|+\|w_n\|)|\alpha_{n+1}-\alpha_n|+(1-\alpha_{n+1})\|w_{n+1}-w_n\|\\
&\le \|x_{n+1}-x_n\|+M_4(|r_{n+1}-r_n|+|s_{n+1}-s_n|+|\lambda_{n+1}-\lambda_n|+|\alpha_{n+1}-\alpha_n|), \qquad (2.8)
\end{aligned}$$

where M_4 is an appropriate constant such that $M_4=\max\{\sup_{n\ge1}\{\|x\|+\|w_n\|\},M_3\}$.

On the other hand, we have

$$\begin{aligned}
\|S_{n+1}y_{n+1}-S_ny_n\| &\le \|S_{n+1}y_{n+1}-S_{n+1}y_n\|+\|S_{n+1}y_n-S_ny_n\|\\
&\le \|y_{n+1}-y_n\|+\|\gamma_{n+1}y_n+(1-\gamma_{n+1})Sy_n-\gamma_ny_n-(1-\gamma_n)Sy_n\|\\
&\le \|y_{n+1}-y_n\|+M_5|\gamma_{n+1}-\gamma_n|, \qquad (2.9)
\end{aligned}$$

where M_5 is an appropriate constant such that $M_5=\sup_{n\ge1}\{\|y_n\|+\|Sy_n\|\}$. Combining (2.8) with (2.9), we arrive at

$$\begin{aligned}
&\|S_{n+1}y_{n+1}-S_ny_n\|-\|x_{n+1}-x_n\|\\
&\qquad\le \|y_{n+1}-y_n\|+M_5|\gamma_{n+1}-\gamma_n|-\|x_{n+1}-x_n\|\\
&\qquad\le M_6(|r_{n+1}-r_n|+|s_{n+1}-s_n|+|\lambda_{n+1}-\lambda_n|+|\alpha_{n+1}-\alpha_n|+|\gamma_{n+1}-\gamma_n|),
\end{aligned}$$

where M_6 is an appropriate constant such that $M_6=\max\{M_4,M_6\}$. In view of (C1), (C3), (C4), and (C5), we see that

$$\limsup_{n\to\infty}\left(\|S_{n+1}y_{n+1}-S_ny_n\|-\|x_{n+1}-x_n\|\right)\le0.$$

It follows from Lemma 1.3, that we obtain $\lim_{n\to\infty}\|S_n y_n - x_n\| = 0$.

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|Sy_n - x_n\| = 0. \tag{2.10}$$

On the other hand, we have

$$\begin{aligned}\|u_n - p\|^2 &\le \|(x_n-p) - r_n(Ax_n - Ap)\|^2\\ &= \|x_n-p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + r_n^2\|Ax_n - Ap\|^2\\ &\le \|x_n-p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2.\end{aligned} \tag{2.11}$$

In a similar way, we can obtain that

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - s_n(2\beta - s_n)\|Bx_n - Bp\|^2. \tag{2.12}$$

In view of (2.11) and (2.12), we see that

$$\begin{aligned}\|w_n - p\|^2 &\le \lambda_n\|u_n - p\|^2 + (1-\lambda_n)\|v_n - p\|^2\\ &\le \|x_n-p\|^2 - r_n(2\alpha - r_n)\lambda_n\|Ax_n - Ap\|^2 - s_n(2\beta - s_n)(1-\lambda_n)\|Bx_n - Bp\|^2.\end{aligned} \tag{2.13}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|S_n y_n - p\|^2\\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2\\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\big(\|\alpha_n(x-p) + (1-\alpha_n)(w_n - p)\|^2\big)\\ &\le \beta_n\|x_n - p\|^2 + \alpha_n\|x-p\|^2 + (1-\beta_n)\|w_n - p\|^2\\ &\le \|x_n - p\|^2 + \alpha_n\|x-p\|^2 - r_n(2\alpha - r_n)\lambda_n(1-\beta_n)\|Ax_n - Ap\|^2\\ &\quad - s_n(2\beta - s_n)(1-\lambda_n)(1-\beta_n)\|Bx_n - Bp\|^2.\end{aligned} \tag{2.14}$$

It implies from (C2), (C4), and (C6) that

$$\begin{aligned}f(2\alpha - g)d(1-b)\|Ax_n - Ap\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|x-p\|^2\\ &\le (\|x_n - p\| - \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \alpha_n\|x-p\|^2.\end{aligned}$$

In view of (2.10), we see that

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{2.15}$$

It also follows from (2.15) that

$$s_n(2\beta - s_n)(1-\lambda_n)(1-\beta_n)\|Bx_n - Bp\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\|x-p\|^2.$$

From (C2), (C4), and (C6) we see that

$$f'(2\beta - g')(1-e)(1-b)\|Bx_n - Bp\|^2 \le (\|x_n - p\| - \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \alpha_n\|x-p\|^2.$$

From (2.10), we can obtain that

$$\lim_{n\to\infty}\|Bx_n - Bp\| = 0. \tag{2.16}$$

On the other hand, we have

$$\begin{aligned}\|u_n-p\|^2&=\|T_{r_n}(I-r_nA)x_n-T_{r_n}(I-r_nA)p\|^2\\&\le\langle(I-r_nA)x_n-(I-r_nA)p,u_n-p\rangle\\&=\frac{1}{2}\Big(\|(I-r_nA)x_n-(I-r_nA)p\|^2+\|u_n-p\|^2-\|(I-r_nA)x_n-(I-r_nA)p-(u_n-p)\|^2\Big)\\&\le\frac{1}{2}\Big(\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n-r_n(Ax_n-Ap)\|^2\Big)\\&=\frac{1}{2}\Big(\|x_n-p\|^2+\|u_n-p\|^2-\big(\|x_n-u_n\|^2-2r_n\langle x_n-u_n,Ax_n-Ap\rangle+r_n^2\|Ax_n-Ap\|^2\big)\Big).\end{aligned}$$

This implies that

$$\|u_n-p\|^2\le\|x_n-p\|^2-\|x_n-u_n\|^2+2r_n\|x_n-u_n\|\|Ax_n-Ap\|. \tag{2.17}$$

In a similar way, we can also obtain that

$$\|v_n-p\|^2\le\|x_n-p\|^2-\|x_n-v_n\|^2+2s_n\|x_n-v_n\|\|Bx_n-Bp\|. \tag{2.18}$$

Note that

$$\begin{aligned}\|w_n-p\|^2&\le\lambda_n\|u_n-p\|^2+(1-\lambda_n)\|v_n-p\|^2\\&\le\|x_n-p\|^2-\lambda_n\|x_n-u_n\|^2+2r_n\lambda_n\|x_n-u_n\|\|Ax_n-Ap\|\\&\quad-(1-\lambda_n)\|x_n-v_n\|^2+2s_n(1-\lambda_n)\|x_n-v_n\|\|Bx_n-Bp\|,\end{aligned} \tag{2.19}$$

from which it follows that

$$\begin{aligned}\|x_{n+1}-p\|^2&\le\beta_n\|x_n-p\|^2+(1-\beta_n)\|S_ny_n-p\|^2\\&\le\beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2\\&\le\beta_n\|x_n-p\|^2+(1-\beta_n)\big(\|\alpha_n(x-p)+(1-\alpha_n)(w_n-p)\|^2\big)\\&\le\beta_n\|x_n-p\|^2+\alpha_n\|x-p\|^2+(1-\beta_n)\|w_n-p\|^2\\&\le\|x_n-p\|^2+\alpha_n\|x-p\|^2-\lambda_n(1-\beta_n)\|x_n-u_n\|^2\\&\quad+2r_n(1-\beta_n)\lambda_n\|x_n-u_n\|\|Ax_n-Ap\|-(1-\lambda_n)(1-\beta_n)\|x_n-v_n\|^2\\&\quad+2s_n(1-\lambda_n)(1-\beta_n)\|x_n-v_n\|\|Bx_n-Bp\|.\end{aligned} \tag{2.20}$$

It implies from (C2) and (C4) that

$$\begin{aligned}d(1-b)\|x_n-u_n\|^2&\le(\|x_n-p\|-\|x_{n+1}-p\|)\|x_{n+1}-x_n\|+\alpha_n\|x-p\|^2\\&\quad+2r_n(1-\beta_n)\lambda_n\|x_n-u_n\|\|Ax_n-Ap\|\\&\quad+2s_n(1-\lambda_n)(1-\beta_n)\|x_n-v_n\|\|Bx_n-Bp\|.\end{aligned}$$

In view of (2.10), (2.15), and (2.16), we arrive at

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{2.21}$$

From (2.20), we also have

$$\begin{aligned}(1-\lambda_n)(1-\beta_n)\|x_n-v_n\|^2&\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|x-p\|^2\\&\quad+2r_n(1-\beta_n)\lambda_n\|x_n-u_n\|\|Ax_n-Ap\|\\&\quad+2s_n(1-\lambda_n)(1-\beta_n)\|x_n-v_n\|\|Bx_n-Bp\|.\end{aligned}$$

By virtue of (C2) and (C4), we obtain that

$$\begin{aligned}(1-e)(1-b)\|x_n-v_n\|^2 \le (\|x_n-p\|-\|x_{n+1}-p\|)\|x_{n+1}-x_n\| + \alpha_n\|x-p\|^2 \\ +2r_n(1-\beta_n)\lambda_n\|x_n-u_n\|\|Ax_n-Ap\| \\ +2s_n(1-\lambda_n)(1-\beta_n)\|x_n-v_n\|\|Bx_n-Bp\|.\end{aligned}$$

It follows from (2.10), (2.15), and (2.16) that

$$\lim_{n\to\infty}\|x_n-v_n\|=0. \tag{2.22}$$

Note that

$$\|y_n-x_n\| \le \alpha_n\|x-x_n\| + (1-\alpha_n)\lambda_n\|u_n-x_n\| + (1-\alpha_n)(1-\lambda_n)\|v_n-x_n\|.$$

From (2.10), (2.21), and (2.22), we see that

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{2.23}$$

Note that

$$\gamma_n(y_n-x_n)+(1-\gamma_n)(Sy_n-x_n)=S_ny_n-x_n\to 0$$

as $n\to\infty$. On the other hand, we have

$$\|y_n-Sy_n\| \le \|y_n-x_n\|+\|x_n-Sy_n\|.$$

From (C3) and (2.23), we obtain that

$$\lim_{n\to\infty}\|y_n-Sy_n\|=0. \tag{2.24}$$

Next, we claim that

$$\limsup_{n\to\infty}\langle x-\bar{x}, y_n-\bar{x}\rangle \le 0.$$

To show this inequality, take a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle x-\bar{x}, y_n-\bar{x}\rangle = \lim_{i\to\infty}\langle x-\bar{x}, y_{n_i}-\bar{x}\rangle. \tag{2.25}$$

We may, without loss of generality, assume that $y_{n_i} \rightharpoonup \eta$. Next, we show that $\eta \in \mathscr{F}$. First, we show that $\eta \in EP(F_1,A)$. In view of $\|y_n-u_n\| \le \|y_n-x_n\|+\|x_n-u_n\|$, we see from (2.21) and (2.23) that $\lim_{n\to\infty}\|y_n-u_n\|=0$. It follows that $u_{n_i} \rightharpoonup \eta$. Note that

$$F_1(u_n,u)+\langle Ax_n, u-u_n\rangle + \frac{1}{r_n}\langle u-u_n, u_n-x_n\rangle \ge 0, \quad \forall u\in C.$$

From (A2), we see that

$$\langle Ax_n, u-u_n\rangle + \frac{1}{r_n}\langle u-u_n, u_n-x_n\rangle \ge F_1(u,u_n), \quad \forall u\in C.$$

Replacing n by n_i, we arrive at

$$\langle Ax_{n_i}, u-u_{n_i}\rangle + \left\langle u-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle \ge F_1(u,u_{n_i}), \quad \forall u\in C. \tag{2.26}$$

For t with $0<t\le 1$ and $u\in C$, let $u_t=tu+(1-t)\eta$. Since $u\in C$ and $\eta\in C$, we have $u_t\in C$. It follows from (2.26) that

$$\begin{aligned}\langle u_t-u_{n_i}, Au_t\rangle &\ge \langle u_t-u_{n_i}, Au_t\rangle - \langle Ax_{n_i}, u_t-u_{n_i}\rangle - \left\langle u_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle + F_1(u_t,u_{n_i}) \\ &= \langle u_t-u_{n_i}, Au_t-Au_{n,i}\rangle + \langle u_t-u_{n_i}, Au_{n,i}-Ax_{n_i}\rangle - \left\langle u_t-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle + F_1(u_t,u_{n_i}). \end{aligned} \tag{2.27}$$

From (2.21), we have $Au_{n_i} - Ax_{n_i} \to 0$ as $i \to \infty$. On the other hand, we obtain from the monotonicity of A that $\langle u_t - u_{n_i}, Au_t - Au_{n,i}\rangle \geq 0$. It follows from (A4) that

$$\langle u_t - \eta, Au_t\rangle \geq F_1(u_t, \eta). \tag{2.28}$$

From (A1), (A4), and (2.28), we obtain that

$$\begin{aligned} 0 = F_1(u_t,u_t) &\leq tF_1(u_t,u) + (1-t)F_1(u_t,\eta) \\ &\leq tF_1(u_t,u) + (1-t)\langle u_t - \eta, Au_t\rangle \\ &= tF_1(u_t,u) + (1-t)t\langle u - \eta, Au_t\rangle, \end{aligned}$$

which yields that $F_1(u_t,u) + (1-t)\langle u - \eta, Au_t\rangle \geq 0$. Letting $t \to 0$ in the above inequality, we arrive at

$$F_1(\eta,u) + \langle u - \eta, A\eta\rangle \geq 0.$$

This shows that $\eta \in EP(F_1,A)$. In a similar way, we can show that $\eta \in EP(F_2,B)$. On the other hand, we obtain from Lemma 1.1 that $\eta \in F(S)$. This proves that $\eta \in \mathscr{F}$. In view of (2.25), we see that

$$\limsup_{n\to\infty}\langle x - \bar{x}, y_n - \bar{x}\rangle \leq 0. \tag{2.29}$$

Finally, we show that $x_n \to \bar{x}$ as $n \to \infty$. Note that

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 &= \beta_n\langle x_n - \bar{x}, x_{n+1} - \bar{x}\rangle + (1-\beta_n)\langle S_n y_n - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &\leq \beta_n\|x_n - \bar{x}\|\,\|x_{n+1} - \bar{x}\| + (1-\beta_n)\|S_n y_n - \bar{x}\|\,\|x_{n+1} - \bar{x}\| \\ &\leq \frac{\beta_n}{2}(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2) + \frac{1-\beta_n}{2}(\|y_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2). \end{aligned}$$

It follows that

$$\|x_{n+1} - \bar{x}\|^2 \leq \beta_n\|x_n - \bar{x}\|^2 + (1-\beta_n)\|y_n - \bar{x}\|^2. \tag{2.30}$$

On the other hand, we have

$$\begin{aligned} \|y_n - \bar{x}\|^2 &= \alpha_n\langle x - \bar{x}, y_n - \bar{x}\rangle + (1-\alpha_n)\langle w_n - \bar{x}, y_n - \bar{x}\rangle \\ &\leq \alpha_n\langle x - \bar{x}, y_n - \bar{x}\rangle + (1-\alpha_n)\|w_n - \bar{x}\|\,\|y_n - \bar{x}\| \\ &\leq \alpha_n\langle x - \bar{x}, y_n - \bar{x}\rangle + (1-\alpha_n)\|x_n - \bar{x}\|\,\|y_n - \bar{x}\| \\ &\leq \alpha_n\langle x - \bar{x}, y_n - \bar{x}\rangle + \frac{1-\alpha_n}{2}(\|x_n - \bar{x}\|^2 + \|y_n - \bar{x}\|^2). \end{aligned}$$

It follows that

$$\|y_n - \bar{x}\|^2 \leq 2\alpha_n\langle x - \bar{x}, y_n - \bar{x}\rangle + (1-\alpha_n)\|x_n - \bar{x}\|^2. \tag{2.31}$$

Substituting (2.31) into (2.30), we arrive at

$$\|x_{n+1} - \bar{x}\|^2 \leq [1 - \alpha_n(1-\beta_n)]\|x_n - \bar{x}\|^2 + 2\alpha_n(1-\beta_n)\langle x - \bar{x}, y_n - \bar{x}\rangle.$$

In view of (2.29), we see from Lemma 1.5 that $x_n \to \bar{x}$ as $n \to \infty$. This completes the proof. □

Remark 2.2. Theorem 2.1 can be viewed as a generalization of Theorem QKC.

If S is a nonexpansive mapping, then Theorem 2.1 is reduced to the following.

Corollary 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $C \times C$ to $\mathbb{R}$ which satisfy* (A1)–(A4). *Let $A : C \to H$ be an α-inverse-strongly monotone* *mapping, $B : C \to H$ a β-inverse-strongly monotone mapping, and $S : C \to C$ a nonexpansive mapping. Assume that $\mathscr{F} := EP(F_1,A) \cap EP(F_2,B) \cap F(S)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner: $x_1 \in C$ and*

$$\begin{cases} F_1(u_n,u) + \langle Ax_n, u-u_n\rangle + \dfrac{1}{r_n}\langle u-u_n, u_n-x_n\rangle \geq 0, \quad \forall u \in C, \\ F_2(v_n,v) + \langle Bx_n, v-v_n\rangle + \dfrac{1}{s_n}\langle v-v_n, v_n-x_n\rangle \geq 0, \quad \forall v \in C, \\ y_n = \alpha_n x + (1-\alpha_n)(\lambda_n u_n + (1-\lambda_n)v_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\gamma_n y_n + (1-\gamma_n)Sy_n), \quad \forall n \geq 1, \end{cases}$$

where $x \in C$ is a fixed element, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ are sequences in $(0,1)$ and $\{r_n\}$, $\{s_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions
(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a \leq \beta_n \leq b < 1$;
(C3) $\lim_{n\to\infty}(\gamma_{n+1} - \gamma_n) = 0$;
(C4) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$ *and* $0 < d \leq \lambda_n \leq e < 1$;
(C5) $\lim_{n\to\infty}(r_{n+1} - r_n) = \lim_{n\to\infty}(s_{n+1} - s_n) = 0$;
(C6) $0 < f \leq r_n \leq g < 2\alpha$ *and* $0 < f' \leq s_n \leq g' < 2\beta$.
Then the sequence $\{x_n\}$ converges strongly to some point $\bar{x}$, where $\bar{x} = P_\Omega u$.

Putting $F_1 = F_2$ and $r_n = s_n$ for each $n \geq 1$, we can obtain the following result immediately.

Corollary 2.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping and $S : C \to C$ a strictly pseudocontractive mapping with the coefficient $k \in [0,1)$. Assume that $\mathscr{F} := EP(F,A) \cap F(S)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner: $x_1 \in C$ and*

$$\begin{cases} F(u_n,u) + \langle Ax_n, u-u_n\rangle + \dfrac{1}{r_n}\langle u-u_n, u_n-x_n\rangle \geq 0, \quad \forall u \in C, \\ y_n = \alpha_n x + (1-\alpha_n)u_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\gamma_n y_n + (1-\gamma_n)Sy_n), \quad \forall n \geq 1, \end{cases}$$

where $x \in C$ is a fixed element, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$, and $\{r_n\}$ is a positive number sequence. Assume that the above control sequences satisfy the following restrictions
(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a \leq \beta_n \leq b < 1$;
(C3) $\lim_{n\to\infty}(\gamma_{n+1} - \gamma_n) = 0$ *and* $0 \leq k \leq \gamma_n < c < 1$;
(C4) $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$;
(C5) $0 < f \leq r_n \leq g < 2\alpha$.
Then the sequence $\{x_n\}$ converges strongly to some point $\bar{x}$, where $\bar{x} = P_{\mathscr{F}} u$.

Remark 2.5. Note that the class of strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings as a special case. Corollary 2.4 can be viewed as a generalization of Theorem TT2.

Theorem 2.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping, $B : C \to H$ a β-inverse-strongly monotone mapping, and $S : C \to C$ a strictly pseudocontractive mapping with the coefficient $k \in [0,1)$. Assume that $\mathscr{F} := VI(C,A) \cap VI(C,B) \cap F(S)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner: $x_1 \in C$ and*

$$\begin{cases} y_n = \alpha_n x + (1-\alpha_n)(\lambda_n P_C(x_n - r_n A x_n) + (1-\lambda_n)P_C(x_n - s_n B x_n)), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\gamma_n y_n + (1-\gamma_n)Sy_n), \quad \forall n \geq 1, \end{cases}$$

where $x \in C$ is a fixed element, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ are sequences in $(0,1)$, and $\{r_n\}$, $\{s_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a \le \beta_n \le b < 1$;
(C3) $\lim_{n\to\infty}(\gamma_{n+1} - \gamma_n) = 0$ *and* $0 \le k \le \gamma_n < c < 1$;
(C4) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$ *and* $0 < d \le \lambda_n \le e < 1$;
(C5) $\lim_{n\to\infty}(r_{n+1} - r_n) = \lim_{n\to\infty}(s_{n+1} - s_n) = 0$;
(C6) $0 < f \le r_n \le g < 2\alpha$ *and* $0 < f' \le s_n \le g' < 2\beta$.

Then the sequence $\{x_n\}$ converges strongly to some point $\bar{x}$, where $\bar{x} = P_{\mathscr{F}} u$.

Proof. Putting $F_1 = 0$, we see that

$$\langle Ax_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \ge 0, \forall u \in C$$

is equivalent to

$$\langle x_n - r_n Ax_n - u_n, u_n - u\rangle \ge 0, \forall u \in C.$$

This implies that $u_n = P_C(x_n - r_n Ax_n)$. We also have $v_n = P_C(x_n - s_n Bx_n)$. We can obtain from Theorem 2.1 the desired results immediately. □

Theorem 2.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $C \times C$ to $\mathbb{R}$ which satisfy (A1)–(A4). Let $T_A : C \to C$ be a ρ_α-strict pseudocontraction, $B : C \to C$ a ρ_β-strict pseudocontraction, and $S : C \to C$ a k-strict pseudocontraction. Assume that $\mathscr{F} := EP(F_1, I - T_A) \cap EP(F_2, I - T_B) \cap F(S)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner: $x_1 \in C$ and*

$$\begin{cases} F_1(u_n, u) + \langle (I - T_A)x_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \ge 0, \quad \forall u \in C, \\ F_2(v_n, v) + \langle (I - T_B)x_n, v - v_n\rangle + \frac{1}{s_n}\langle v - v_n, v_n - x_n\rangle \ge 0, \quad \forall v \in C, \\ y_n = \alpha_n x + (1 - \alpha_n)(\lambda_n u_n + (1 - \lambda_n)v_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)(\gamma_n y_n + (1 - \gamma_n)Sy_n), \quad \forall n \ge 1, \end{cases}$$

where $x \in C$ is a fixed element, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ are sequences in $(0,1)$, and $\{r_n\}$, $\{s_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < a \le \beta_n \le b < 1$;
(C3) $\lim_{n\to\infty}(\gamma_{n+1} - \gamma_n) = 0$ *and* $0 \le k \le \gamma_n < c < 1$;
(C4) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$ *and* $0 < d \le \lambda_n \le e < 1$;
(C5) $\lim_{n\to\infty}(r_{n+1} - r_n) = \lim_{n\to\infty}(s_{n+1} - s_n) = 0$;
(C6) $0 < f \le r_n \le g < 1 - \rho_\alpha$ *and* $0 < f' \le s_n \le g' < 1 - \rho_\beta$.

Then the sequence $\{x_n\}$ converges strongly to some point $\bar{x}$, where $\bar{x} = P_{\mathscr{F}} u$.

3. CONCLUSION

The iterative process (2.1) which can be employed to approximate a common element in the common solution set of two equilibrium problems and in the fixed point set of a strict pseudocontraction is general. The main results presented in this paper include the corresponding results announced by Qin, Kang, and Cho [11]. It would be of interest to improve the results to certain Banach spaces.

ACKNOWLEDGEMENTS

This project was supported by the National Natural Science Foundation of China (No. 10901140). The author is extremely grateful to the editor for useful suggestions that improved the contents of the paper.

REFERENCES

1. Browder, F. E. and Petryshyn, W. V. Construction of fixed points of nonlinear mappings in Hilbert space. *J. Math. Anal. Appl.*, 1967, **20**, 197–228.
2. Blum, E. and Oettli, W. From optimization and variational inequalities to equilibrium problems. *Math. Stud.*, 1964, **63**, 123–145.
3. Combettes, P. L. and Hirstoaga, S. A. Equilibrium programming in Hilbert spaces. *J. Nonlinear Convex Anal.*, 2005, **6**, 117–136.
4. Ceng, L. C. and Yao, J. C. Hybrid viscosity approximation schemes for equilibrium problems and fixed point problems of infinitely many nonexpansive mappings. *Appl. Math. Comput.*, 2008, **198**, 729–741.
5. Colao, V., Marino, G., and Xu, H. K. An iterative method for finding common solutions of equilibrium and fixed point problems. *J. Math. Anal. Appl.*, 2008, **344**, 340–352.
6. Chang, S. S., Lee, H. W. J., and Chan, C. K. A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization. *Nonlinear Anal.*, 2009, **70**, 3307–3319.
7. Cho, Y. J., Qin, X., and Kang, J. I. Convergence theorems based on hybrid methods for generalized equilibrium problems and fixed point problems. *Nonlinear Anal.*, 2009, **71**, 4203–4214.
8. Moudafi, A. Weak convergence theorems for nonexpansive mappings and equilibrium problems. *J. Nonlinear Convex Anal.*, 2008, **9**, 37–43.
9. Plubtieng, S. and Punpaeng, R. A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings. *Appl. Math. Comput.*, 2008, **197**, 548–558.
10. Qin, X., Cho, Y. J., and Kang, S. M. Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces. *J. Comput. Appl. Math.*, 2009, **225**, 20–30.
11. Qin, X., Kang, S. M., and Cho, Y. J. Convergence theorems on generalized equilibrium problems and fixed point problems with applications. *Proc. Estonian Acad. Sci.*, 2009, **58**, 170–318.
12. Takahashi, S. and Takahashi, T. Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. *J. Math. Anal. Appl.*, 2007, **331**, 506–515.
13. Takahashi, S. and Takahashi, W. Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space. *Nonlinear Anal.*, 2008, **69**, 1025–1033.
14. Marino, G. and Xu, H. K. Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. *J. Math. Anal. Appl.*, 2007, **329**, 336–346.
15. Suzuki, T. Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochne integrals. *J. Math. Anal. Appl.*, 2005, **305**, 227–239.
16. Zhou, H. Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert spaces. *Nonlinear Anal.*, 2008, **69**, 456–462.
17. Xu, H. K. Iterative algorithms for nonlinear operators. *J. London Math. Soc.*, 2002, **66**, 240–256.

Tugev koonduvusteoreem üldistatud tasakaalu ülesannete ja rangelt pseudoahendavate kujutuste kohta

Yan Hao

On uuritud üldist iteratiivset protsessi üldistatud tasakaalu ülesandel ja rangelt pseudoahendavatel kujutustel. On tõestatud tugev koonduvusteoreem rangelt pseudoahendavate kujutuste ühiste püsipunktihulkade ja üldistatud tasakaalu ülesande lahendihulkade kohta Hilberti ruumides, parandades mitmete eelnevate autorite tulemusi.