Proceedings of the Estonian Academy of Sciences,
2011, **60**, 1, 1–11
doi: 10.3176/proc.2011.1.01
Available online at www.eap.ee/proceedings

MATHEMATICS

On pseudo-slant submanifolds of trans-Sasakian manifolds

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Received 30 January 2009, accepted 24 August 2010

Abstract. The object of the present paper is to study pseudo-slant submanifolds of trans-Sasakian manifolds. Integrability conditions of the distributions on these submanifolds are worked out. Some interesting results regarding such manifolds have also been deduced. An example of a pseudo-slant submanifold of a trans-Sasakian manifold is given.

Key words: trans-Sasakian manifolds, contact metric manifolds, pseudo-slant submanifolds.

1. INTRODUCTION

In 1990, Chen [6] introduced the concept of slant immersions as a generalization of both holomorphic and totally real immersions. Many authors have studied slant immersions in Hermitian manifolds. Lotta [9], introduced the notion of slant immersions in contact manifolds. In papers [3,4], slant submanifolds of K-contact and Sasakian manifolds have been characterized by Cabrerizo et al. Recently, Carriazo [5] defined and studied bi-slant immersions in almost Hermitian manifolds and simultaneously gave the notion of pseudo-slant submanifolds in almost Hermitian manifolds. The contact version of pseudo-slant submanifolds has been defined and studied by V. A. Khan and M. A. Khan [8]. Slant submanifolds of trans-Sasakian manifolds have been studied by Gupta et al. [7]. In an analogous way we would like to extend the notion of pseudo-slant submanifolds in trans-Sasakian manifolds. The present paper is organized as follows.

Preliminaries are given in Section 2. In Section 3, we define pseudo-slant submanifolds of trans-Sasakian manifolds. Section 4 deals with the integrability conditions of the distributions of such submanifolds and some other geometric results. Section 5 contains an example of a pseudo-slant submanifold of a trans-Sasakian manifold.

2. PRELIMINARIES

Let $\tilde{M}$ be a $(2n+1)$-dimensional C^∞-differentiable manifold endowed with the almost contact metric structure (ϕ,ξ,η,g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g a Riemannian metric on $\tilde{M}$, all these tensor fields satisfying [1]

$$\phi^2(X) = -X+\eta(X)\xi,\ \ \eta(\xi)=1,\ \ g(X,\xi)=\eta(X), \tag{2.1}$$

$$g(\phi X,\phi Y) = g(X,Y)-\eta(X)\eta(Y), \tag{2.2}$$

$$\phi\xi=0,\ \ \eta\phi=0,\ \ g(X,\phi Y)=-g(\phi X,Y), \tag{2.3}$$

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for any $X,Y \in T\tilde{M}$. Here $T\tilde{M}$ is the standard notation for the tangent bundle of $\tilde{M}$. The two-form Φ denotes the fundamental two-form and is given by $g(X,\phi Y) = \Phi(X,Y)$. The manifold is said to be contact if $\Phi = d\eta$.

If ξ is a Killing vector field with respect to g, the contact metric structure is called a K-contact structure. It is known that a contact metric manifold is K-contact if and only if $\tilde{\nabla}_X \xi = -\phi X$, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$. The almost contact structure $\tilde{M}$ is said to be normal if $[\phi,\phi] + 2d\eta \otimes \xi = 0$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ. A Sasakian manifold is a normal contact metric manifold. Every Sasakian manifold is K-contact. A three-dimensional K-contact manifold is Sasakian. An almost contact metric manifold is Sasakian if and only if

$$(\tilde{\nabla}_X \phi)Y = g(X,Y)\xi - \eta(Y)X. \tag{2.4}$$

Moreover, on a Sasakian manifold

$$\tilde{\nabla}_X \xi = -\phi X, \tag{2.5}$$

for any $X \in T\tilde{M}$ and ξ is the structure vector field.

An almost contact metric structure (ϕ,ξ,η,g) on $\tilde{M}$ is called a trans-Sasakian structure of type (α,β) if it satisfies

$$(\tilde{\nabla}_X \phi)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X,Y)\xi - \eta(Y)\phi X\}, \tag{2.6}$$

for certain functions α and β on $\tilde{M}$, where $\tilde{\nabla}$ means the covariant differentiation with respect to g. In particular, it is normal and generalizes cosymplectic, α-Sasakian, and β-Kenmotsu manifolds. If $\beta = 0$, then the structure is called α-Sasakian. If $\alpha = 0$, then the structure is called β-Kenmotsu. If both α and β are zero, then the manifold reduces to a cosymplectic manifold [2]. If α and β are not simultaneously zero, then we shall call a trans-Sasakian manifold a proper trans-Sasakian manifold.

Again, it is known that a trans-Sasakian manifold of dimension ≥ 5 is either α-Sasakian or β-Kenmotsu or cosymplectic [10]. We know that a trans-Sasakian structure satisfies

$$\tilde{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.7}$$

for any $X \in T\tilde{M}$ and ξ is the structure vector field.

Let M be a submanifold immersed in a $(2n+1)$-dimensional contact metric manifold $\tilde{M}$; we denote by the same symbol g the induced metric on M. TM is the tangent bundle of the manifold M and $T^{\perp}M$ is the set of vector fields normal to M. Then the Gauss and Weingarten formula is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \ \ (X,Y \in TM), \tag{2.8}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \ \ (N \in T^{\perp}M), \tag{2.9}$$

for any $X,Y \in TM$ and $N \in T\tilde{M}$, where $\nabla^{\perp}$ is the connection in the normal bundle. The second fundamental form h and A_N are related by

$$g(A_N X,Y) = g(h(X,Y),N). \tag{2.10}$$

For any $X \in TM$, $N \in T^{\perp}M$, we write

$$\phi X = TX + NX, \ \ (TX \in TM, NX \in T^{\perp}M), \tag{2.11}$$

$$\phi N = tN + nN, \ \ (tN \in TM, nN \in T^{\perp}M). \tag{2.12}$$

The submanifold M is invariant if N is identically zero. On the other hand, M is anti-invariant if T is identically zero. From (2.3) and (2.11), we have

$$g(X,TY) = -g(TX,Y), \tag{2.13}$$

for any $X,Y \in TM$.

From now on, we put $Q = T^2$. We define

$$(\tilde{\nabla}_X Q)Y = \nabla_X QY - Q\nabla_X Y, \tag{2.14}$$

$$(\tilde{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.15}$$

$$(\tilde{\nabla}_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y, \tag{2.16}$$

for any $X,Y \in TM$. In view of (2.8), (2.11), and (2.7) it follows that

$$\nabla_X \xi = -\alpha TX + \beta(X - \eta(X)\xi), \tag{2.17}$$

$$h(X,\xi) = -\alpha NX. \tag{2.18}$$

3. PSEUDO-SLANT SUBMANIFOLDS OF TRANS-SASAKIAN MANIFOLDS

Definition 3.1. *We say that M is a pseudo-slant submanifold of a trans-Sasakian manifold $\tilde{M}$ if there exist two orthogonal distributions D_1 and D_2 on M such that* [8]
(i) *TM admits the orthogonal direct decomposition*

$$TM = D_1 \oplus D_2 \oplus <\xi>,$$

(ii) *the distribution D_1 is anti-invariant, that is,*

$$\phi D_1 \subseteq T^{\perp}M,$$

(iii) *the distribution D_2 is slant with slant angle $\theta \neq \frac{\pi}{2}$, that is, the angle between D_2 and $\phi(D_2)$ is a constant θ.*

From the above definition it is clear that if $\theta = 0$, then the pseudo-slant submanifold is a semi-invariant submanifold. On the other hand, if we denote the dimension of D_i by d_i, for $i = 1,2$, then we find the following cases:
(a) If $d_2 = 0$, then M is an anti-invariant submanifold.
(b) If $d_1 = 0$ and $\theta = 0$, then M is an invariant submanifold.
(c) If $d_1 = 0$ and $\theta \neq 0$, then M is a proper slant submanifold, with the slant angle $\theta \neq 0$.
A pseudo-slant submanfold is proper if $d_1 d_2 \neq 0$ and $\theta \neq 0$.

4. INTEGRABILITY OF THE DISTRIBUTIONS

Theorem 4.1. *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\tilde{M}$. Then*

$$A_{\phi Y}X = A_{\phi X}Y, \tag{4.1}$$

for all $X,Y \in D_1$.

Proof. In view of (2.10),

$$g(A_{\phi Y}X,Z) = g(h(X,Z),\phi Y) = -g(\phi h(X,Z),Y). \tag{4.2}$$

By virtue of (2.8), (4.2) reduces to

$$\begin{aligned} g(A_{\phi Y}X,Z) &= -g(\phi\tilde{\nabla}_Z X,Y) + g(\phi\nabla_Z X,Y) \\ &= -g(\phi\tilde{\nabla}_Z X,Y), \ \text{[since } \phi\nabla_Z X \in T^{\perp}M] \\ &= g((\tilde{\nabla}_Z\phi)X,Y) - g(\tilde{\nabla}_Z \phi X,Y). \end{aligned} \tag{4.3}$$

Now, for $X \in D_1$, $\phi X \in T^{\perp}M$. Hence, from (2.9) we have

$$\tilde{\nabla}_Z \phi X = -A_{\phi X} Z + \nabla^{\perp}_Z \phi X. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain

$$g(A_{\phi Y} X, Z) = g((\tilde{\nabla}_Z \phi)X, Y) + g(A_{\phi X} Z, Y). \tag{4.5}$$

Since $h(X,Y) = h(Y,X)$, it follows from (2.10) that

$$g(A_{\phi X} Z, Y) = g(A_{\phi X} Y, Z).$$

Hence, from (4.5) we obtain, with the help of (2.6),

$$\begin{aligned} g(A_{\phi Y} X, Z) - g(A_{\phi X} Y, Z) &= g(\alpha(g(Z,X)\xi - \eta(X)Z), Y) \\ &\quad + \beta(g(\phi Z, X)\xi - \eta(X)\phi Z), Y) \\ &= \alpha\eta(Y)g(Z,X) - \alpha\eta(X)g(Z,Y) \\ &\quad + \beta\eta(Y)g(\phi Z, X) - \beta\eta(X)g(\phi Z, Y). \end{aligned} \tag{4.6}$$

The above equation yields

$$A_{\phi Y} X - A_{\phi X} Y = \alpha(\eta(Y)X - \eta(X)Y) - \beta(\eta(Y)\phi X - \eta(X)\phi Y).$$

Since $X, Y, Z \in D_1$, an orthonormal distribution to the distribution $< \xi >$, it follows that $\eta(X) = \eta(Y) = 0$. Therefore, the above equation reduces to

$$A_{\phi Y} X = A_{\phi X} Y. \qquad \square$$

Remark 4.1. As particular cases the above result holds for α-Sasakian, β-Kenmotsu, and cosymplectic manifolds. For the Sasakian case the above result has been proved in [8].

Theorem 4.2. *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\tilde{M}$. Then the distribution $D_1 \oplus < \xi >$ is integrable.*

Proof. Since $h(X,Y) = h(Y,X)$, in view of (2.8) we see that

$$\nabla_X Y - \nabla_Y X = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X. \tag{4.7}$$

Let $X \in D_1, Y \in D_2$, then

$$(\tilde{\nabla}_X g)(Y,Z) = \tilde{\nabla}_X g(Y,Z) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z)$$

or,

$$0 = 0 - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z).$$

Hence

$$g(\tilde{\nabla}_X Y, Z) = -g(Y, \tilde{\nabla}_X Z). \tag{4.8}$$

Now

$$\begin{aligned} g([X,\xi], TZ) &= g(\nabla_X \xi - \nabla_\xi X, TZ) \\ &= g(\tilde{\nabla}_X \xi - \tilde{\nabla}_\xi X, TZ) \\ &= g(\tilde{\nabla}_X \xi, TZ) - g(\tilde{\nabla}_\xi X, TZ). \end{aligned} \tag{4.9}$$

Since $X \in D_1$ and $Z \in D_2$, where D_1 and D_2 are two orthogonal distributions and D_1 is anti-invariant, in view of (2.7), (4.8) we obtain from (4.9)

$$g([X,\xi],TZ) = g(\tilde{\nabla}_\xi TZ, X). \tag{4.10}$$

In view of (2.6),

$$(\tilde{\nabla}_\xi \phi)Y = 0. \tag{4.11}$$

In virtue of (2.11) and (4.11), equation (4.10) yields

$$g([X,\xi],TZ) = 0.$$

Hence $[X,\xi] \in D_1$ for $X \in D_1$. Therefore, the distribution $D_1 \oplus < \xi >$ is integrable. □

Remark 4.2. As particular cases the above result holds for α-Sasakian, β-Kenmotsu, and cosymplectic manifolds. For the Sasakian case the above result has been proved in [8].

Theorem 4.3. *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\tilde{M}$. Then for any $X,Y \in D_1 \oplus D_2$*

$$g([X,Y],\xi) = 2\alpha g(X,TY). \tag{4.12}$$

Proof.

$$g([X,Y],\xi) = g(\nabla_X Y,\xi) - g(\nabla_Y X,\xi). \tag{4.13}$$

In view of (4.8) we have from above

$$g([X,Y],\xi) = -g(\nabla_X \xi,Y) + g(\nabla_Y \xi,X). \tag{4.14}$$

By (2.17), (4.14) yields

$$g([X,Y],\xi) = 2\alpha g(X,TY).$$ □

The above equation gives the following:

Corollary 4.1. *In a proper trans-Sasakian manifold and α-Sasakian manifold the distribution $D_1 \oplus D_2$ is not integrable.*

Suppose $\alpha = 0$, that is, the manifold is β-Kenmotsu. Then $g([X,Y],\xi) = 0$. This implies that $[X,Y] \in D_1 \oplus D_2$, for $X,Y \in D_1 \oplus D_2$. In other words, we have the following

Corollary 4.2. *In a β-Kenmotsu manifold the distribution $D_1 \oplus D_2$ is integrable.*

Again, in a similar manner we have

Corollary 4.3. *In a cosymplectic manifold the distribution $D_1 \oplus D_2$ is integrable.*

Theorem 4.4. *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\tilde{M}$. Then the anti-invariant distribution D_1 is integrable.*

Proof. For any $X \in TM$, let

$$X = P_1 X + P_2 X + \eta(X)\xi, \tag{4.15}$$

where P_i, $i = 1,2$ are projection maps on the distribution D_i. From (4.15) it follows that

$$\phi X = NP_1 X + TP_2 X + NP_2 X,$$

$$TX = TP_2 X, \quad NX = NP_1 X + NP_2 X.$$

Now for any $X,Y \in D_1$ and $Z \in D_2$,

$$g([X,Y],TZ) = g([X,Y],TP_2Z) = -g(\phi[X,Y],P_2Z). \tag{4.16}$$

Now

$$\begin{aligned} \phi[X,Y] &= \phi\nabla_X Y - \phi\nabla_Y X \\ &= \phi\tilde{\nabla}_X Y - \phi\tilde{\nabla}_Y X \\ &= \tilde{\nabla}_X \phi Y - (\tilde{\nabla}_X \phi)Y - \tilde{\nabla}_Y \phi X + (\tilde{\nabla}_Y \phi)X. \end{aligned} \tag{4.17}$$

In view of (2.6) and (2.9) and keeping in mind that $g(U,V) = 0$ for $U \in D_1$ and $V \in D_2$, we obtain from (4.16)

$$\begin{aligned} g([X,Y],TP_2Z) = &- g(A_{\phi X}Y - A_{\phi Y}X - \alpha(\eta(Y)X - \eta(X)Y) \\ &+ \beta(\eta(Y)\phi X - \eta(X)\phi Y), P_2Z). \end{aligned} \tag{4.18}$$

For $X,Y \in D_1$, we get $\eta(X) = \eta(Y) = 0$. Hence Theorem 4.1 and the above equation yield $g([X,Y],TZ) = 0$, that is, $[X,Y] \in D_1$ for $X,Y \in D_1$. Therefore the distribution D_1 is integrable. □

The immediate consequence of the above theorem is the following:

Corollary 4.4. *On a pseudo-slant submanifold M of a trans-Sasakian manifold $\tilde{M}$, the distribution $D_1 \oplus < \xi >$ is integrable.*

Remark 4.3. The above result also holds for cosymplectic, α-Sasakian, and β-Kenmotsu manifolds.

For a Sasakian manifold the above result was proved by V. A. Khan and M. A. Khan [8].

Theorem 4.5. *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\tilde{M}$. Then the slant distribution D_2 is not integrable.*

Proof. Since $g([X,Y],\xi) = 2\alpha g(X,TY)$, by the definition of the pseudo-slant submanifold the proof follows. □

The above theorem produces

Corollary 4.5. *In an α-Sasakian manifold the slant distribution D_2 is not integrable.*

Theorem 4.6. *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$, such that $\xi \in TM$. Then M is a pseudo-slant submanifold if and only if there exists a constant $\lambda \in (0,1]$ such that*
(a) *$D = \{X \in TM | T^2X = -\lambda X\}$ is a distribution on M.*
(b) *For any $X \in TM$, orthogonal to D, $TX = 0$.*

Furthermore, in this case $\lambda = \cos^2\theta$, where θ denotes the slant angle of D.

Proof. Follows from [8]. □

Theorem 4.7. *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\tilde{M}$. Then $\nabla Q = 0$ if and only if M is an anti-invariant submanifold.*

Proof. If we consider the distribution $D_2 \oplus < \xi >$, then from Theorem 4.6 we can write

$$T^2X = -\lambda(X - \eta(X)\xi). \tag{4.19}$$

Denote by θ the slant angle of M. Then, replacing X by $\nabla_X Y$, we get from (4.19)

$$Q(\nabla_X Y) = -\cos^2\theta(\nabla_X Y) + \cos^2\theta\,\eta(\nabla_X Y)\xi, \tag{4.20}$$

for any $X,Y \in D_2 \oplus < \xi >$.

Equation (4.19) also gives

$$\begin{aligned}\nabla_X QY = & - \cos^2\theta(\nabla_X Y) + \cos^2\theta\eta(\nabla_X Y)\xi \\ & + \cos^2\theta g(Y, \nabla_X \xi)\xi + \cos^2\theta\eta(Y)\nabla_X \xi,\end{aligned} \tag{4.21}$$

because $X\eta(Y) = \eta(\nabla_X Y) + g(Y, \nabla_X \xi)$. Now, since M is a submanifold of a trans-Sasakian manifold $\tilde{M}$,

$$\nabla_X \xi = -\alpha TX + \beta(X - \eta(X)\xi), \tag{4.22}$$

for any $X \in TM$. Putting the value of $\nabla_X \xi$ in (4.21), we obtain

$$\begin{aligned}\nabla_X QY = & - \cos^2\theta(\nabla_X Y) + \cos^2\theta\eta(\nabla_X Y)\xi \\ & - \alpha\cos^2\theta g(Y, TX)\xi - \alpha\cos^2\theta\eta(Y)TX \\ & + \beta\cos^2\theta(g(X,Y)\xi - \eta(X)\eta(Y)\xi) \\ & + \beta\cos^2\theta(\eta(Y)X - \eta(X)\eta(Y)\xi).\end{aligned} \tag{4.23}$$

Combining (4.20) and (4.23), we find

$$\begin{aligned}(\nabla_X Q)Y = & - \alpha\cos^2\theta(g(X, TX)\xi - \eta(Y)TX) \\ & + \beta\cos^2\theta(g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X),\end{aligned} \tag{4.24}$$

for any $X, Y \in D_2 \oplus < \xi >$. Here, we note that

$$g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X \neq 0.$$

Hence $\nabla Q = 0$ if and only if $\theta = \frac{\pi}{2}$ holds in $D_2 \oplus < \xi >$. Again, D_1 is anti-invariant by definition. Thus, the theorem follows. □

As a consequence of Theorem 4.7 we immediately obtain

Corollary 4.6. *In a pseudo-slant submanifold of an α-Sasakian manifold $\nabla Q = 0$ if and only if the submanifold is anti-invariant.*

Corollary 4.7. *In a pseudo-slant submanifold of a β-Kenmotsu manifold $\nabla Q = 0$ if and only if the submanifold is anti-invariant.*

But for a cosymplectic manifold we have

Corollary 4.8. *In a submanifold of a cosymplectic manifold ∇Q is always zero, whether the submanifold is anti-invariant or not.*

Theorem 4.8. *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ with a slant angle θ. Then, at each point $x \in M$, $Q_{|D}$ has only one eigenvalue $\lambda_1 = \cos^2\theta$, for the slant distribution D of M.*

Proof. Follows from [9]. □

Theorem 4.9. *Let M be a submanifold of a trans-Sasakian manifold $\tilde{M}$ with $TM = D_1 \oplus D_2 \oplus < \xi >$. Then $\tilde{M}$ is pseudo-slant if and only if*
(a) *the endomorphism $Q_{|D_2}$ has only one eigenvalue at each point of M,*
(b) *there exists a function $\lambda : M \to [0,1]$ such that*

$$\begin{aligned}(\nabla_X Q)Y \quad = \quad & \lambda\{\alpha(g(X, PY)\xi - \eta(Y)PX) \\ & + \beta(g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X)\},\end{aligned} \tag{4.25}$$

for any $X, Y \in D_2 \oplus < \xi >$. Moreover, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

Proof. Statements (a) and (b) follow from (4.24) and Theorem 4.8.

Conversely, suppose that (a) and (b) hold. Let $\lambda_1(x)$ be the eigenvalue of $Q_{|D_2}$ at each point x of M and $Y \in D_2$ be a unit eigenvector associated with λ_1, that is, $QY = \lambda_1 Y$. Then from (b) we get

$$\begin{aligned} X(\lambda_1 Y) + \lambda_1 \nabla_X Y &= (\nabla_X Q)Y + Q(\nabla_X)Y \\ &= \nabla_X(QY) = Q(\nabla_X Y) + \lambda(\alpha g(X,TY)\xi) + \beta(g(X,Y)\xi), \end{aligned} \tag{4.26}$$

for any $X \in TM$. Since both $\nabla_X Y$ and $Q(\nabla_X Y)$ are perpendicular to Y, we conclude that λ_1 is a constant on M.

Now we want to prove that M is pseudo-slant. To fulfill our purpose, in view of Theorem 4.6, it is sufficient to show that there exists a constant μ such that

$$T^2 = Q = -\mu I,$$

holds in D_2. To this end, let X be in $D_2 \subset TM$. Then by condition (a)

$$\overline{X} = X - \eta(X)\xi \in D_2.$$

Hence,

$$QX = \lambda_1 X,$$

where λ_1 is the eigenvalue of $Q_{|D_2}$. Putting $\lambda_1 = -\mu$, we see that condition (a) of Theorem 4.6 is satisfied. Now, $\phi\xi = 0$ implies $T\xi = 0$. Again, for all $Y \in D_1$, $g(\phi Y, Y) = 0$. This gives that $\phi Y \in T^{\perp}D_1$. For $X \in D_2$, the slant distribution, $g(\phi Y, X) = g(Y, \phi X) = g(Y, kX) = kg(Y, X) = 0$, $k \in (0,1)$. This shows that $\phi Y \in T^{\perp}M$. Therefore $TY = 0$.

Thus, by the use of Theorem 4.6, the above theorem is proved. Since $Q_{|D_2} = \lambda_1 I$, we have $Q\overline{X} = \lambda_1 \overline{X}$ and so $\lambda_1 \overline{X} = \lambda_1(X - \eta(X)\xi)$. By taking $\mu = -\lambda_1$, we get M is pseudo-slant.

Moreover, since M is pseudo-slant, then from (4.25), $\lambda = -\lambda_1 = \mu = \cos^2\theta$,where θ denotes the slant angle of M. □

For an α-Sasakian manifold the above theorem yields the following:

Corollary 4.9. *Let M be a submanifold of an α-Sasakian manifold $\tilde{M}$ with $TM = D_1 \oplus D_2 \oplus < \xi >$. Then $\tilde{M}$ is pseudo-slant if and only if*
(a) *the endomorphism $Q_{|D_2}$ has only one eigenvalue at each point of M,*
(b) *there exists a function $\lambda : M \to [0,1]$ such that*

$$(\nabla_X Q)Y = \lambda\{\alpha(g(X,PY)\xi - \eta(Y)PX)\} \tag{4.27}$$

for any $X, Y \in D_2 \oplus < \xi >$. Moreover, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

In the β-Kenmotsu case Theorem 4.9 takes the following form:

Corollary 4.10. *Let M be a submanifold of a β-Kenmotsu manifold $\tilde{M}$ with $TM = D_1 \oplus D_2 \oplus < \xi >$. Then $\tilde{M}$ is pseudo-slant if and only if*
(a) *the endomorphism $Q_{|D_2}$ has only one eigenvalue at each point of M,*
(b) *there exists a function $\lambda : M \to [0,1]$ such that*

$$(\nabla_X Q)Y = \beta(g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X), \tag{4.28}$$

for any $X, Y \in D_2 \oplus < \xi >$. Moreover, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

For the cosymplectic case we obtain the following:

Corollary 4.11. *Let M be a submanifold of a cosymplectic manifold $\tilde{M}$ with $TM = D_1 \oplus D_2 \oplus <\xi>$. Then $\tilde{M}$ is pseudo-slant if and only if*
(a) *the endomorphism $Q_{|D_2}$ has only one eigenvalue at each point of M,*
(b) *there exists a function $\lambda : M \to [0,1]$ such that*

$$(\nabla_X Q)Y = 0, \tag{4.29}$$

for any $X, Y \in D_2 \oplus <\xi>$.

Theorem 4.10. *In a pseudo-slant submanifold of a trans-Sasakian manifold*

$$\begin{aligned}(\nabla_X T)Y &= A_{NY}X + th(X,Y)\\ &+ \alpha(g(X,Y)\xi - \eta(Y)X)\\ &+ \beta(g(TX,Y)\xi - \eta(Y)TX).\end{aligned} \tag{4.30}$$

Proof. For any $X, Y \in TM$ we have

$$\tilde{\nabla}_X \phi Y = (\tilde{\nabla}_X \phi Y) + \phi \tilde{\nabla}_X Y.$$

By (2.8) and (2.11) we have from above

$$\tilde{\nabla}_X TY + \tilde{\nabla}_X NY = (\tilde{\nabla}_X \phi)Y + \phi(\nabla_X Y + h(X,Y)).$$

Again, by (2.11) and (2.12)

$$\tilde{\nabla}_X TY + \tilde{\nabla}_X NY = (\tilde{\nabla}_X \phi)Y + T\nabla_X Y + N\nabla_X Y + th(X,Y) + nh(X,Y).$$

Using (2.8) and (2.9) from above, we get

$$\begin{aligned}\nabla_X TY + h(X,TY) - A_{NY}X + \nabla^{\perp}_X NY &= \alpha(g(X,Y)\xi - \eta(Y)X)\\ &+ \beta(g(\phi X,Y)\xi - \eta(Y)\phi X)\\ &+ T\nabla_X Y + N\nabla_X Y\\ &+ th(X,Y) + nh(X,Y).\end{aligned} \tag{4.31}$$

Comparing tangential and normal parts, we have

$$\begin{aligned}\nabla_X TY - A_{NY}X &= \alpha(g(X,Y)\xi - \eta(Y)X)\\ &+ \beta(g(TX,Y)\xi - \eta(Y)TX)\\ &+ T\nabla_X Y + th(X,Y).\end{aligned} \tag{4.32}$$

That is,

$$\begin{aligned}(\nabla_X T)Y &= A_{NY}X + th(X,Y)\\ &+ \alpha(g(X,Y)\xi - \eta(Y)X)\\ &+ \beta(g(TX,Y)\xi - \eta(Y)TX).\end{aligned} \tag{4.33}$$

□

As a consequence of the above theorem we obtain the following:

Corollary 4.12. *In a pseudo-slant submanifold of an α-Sasakian manifold*

$$\begin{aligned}(\nabla_X T)Y &= A_{NY}X + th(X,Y)\\ &+ \alpha(g(X,Y)\xi - \eta(Y)X).\end{aligned} \tag{4.34}$$

Corollary 4.13. *In a pseudo-slant submanifold of a β-Kenmotsu manifold*

$$\begin{aligned}(\nabla_X T)Y &= A_{NY}X + th(X,Y)\\ &+ \beta(g(TX,Y)\xi - \eta(Y)TX).\end{aligned} \tag{4.35}$$

Corollary 4.14. *In a pseudo-slant submanifold of a cosymplectic manifold*

$$(\nabla_X T)Y = A_{NY}X + th(X,Y). \tag{4.36}$$

5. EXAMPLE

From [11] we know that $\mathbb{R}^{2n+1}$ admits a trans-Sasakian structure. Now consider an example of a three-dimensional submanifold of a trans-Sasakian manifold.

Let (x,y,z) be Cartesian coordinates of $\mathbb{R}^3$ and put

$$\xi = \frac{\partial}{\partial z}, \qquad \eta = dz - ydx,$$

$$\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\delta\Phi(\xi) = -1$, $\delta\eta = -1$ and (ϕ,ξ,η,g) is a trans-Sasakian structure on $\mathbb{R}^3$ of type $(-\frac{1}{2},\frac{1}{2})$ [2]. The vector fields

$$e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial x}$$

form an orthonormal frame of TM. We see that $\phi e_1 = 0$, $\phi e_3 = e_2$, $\phi e_2 = -e_3$.

Let $D_1 =< e_2 >$, $D_2 =< e_3 >$, $< \xi >=< e_1 >$. Suppose $X \in D_1$, and $Y \in TM$. Then we can write $X = ke_2$, k is a scalar and $Y = re_1 + se_2 + te_3$, r, s, t are scalars. Now $\cos\angle(\phi X,Y) = \frac{g(\phi X,Y)}{|\phi X||Y|}$. From the components of the metric g see that $g(\phi X,Y) = krg(\phi e_2,e_1) + ksg(\phi e_2,e_2) + ktg(\phi e_2,e_3) = 0$. Hence, the distribution D_1 is anti-invariant.

Again, let us suppose $U \in D_2$, and $V \in TM$. Then we can write $U = ce_3$, c is a scalar and $V = ke_1 + le_2 + me_3$, k, l, m are scalars. Now $\cos\angle(\phi U,V) = \frac{g(\phi U,V)}{|\phi U||V|}$. We see that $g(\phi U,V) = ckg(\phi e_3,e_1) + clg(\phi e_3,e_2) + cmg(\phi e_3,e_3) = 1$. Therefore, $\cos\angle(\phi U,V) = \frac{1}{|c\phi e_3||ke_1+le_2+me_3|}$ which is constant. We see that the distribution D_2 is slant.

In this case, the distribution D_1 is anti-invariant while the distribution D_2 is slant. Hence the submanifold under consideration is pseudo-slant.

6. CONCLUSION

Trans-Sasakian manifolds generalize both α-Sasakian and β-Kenmotsu manifolds. Pseudo-slant submanifolds mainly extend the notion of semi-invariant submanifolds. The integrability of the distributions of the tangent bundle of a submanifold determines the nature of the submanifold. In the present paper we consider the direct orthogonal decomposition of the tangent bundle TM of the pseudo-slant submanifold M of a trans-Sasakian manifold $\tilde{M}$ as $TM = D_1 \oplus D_2 \oplus < \xi >$, where D_1 is the anti-invariant distribution and D_2 is the slant distribution. We mainly show that the distributions $D_1 \oplus < \xi >$ and D_1 are integrable but the distribution D_2 is not integrable. A necessary and sufficient condition for a pseudo-slant submanifold to be anti-invariant is obtained. An example of a pseudo-slant submanifold of a trans-Sasakian manifold is constructed.

Submanifold theory has an important role in many branches of applied mathematics. The results obtained in this paper can be used in many problems of dynamical system and critical point theory.

ACKNOWLEDGEMENTS

The authors are thankful to the referee for pointing out the typographical errors and for linguistic corrections. The second author was partially supported by UGC Minor Research Project, India.

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Trans-Sasaki muutkondade pseudo-längus alammuutkondadest

Uday Chand De ja Avijit Sarkar

Artikkel on uurimus trans-Sasaki muutkondade pseudo-längus alammuutkondadest. On välja töötatud jaotuste integreeruvustingimused sellistel alammuutkondadel, loetletud teisi geomeetrilisi omadusi ja toodud ühe pseudo-längus trans-Sasaki alammuutkonna näide.