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MATHEMATICS

Transformations of semiholonomic 2- and 3-jets and semiholonomic prolongation of connections

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Abstract. We recall the description of natural transformations of semiholonomic jet functors $\overline{J}^r$ defined on the categories $\mathcal{M}f_m \times \mathcal{M}f$ and $\mathcal{FM}_{m,n}$. Up to order three, exact coordinate formulae are known, for general order several related results are reminded. We also show an application of semiholonomic jet transformations to prolongation of general connections.

Key words: jet, natural transformation, connection, Ehresmann prolongation.

1. INTRODUCTION

Let $\mathcal{FM}_{m,n}$ be the category of fibred manifolds with m-dimensional bases, n-dimensional fibres, and locally invertible fibre-preserving mappings. Further, let $\mathcal{M}f_m$ and $\mathcal{M}f$ be the category of m-dimensional manifolds endowed with local diffeomorphisms and the category of all manifolds and all smooth mappings, respectively. It is well known that suitable models for many physical phenomena can be found among the objects of $\mathcal{FM}_{m,n}$. Important characteristics of physical laws can be expressed by means of geometrical objects like jets, connections, and natural operators. This has been widely studied and can be found in e.g. [7].

This paper is devoted to natural transformations of semiholonomic jet functors $\overline{J}^r$ defined on categories $\mathcal{M}f_m \times \mathcal{M}f$ and $\mathcal{FM}_{m,n}$ and their applications to prolongation of connections. For $r \leq 3$, exact coordinate formulae are known, for general order r we recall several results concerning transformations of nonholonomic, semiholonomic, and holonomic jet functors and their combinations.

In the last section, we show a direct application of jet transformations to prolongation of general connections. For order two, we recall a formula of all natural operators transforming first-order general connection into second-order general connection by means of the so-called Ehresmann prolongation. For higher orders the problem becomes technically complicated and remains open.

We note that differential prolongations of different objects, including connections, are widely studied. Some other procedures like immersion of connections in the space of infinite jets and further applications can be found in [1,2].

2. FOUNDATIONS

Let $p : Y \to M$ be a fibred manifold. By (x^i), $i = 1,\dots,m$ we denote local coordinates on M and by (x^i, y^p), $i = 1,\dots,m$, $p = 1,\dots,n$ local coordinates on Y. Denote by $J^rY \to M$ the r-th jet prolongation of $p : Y \to M$, that is the space of r-jets of local sections $M \to Y$. In what follows, J^rY will be called the r-th holonomic prolongation of Y.

Recall that r-th nonholonomic prolongation $\widetilde{J}^rY$ of Y is defined by iteration

$$\widetilde{J}^1Y = J^1Y, \qquad \widetilde{J}^rY = J^1(\widetilde{J}^{r-1}Y \to M).$$

Clearly, we have an inclusion $J^rY \subset \widetilde{J}^rY$ given by $j^r_x\gamma \mapsto j^1_x(j^{r-1}\gamma)$. Further, r-th semiholonomic prolongation $\overline{J}^rY \subset \widetilde{J}^rY$ is defined by the following induction. First, by $\beta_1 = \beta_Y$ we denote the projection $J^1Y \to Y$ and by $\beta_r = \beta_{\widetilde{J}^{r-1}Y}$ the projection $\widetilde{J}^rY = J^1\widetilde{J}^{r-1}Y \to \widetilde{J}^{r-1}Y$, $r = 2,3,\dots$. If we set $\overline{J}^1Y = J^1Y$ and assume we have $\overline{J}^{r-1}Y \subset \widetilde{J}^{r-1}Y$ such that the restriction of the projection $\beta_{r-1} : \widetilde{J}^{r-1}Y \to \widetilde{J}^{r-2}Y$ maps $\overline{J}^{r-1}Y$ into $\overline{J}^{r-2}Y$, we can construct $J^1\beta_{r-1} : J^1\overline{J}^{r-1}Y \to J^1\overline{J}^{r-2}Y$ and define

$$\overline{J}^rY = \{A \in J^1\overline{J}^{r-1}Y;\ \beta_r(A) \in \overline{J}^{r-1}Y\}.$$

We recall that the induced coordinates on the holonomic prolongation J^rY are given by (x^i, y^p_α), where α is a multiindex of range m satisfying $|\alpha| \le r$. Clearly, the coordinates y^p_α on J^rY are characterized by full symmetry in the indices of α. Having the nonholonomic prolongation $\widetilde{J}^rY$ constructed by iteration, we define local coordinates inductively as follows:
1. Suppose that the coordinates on $\widetilde{J}^{r-1}Y$ are of the form

$$(x^i, y^p_{k_1\dots k_{r-1}}),\ k_1,\dots,k_{r-1} = 0,1,\dots,m.$$

2. We define the induced coordinates on $\widetilde{J}^rY$ by

$$(x^i, y^p_{k_1\dots k_{r-1}0} = y^p_{k_1\dots k_{r-1}}, y^p_{k_1\dots k_{r-1}i} = \frac{\partial}{\partial x^i} y^p_{k_1\dots k_{r-1}}).$$

It remains to describe coordinates on the semiholonomic prolongation $\overline{J}^rY$. Let $(k_1,\dots,k_r)$, $k_1,\dots,k_r = 0,1,\dots,m$ be a sequence of indices and denote by $\langle k_1,\dots,k_s\rangle$, $s \le r$ the sequence of non-zero indices in $(k_1,\dots,k_r)$ respecting the order. Obviously, $J^r, \overline{J}^r$, and $\widetilde{J}^r$ are bundle functors on the category $\mathscr{FM}_{m,n}$.

On the other hand, $\widetilde{J}^r$, $\overline{J}^r$, and J^r can be also considered as bundle functors on the product category $\mathscr{M}f_m \times \mathscr{M}f$. Indeed, the space $\widetilde{J}^r(M,N)$ of nonholonomic r-jets of M into N is exactly the r-th nonholonomic prolongation of a product fibred manifold $M \times N \to M$. For every local diffeomorphism $f : M \to \overline{M}$ and every smooth map $g : N \to \overline{N}$ we define

$$\overline{J}^r(f,g) : \overline{J}^r(M,N) \to \overline{J}^r(\overline{M},\overline{N}) \quad \text{by} \quad \overline{J}^r(f,g)(X) = (j^r_yg) \circ X \circ (j^r_xf)^{-1},$$

where $x = \alpha X$ and $y = \beta X$ are the source and target of $X \in \overline{J}^r(M,N)$, respectively, see [7].

According to [7], two maps $f,g : M \to N$ determine the same r-jet at the point $x \in M$, i.e. $j^r_xf = j^r_xg$, if and only if all partial derivatives up to order r of the components f^p and g^p of their coordinate expressions coincide at x. Thus if we use the notation $z^p_{i_1\dots i_m} = \frac{\partial^m f}{\partial x_{i_1}\cdots\partial x_{i_m}}$, local coordinates on $J^r(M,N)$ are given by (x^i, y^p, z^p_α), where α is a multiindex of range $\dim M$ satisfying $|\alpha| \le r$, x^i and y^p are local coordinates on M and N, respectively. Similarly to the jet prolongations of a fibred manifold, local coordinates on $\widetilde{J}^r(M,N)$ include zero indices in the subscripts and there is no symmetry involved, while in the semiholonomic case the zero indices are eliminated. For example, for $r = 2$ we have the coordinate chart $(x^i, y^p, z^p_{0i}, z^p_i, z^p_{ij})$ on $\widetilde{J}^2(M,N)$. Then $\overline{J}^2(M,N)$ is characterized by $z^p_{0i} = z^p_i$ and $J^2(M,N)$ by $z^p_{0i} = z^p_i$, $z^p_{ij} = z^p_{ji}$.

3. JETS ON THE CATEGORY $\mathcal{M}f_m \times \mathcal{M}f$

We first recall some results about second- and third-order semiholonomic jet transformations. For the functor $\overline{J}^2$ defined on the category $\mathcal{M}f_m \times \mathcal{M}f$, we consider an involutory map $i : \overline{J}^2 \to \overline{J}^2$ introduced by Pradines [9]. The coordinate effect of i consists in the exchange of the subscripts of z^p_{ij}. Kolář and Vosmanská proved in [6]

Proposition 3.1. *All natural transformations $\overline{J}^2 \to \overline{J}^2$ form two one-parameter families*

$$t(X - i(X)) + i(X), \quad k(X - i(X)), \quad k, t \in \mathbb{R}.$$

The following description of all natural transformations $\overline{J}^3 \to \overline{J}^3$ between semiholonomic 3-jet functors defined on the product category $\mathcal{M}f_m \times \mathcal{M}f$ can be found in [12].

Proposition 3.2. *The only natural transformations $\overline{J}^3 \to \overline{J}^3$ are the identity and the contraction, i.e. the map*

$$X \mapsto j^3_{\alpha X}\widehat{\beta X},$$

where $\widehat{\beta X}$ denotes the constant map of M into $\beta X \in N$, βX and αX being the target and source of $X \in \overline{J}^3(M, N)$, respectively.

Remark. According to [12], transformations $\overline{J}^r \to \overline{J}^r, r \geq 4$ of the functor $\overline{J}^r$ defined on the category $\mathcal{M}f_m \times \mathcal{M}f$ are not generally trivial and exact results are not known.

Finally, we recall the transformations of holonomic jet functor J^r defined on the category $\mathcal{M}f_m \times \mathcal{M}f$, see [7].

Proposition 3.3. *For $r \geq 2$ the only natural transformations $J^r \to J^r$ are the identity and the contraction. For $r = 1$, all natural transformations $J^1 \to J^1$ form the one-parameter family of homotheties $X \mapsto cX$, $c \in \mathbb{R}$.*

4. JETS ON THE CATEGORY $\mathcal{FM}_{m,n}$

Now we consider the functor $\overline{J}^2$ on the category $\mathcal{FM}_{m,n}$, i.e. we discuss semiholonomic 2-jets of local sections of a fibred manifold $Y \to M$. For the iteration of functor J^1, Modugno introduced an exchange map $e_\Lambda : J^1J^1Y \to J^1J^1Y$ depending on the linear connection Λ defined on the base manifold M, see [8]. If we consider the restriction of map e_Λ to the subbundle $\overline{J}^2Y \subset J^1J^1Y$, we obtain a natural map $e : \overline{J}^2Y \to \overline{J}^2Y$. From the coordinate formula of e_Λ it is easy to derive that e does not depend on any linear connection on the base manifold, see [7]. Given local coordinates $(x^i, y^p, y^p_i, y^p_{ij})$ on $\overline{J}^2Y$, the map $e : \overline{J}^2Y \to \overline{J}^2Y$ has the coordinate expression

$$y^p_i = y^p_i, \qquad y^p_{ij} = y^p_{ji}.$$

Kolář and Modugno proved in [5]

Proposition 4.1. *All natural transformations $\overline{J}^2 \to \overline{J}^2$ form a one-parameter family*

$$X \mapsto t(X - e(X)) + e(X), \qquad t \in \mathbb{R}.$$

Now we use the fact that the space $\overline{J}^3(M, N)$ coincides with the third semiholonomic prolongation of the product manifold $M \times N \to M$ and, consequently, all natural transformations $\overline{J}^3 \to \overline{J}^3$ of the functor $\overline{J}^3$ defined on the category $\mathcal{FM}_{m,n}$ are just restrictions of the transformations from Proposition 3.2. As the contraction cannot be restricted we derive

Corollary 4.1. *The only natural transformation $\overline{J}^3 \to \overline{J}^3$ of the functor $\overline{J}^3$ defined on the category $\mathcal{FM}_{m,n}$ is the identity.*

Now we show several consequences on the transformations of semiholonomic onto holonomic jets. Corollary 4.1 and the fact that $\overline{J}^3 Y \subset\!\to J^3 Y$ directly implies

Corollary 4.2. *There is no natural transformation $\overline{J}^3 \to J^3$ between third-order semiholonomic and holonomic jet functors.*

On the other hand, there exists a natural transformation $\overline{J}^2 \to J^2$ given by symmetrization. More precisely, [3] reads that all $\mathcal{FM}_{m,n}$-natural transformations $\overline{J}^2 Y \to J^2 Y$ with $\dim M \geq 2$ are of the form

$$c^{(2)}(\sigma) := \sigma' + \mathrm{Sym}(\sigma - \sigma'),$$

where $\sigma \in \overline{J}^2 Y$ is arbitrary, $\sigma' \in \overline{J}^2 Y$ is such an element that the jet projections $\overline{\pi}^2_1(\sigma)$ and $\pi^2_1(\sigma')$ coincide, and $\mathrm{Sym} : \otimes^2 T^*M \otimes VY \to S^2 T^*M \otimes VY$ is induced by symmetrization.

Remark. A complete description of all transformations $\overline{J}^r \to J^r$ for any r can be found in [3]. The authors proved in a very general way that for $\dim M \geq 2$ and $r \geq 3$ there is no natural transformation of given type and, thus, Corollary 4.2 and Corollary 4.3 are consequences of their considerations.

Finally, there is an easy consequence on the transformation of nonholonomic onto holonomic jets. The inclusion $\overline{J}^3 Y \subseteq \widetilde{J}^3 Y$ together with Corollary 4.2 directly implies

Corollary 4.3. *There is no natural transformation $\widetilde{J}^3 \to J^3$ between third-order nonholonomic and holonomic jet functors.*

5. PROLONGATION OF CONNECTIONS

We recall that a general connection on the fibre bundle $Y \to M$ is a vector-valued 1-form Γ with values in the vertical bundle VY such that $\Gamma \circ \Gamma = \Gamma$ and $\mathrm{Im}\Gamma = VY$. Thus any connection on the fibre bundle $Y \to M$ is determined by horizontal projection $\chi = \mathrm{id}_{TY} - \Gamma$, or by horizontal subspaces $\chi(T_y Y) \subset T_y Y$ in the individual tangent spaces, i.e. by horizontal distribution. But every horizontal subspace $\chi(T_y Y)$ is complementary to the vertical subspace $V_y Y$ and therefore it is canonically identified with a unique element $j^1_y s \in J^1_y Y$. On the other hand, each $j^1_y s \in J^1_y Y$ determines a subspace in $T_y Y$ complementary to $V_y Y$. This leads us to the equivalent definition: a general connection on a fibred manifold $p : Y \to M$ is a section $\Gamma : Y \to J^1 Y$ of the first jet prolongation $J^1 Y \to Y$. Analogously, a higher-order general connection is a section $Y \to J^r Y$, $r > 1$, where holonomic jet prolongation $J^r Y$ can be replaced by semiholonomic or nonholonomic one. Consequently, a higher-order connection is called holonomic, semiholonomic, or nonholonomic according to the type of target space.

Given two higher-order connections $\Gamma : Y \to \widetilde{J}^r Y$ and $\overline{\Gamma} : Y \to \widetilde{J}^s Y$, the product of Γ and $\overline{\Gamma}$ is the $(r+s)$-th order connection $\Gamma * \overline{\Gamma} : Y \to \widetilde{J}^{r+s} Y$ defined by

$$\Gamma * \overline{\Gamma} = \widetilde{J}^s \Gamma \circ \overline{\Gamma}.$$

As an example we show the coordinate expression of an arbitrary nonholonomic second-order connection and of the product of two first-order connections. The coordinate form of $\Delta : Y \to \widetilde{J}^2 Y$ is

$$y^p_i = F^p_i(x,y), \quad y^p_{0i} = G^p_i(x,y), \quad y^p_{ij} = H^p_{ij}(x,y),$$

where F,G,H are arbitrary smooth functions. Further, if the coordinate expressions of two first-order connections $\Gamma,\overline{\Gamma}:Y\to J^1Y$ are

$$\Gamma: \quad y_i^p = F_i^p(x,y), \qquad \overline{\Gamma}: \quad y_i^p = G_i^p(x,y),$$

then the second-order connection $\Gamma*\overline{\Gamma}:Y\to\widetilde{J}^2Y$ has equations

$$y_i^p = F_i^p, \quad y_{0i}^p = G_i^p, \quad y_{ij}^p = \frac{\partial F_i^p}{\partial x^j} + \frac{\partial F_i^p}{\partial y^q}G_j^q.$$

If both Γ and $\overline{\Gamma}$ are of the first order, then $\Gamma*\overline{\Gamma}:Y\to\widetilde{J}^2Y$ is semiholonomic if and only if $\Gamma=\overline{\Gamma}$, and $\Gamma*\Gamma$ is holonomic if and only if Γ is curvature free, [4,11].

Considering a connection $\Gamma : Y \to J^1Y$, we can define an r-th order connection $\Gamma^{(r-1)}:Y\to\widetilde{J}^rY$ by

$$\Gamma^{(1)} := \Gamma*\Gamma = J^1\Gamma\circ\Gamma, \qquad \Gamma^{(r-1)} := \Gamma^{(r-2)}*\Gamma = J^1\Gamma^{(r-2)}\circ\Gamma.$$

The connection $\Gamma^{(r-1)}$ is called the $(r-1)$-st prolongation of Γ in the sense of Ehresmann, shortly $(r-1)$-st Ehresmann prolongation. By [4], the values of $\Gamma^{(r-1)}$ lie in the semiholonomic prolongation $\overline{J}^rY$, and $\Gamma^{(r-1)}$ is holonomic if and only if Γ is curvature free, [11].

Now, let $e:\overline{J}^2Y\to\overline{J}^2Y$ be the map described in Section 3. Then we recall, see [10]:

Proposition 5.1. *All natural operators transforming first-order connection $\Gamma:Y\to J^1Y$ into second-order semiholonomic connection $Y\to\overline{J}^2Y$ form a one-parameter family*

$$\Gamma\mapsto k\cdot(\Gamma*\Gamma)+(1-k)\cdot e(\Gamma*\Gamma), \qquad k\in\mathbb{R}.$$

In other words, all natural operators from Proposition 5.1 can be obtained from the Ehresmann prolongation $\Gamma*\Gamma$ by applying all natural transformations $\overline{J}^2\to\overline{J}^2$ from Proposition 4.1.

Taking into account the natural operators transforming first-order connections into r-th order semiholonomic connections with $r\geq 3$, full classification becomes technically complicated and thus it remains open. As an example, if a first-order connection Γ is given by $y_i^p=G_i^p$, the coordinate expression of $\Gamma^{(2)}=\Gamma*\Gamma*\Gamma$ is of the form

$$y_i^p = G_i^p, \quad y_{ij}^p = \frac{\partial G_i^p}{\partial x^j} + \frac{\partial G_i^p}{\partial y^q}G_j^q,$$

$$y_{ijk}^p = \frac{\partial^2 G_i^p}{\partial x^j\partial x^k} + \frac{\partial^2 G_i^p}{\partial x^j\partial y^q}G_k^q + \frac{\partial^2 G_i^p}{\partial y^q\partial x^k}G_j^q + \frac{\partial^2 G_i^p}{\partial y^q\partial y^r}G_k^rG_j^q + \frac{\partial G_i^p}{\partial y^q}\frac{\partial G_j^q}{\partial x^k}.$$

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Poolholonoomsete 2- ja 3-džettide teisendused ning seostuste poolholonoomne jätkamine

Petr Vašík

On meenutatud poolholonoomsete džettide funktori $\overline{J}^r$ naturaalsete teisenduste kirjeldust, kusjuures funktor on määratud kategooriatel $\mathscr{M}f_m \times \mathscr{M}f$ ja $\mathscr{F}\mathscr{M}_{m,n}$. Kuni kolmandat järku täpsed valemid koordinaatides on teada, on meenutatud mõned sellega seotud tulemused suvalise järgu korral. On näidatud, kuidas saab džettide poolholonoomseid teisendusi kasutada üldiste seostuste jätkamiseks.