



## Quantum counterparts of $VII_a$ , $III_{a=1}$ , $VI_{a \neq 1}$ over the harmonic oscillator in semiclassical approximation

Eugen Paal\* and Jüri Virkepu

Department of Mathematics, Tallinn University of Technology, Ehitajate tee 5, 19086 Tallinn, Estonia; [jvirkepu@staff.ttu.ee](mailto:jvirkepu@staff.ttu.ee)

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**Abstract.** Operadic Lax representations for the harmonic oscillator are used to construct the quantum counterparts of some real 3-dimensional Lie algebras. The Jacobi operators of these quantum algebras are studied in semiclassical approximation.

**Key words:** operad, 3d real Lie algebras, operadic Lax equation, Jacobi operator.

### 1. INTRODUCTION AND OUTLINE OF THE PAPER

In Hamiltonian formalism a mechanical system is described by the canonical variables  $q^i, p_i$  and their time evolution is prescribed by the Hamiltonian equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}. \quad (1.1)$$

By a Lax representation [3] of a mechanical system one means such a pair  $(L, M)$  of matrices (linear operators)  $L, M$  that the above Hamiltonian system may be represented as the Lax equation

$$\frac{dL}{dt} = ML - LM. \quad (1.2)$$

Thus, from the algebraic point of view, mechanical systems may be represented by linear operators, i.e. by linear maps  $V \rightarrow V$  of a vector space  $V$ . As a generalization of this one can pose the following question [4]: how to describe the time evolution of the linear operations (multiplications)  $V^{\otimes n} \rightarrow V$ ?

The algebraic operations (multiplications) can be seen as an example of the *operadic* variables [1]. If an operadic system depends on time, one can speak about *operadic dynamics* [4]. The latter may be introduced by simple and natural analogy with the Hamiltonian dynamics. In particular, the time evolution of the operadic variables may be given by the operadic Lax equation. In [5–7] the low-dimensional binary operadic Lax representations for the harmonic oscillator (HO) were constructed. In [8] it was shown how the operadic Lax representations are related to the conservation of energy.

In this paper the operadic Lax representations for the HO are used to construct the quantum counterparts of some real 3-dimensional Lie algebras. The Jacobi operators of these quantum algebras are studied in semiclassical approximation.

\* Corresponding author, [eugen.paal@ttu.ee](mailto:eugen.paal@ttu.ee)

The paper is organized as follows. In Sec. 2 we recall the definition of the endomorphism operad and introduce the Gerstenhaber brackets as a main tool for constructing the operadic Lax pairs. In Sec. 3 we describe the idea of operadic dynamics by using the operadic Lax equation. This idea is illustrated in Sec. 4 and Sec. 5 by using the HO and the natural initial conditions. In Sec. 6 and Sec. 7 the operadic Lax representations are used to construct the dynamical deformations of real 3-dimensional Lie algebras  $VII_a$ ,  $III_{a=1}$ ,  $VI_{a \neq 1}$  from the Bianchi classification. The quantum counterparts of these Lie algebras are constructed in Sec. 8. The quantum multiplication is defined in a natural way in the state space of the quantum HO. It turns out that the Jacobi identity is violated in these quantum algebras, and to inquire the situation, we use in Sec. 9 and Sec. 10 the semiclassical approximation. In this approximation one can explicitly see how the quantum mechanical fundamental canonical commutation relations spoil the Jacobi identity on the quantum algebras.

## 2. ENDOMORPHISM OPERAD AND GERSTENHABER BRACKETS

Let  $K$  be a unital associative commutative ring,  $V$  be a unital  $K$ -module, and  $\mathcal{E}_V^n := \mathcal{E}nd_V^n := \text{Hom}(V^{\otimes n}, V)$  ( $n \in \mathbb{N}$ ). For an operation  $f \in \mathcal{E}_V^n$  we refer to  $n$  as the *degree* of  $f$  and often write (when it does not cause confusion)  $f$  instead of  $\text{deg } f$ . For example,  $(-1)^f := (-1)^n$ ,  $\mathcal{E}_V^f := \mathcal{E}_V^n$ , and  $\circ_f := \circ_n$ . Also, it is convenient to use the *reduced degree*  $|f| := n - 1$ . Throughout this paper we assume that  $\otimes := \otimes_K$ .

**Definition 2.1** (endomorphism operad [1]). For  $f \otimes g \in \mathcal{E}_V^f \otimes \mathcal{E}_V^g$  define the partial compositions

$$f \circ_i g := (-1)^{|g|} f \circ (\text{id}_V^{\otimes i} \otimes g \otimes \text{id}_V^{\otimes (|f|-i)}) \in \mathcal{E}_V^{f+|g|}, \quad 0 \leq i \leq |f|.$$

The sequence  $\mathcal{E}_V := \{\mathcal{E}_V^n\}_{n \in \mathbb{N}}$ , equipped with the partial compositions  $\circ_i$ , is called the endomorphism operad of  $V$ .

**Definition 2.2** (total composition [1]). The total composition  $\circ : \mathcal{E}_V^f \otimes \mathcal{E}_V^g \rightarrow \mathcal{E}_V^{f+|g|}$  is defined by

$$f \circ g := \sum_{i=0}^{|f|} f \circ_i g \in \mathcal{E}_V^{f+|g|}, \quad |\circ| = 0.$$

The pair  $\text{Com } \mathcal{E}_V := \{\mathcal{E}_V, \circ\}$  is called the composition algebra of  $\mathcal{E}_V$ .

**Definition 2.3** (Gerstenhaber brackets [1]). The Gerstenhaber brackets  $[\cdot, \cdot]$  are defined in  $\text{Com } \mathcal{E}_V$  as a graded commutator by

$$[f, g] := f \circ g - (-1)^{|f||g|} g \circ f = -(-1)^{|f||g|} [g, f], \quad [[\cdot, \cdot]] = 0.$$

The commutator algebra of  $\text{Com } \mathcal{E}_V$  is denoted as  $\text{Com}^- \mathcal{E}_V := \{\mathcal{E}_V, [\cdot, \cdot]\}$ . One can prove (e.g. [1]) that  $\text{Com}^- \mathcal{E}_V$  is a graded Lie algebra. The Jacobi identity reads

$$(-1)^{|f||h|} [f, [g, h]] + (-1)^{|g||f|} [g, [h, f]] + (-1)^{|h||g|} [h, [f, g]] = 0.$$

## 3. OPERADIC DYNAMICS AND LAX EQUATION

Assume that  $K := \mathbb{R}$  or  $K := \mathbb{C}$  and operations are differentiable. Dynamics in operadic systems (operadic dynamics) may be introduced by

**Definition 3.1** (operadic Lax pair [4]). Allow a classical dynamical system to be described by the Hamiltonian system (1.1). An operadic Lax pair is a pair  $(\mu, M)$  of homogeneous operations  $\mu, M \in \mathcal{E}_V$ , such that the Hamiltonian system (1.1) may be represented as the operadic Lax equation

$$\frac{d\mu}{dt} = [M, \mu] := M \circ \mu - (-1)^{|M||\mu|} \mu \circ M.$$

The pair  $(L, M)$  is also called an operadic Lax representation of Hamiltonian system (1.1). Evidently, the degree constraints  $|M| = |L| = 0$  give rise to ordinary Lax equation (1.2) [3]. In this paper we assume that  $|M| = 0$ .

The Hamiltonian of the HO is

$$H(q, p) = \frac{1}{2}(p^2 + \omega^2 q^2).$$

Thus, the Hamiltonian system of HO reads

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = p, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\omega^2 q.$$

If  $\mu$  is a linear algebraic operation, we can use the above Hamilton equations to obtain

$$\frac{d\mu}{dt} = \frac{\partial \mu}{\partial q} \frac{dq}{dt} + \frac{\partial \mu}{\partial p} \frac{dp}{dt} = p \frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p} = [M, \mu].$$

Therefore, we get the following linear partial differential equation for  $\mu(q, p)$ :

$$p \frac{\partial \mu}{\partial q} - \omega^2 q \frac{\partial \mu}{\partial p} = [M, \mu]. \tag{3.1}$$

By integrating (3.1), one can get collections of operations called [4] the *operadic* (Lax representations for/of) HO.

### 4. 3-DIMENSIONAL BINARY ANTI-COMMUTATIVE OPERADIC LAX REPRESENTATIONS FOR THE HARMONIC OSCILLATOR

**Lemma 4.1.** *The matrices*

$$L := \begin{pmatrix} p & \omega q & 0 \\ \omega q & -p & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M := \frac{\omega}{2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

give a 3-dimensional Lax representation for the HO.

**Definition 4.2** (quasi-canonical coordinates). For the HO, define its quasi-canonical coordinates  $Q$  and  $P$  by

$$P^2 - Q^2 = 2p, \quad QP = \omega q. \tag{4.1}$$

**Remark 4.3.** Note that these constraints easily imply

$$P^2 + Q^2 = 2\sqrt{2H}.$$

**Theorem 4.4** ([7]). Let  $C_v \in \mathbb{R}$  ( $v = 1, \dots, 9$ ) be arbitrary real-valued parameters, such that

$$C_2^2 + C_3^2 + C_5^2 + C_6^2 + C_7^2 + C_8^2 \neq 0.$$

Let  $M$  be defined as in Lemma 4.1 and  $\mu : V \otimes V \rightarrow V$  be a binary operation in a 3-dimensional real vector space  $V$  with the coordinates

$$\begin{cases} \mu_{11}^1 = \mu_{22}^1 = \mu_{33}^1 = \mu_{11}^2 = \mu_{22}^2 = \mu_{33}^2 = \mu_{11}^3 = \mu_{22}^3 = \mu_{33}^3 = 0, \\ \mu_{23}^1 = -\mu_{32}^1 = C_2 p - C_3 \omega q - C_4, \\ \mu_{13}^2 = -\mu_{31}^2 = C_2 p - C_3 \omega q + C_4, \\ \mu_{31}^1 = -\mu_{13}^1 = C_2 \omega q + C_3 p - C_1, \\ \mu_{23}^2 = -\mu_{32}^2 = C_2 \omega q + C_3 p + C_1, \\ \mu_{12}^1 = -\mu_{21}^1 = C_5 P + C_6 Q, \\ \mu_{12}^2 = -\mu_{21}^2 = C_5 Q - C_6 P, \\ \mu_{13}^3 = -\mu_{31}^3 = C_7 P + C_8 Q, \\ \mu_{23}^3 = -\mu_{32}^3 = C_7 Q - C_8 P, \\ \mu_{12}^3 = -\mu_{21}^3 = C_9. \end{cases} \quad (4.2)$$

Then  $(\mu, M)$  is an operadic Lax pair for the HO.

### 5. INITIAL CONDITIONS

Specify the coefficients  $C_v$  in Theorem 4.4 by the initial conditions

$$\mu|_{t=0} = \overset{\circ}{\mu}, \quad p|_{t=0} = p_0, \quad q|_{t=0} = 0.$$

If we denote  $E := H|_{t=0}$ , the latter together with (4.1) yields the initial conditions for  $Q$  and  $P$ :

$$\begin{cases} (P^2 + Q^2)|_{t=0} = 2\sqrt{2E} \\ (P^2 - Q^2)|_{t=0} = 2p_0 \\ PQ|_{t=0} = 0 \end{cases} \iff \begin{cases} p_0 > 0 \\ P^2|_{t=0} = 2p_0 \\ Q|_{t=0} = 0 \end{cases} \vee \begin{cases} p_0 < 0 \\ P|_{t=0} = 0 \\ Q^2|_{t=0} = -2p_0 \end{cases}.$$

In what follows assume that  $p_0 > 0$  and  $P|_{t=0} = \sqrt{2p_0}$ . The other cases can be treated similarly. Note that in this case  $p_0 = \sqrt{2E}$ . From (4.2) we get the following linear system:

$$\begin{cases} C_1 = \frac{1}{2} (\overset{\circ}{\mu}_{23}^2 - \overset{\circ}{\mu}_{31}^1), & C_2 = \frac{1}{2p_0} (\overset{\circ}{\mu}_{13}^2 + \overset{\circ}{\mu}_{23}^1), & C_3 = \frac{1}{2p_0} (\overset{\circ}{\mu}_{23}^2 + \overset{\circ}{\mu}_{31}^1), \\ C_4 = \frac{1}{2} (\overset{\circ}{\mu}_{13}^2 - \overset{\circ}{\mu}_{23}^1), & C_5 = \frac{1}{\sqrt{2p_0}} \overset{\circ}{\mu}_{12}^1, & C_6 = -\frac{1}{\sqrt{2p_0}} \overset{\circ}{\mu}_{12}^2, \\ C_7 = \frac{1}{\sqrt{2p_0}} \overset{\circ}{\mu}_{13}^3, & C_8 = -\frac{1}{\sqrt{2p_0}} \overset{\circ}{\mu}_{23}^3, & C_9 = \overset{\circ}{\mu}_{12}^3. \end{cases} \quad (5.1)$$

### 6. VII<sub>a</sub>, III<sub>a=1</sub>, VI<sub>a≠1</sub>

We study only the algebras VII<sub>a</sub>, III<sub>a=1</sub>, VI<sub>a≠1</sub> from the Bianchi classification of the real 3-dimensional Lie algebras [2]. The structure equations of the 3-dimensional real Lie algebras can be presented as follows:

$$[e_1, e_2] = -\alpha e_2 + n^3 e_3, \quad [e_2, e_3] = n^1 e_1, \quad [e_3, e_1] = n^2 e_2 + \alpha e_3.$$

The values of the parameters  $\alpha, n^1, n^2, n^3$  and the corresponding structure constants for II, VII<sub>a</sub>, III<sub>a=1</sub>, VI<sub>a≠1</sub> are presented in Table 1. Note that II is the real 3-dimensional Heisenberg algebra.

**Table 1.** VII<sub>a</sub>, III<sub>a=1</sub>, VI<sub>a≠1</sub>. Here a > 0

Bianchi type	α	n <sup>1</sup>	n <sup>2</sup>	n <sup>3</sup>	μ <sup>1</sup> <sub>12</sub>	μ <sup>2</sup> <sub>12</sub>	μ <sup>3</sup> <sub>12</sub>	μ <sup>1</sup> <sub>23</sub>	μ <sup>2</sup> <sub>23</sub>	μ <sup>3</sup> <sub>23</sub>	μ <sup>1</sup> <sub>31</sub>	μ <sup>2</sup> <sub>31</sub>	μ <sup>3</sup> <sub>31</sub>
VII <sub>a</sub>	a	0	1	1	0	-a	1	0	0	0	0	1	a
III <sub>a=1</sub>	1	0	1	-1	0	-1	-1	0	0	0	0	1	1
VI <sub>a≠1</sub>	a	0	1	-1	0	-a	-1	0	0	0	0	1	a

**7. VII<sub>a</sub><sup>t</sup>, III<sub>a=1</sub><sup>t</sup>, VI<sub>a≠1</sub><sup>t</sup>**

By using the structure constants of the 3-dimensional Lie algebras in the Bianchi classification, Theorem 4.4 and relations (5.1), one can propose that evolution of VII<sub>a</sub>, III<sub>a=1</sub>, VI<sub>a≠1</sub> can be prescribed [8] as given in Table 2.

**Table 2.** VII<sub>a</sub><sup>t</sup>, III<sub>a=1</sub><sup>t</sup>, VI<sub>a≠1</sub><sup>t</sup>. Here p<sub>0</sub> = √2E

Dynamical Bianchi type	μ <sup>1</sup> <sub>12</sub>	μ <sup>2</sup> <sub>12</sub>	μ <sup>3</sup> <sub>12</sub>	μ <sup>1</sup> <sub>23</sub>	μ <sup>2</sup> <sub>23</sub>	μ <sup>3</sup> <sub>23</sub>	μ <sup>1</sup> <sub>31</sub>	μ <sup>2</sup> <sub>31</sub>	μ <sup>3</sup> <sub>31</sub>
VII <sub>a</sub> <sup>t</sup>	$\frac{aQ}{\sqrt{2p_0}}$	$\frac{-aP}{\sqrt{2p_0}}$	1	$\frac{p-p_0}{-2p_0}$	$\frac{\omega q}{-2p_0}$	$\frac{-aQ}{\sqrt{2p_0}}$	$\frac{\omega q}{-2p_0}$	$\frac{p+p_0}{2p_0}$	$\frac{aP}{\sqrt{2p_0}}$
III <sub>a=1</sub> <sup>t</sup>	$\frac{Q}{\sqrt{2p_0}}$	$\frac{-P}{\sqrt{2p_0}}$	-1	$\frac{p-p_0}{-2p_0}$	$\frac{\omega q}{-2p_0}$	$\frac{-Q}{\sqrt{2p_0}}$	$\frac{\omega q}{-2p_0}$	$\frac{p+p_0}{2p_0}$	$\frac{P}{\sqrt{2p_0}}$
VI <sub>a≠1</sub> <sup>t</sup>	$\frac{aQ}{\sqrt{2p_0}}$	$\frac{-aP}{\sqrt{2p_0}}$	-1	$\frac{p-p_0}{-2p_0}$	$\frac{\omega q}{-2p_0}$	$\frac{-aQ}{\sqrt{2p_0}}$	$\frac{\omega q}{-2p_0}$	$\frac{p+p_0}{2p_0}$	$\frac{aP}{\sqrt{2p_0}}$

**8. VII<sub>a</sub><sup>ħ</sup>, III<sub>a=1</sub><sup>ħ</sup>, VI<sub>a≠1</sub><sup>ħ</sup>, AND QUANTUM JACOBI OPERATORS**

By using the algebras VII<sub>a</sub><sup>t</sup>, III<sub>a=1</sub><sup>t</sup>, VI<sub>a≠1</sub><sup>t</sup> from Table 2, one can propose [9] their quantum counterparts VII<sub>a</sub><sup>ħ</sup>, III<sub>a=1</sub><sup>ħ</sup>, VI<sub>a≠1</sub><sup>ħ</sup> as follows.

Let  $\mathcal{A}_{HO}$  denote the state space of the quantum HO and {e<sub>1</sub>, e<sub>2</sub>, ...} be its basis. By using Table 3, we define the structure equations in  $\mathcal{A}_{HO}$  by

$$[e_i, e_j]_{\hbar} := \hat{\mu}_{ij}^s e_s,$$

where the structure operators  $\hat{\mu}_{ij}^s$  for i, j, s ≤ 3 are defined by Table 3 and  $\hat{\mu}_{ij}^s := 0$  for i, j, s > 3. For x, y ∈  $\mathcal{A}_{HO}$ , their quantum multiplication is defined by

$$[x, y]_{\hbar} := \hat{\mu}_{jk}^i x^j y^k e_i = \hat{\mu}_{jk}^1 x^j y^k e_1 + \hat{\mu}_{jk}^2 x^j y^k e_2 + \hat{\mu}_{jk}^3 x^j y^k e_3,$$

where we omitted the trivial terms, because  $\hat{\mu}_{jk}^i = 0$  for i > 3. Then the quantum Jacobi operator is defined by

$$\begin{aligned} \hat{J}_{\hbar}(x; y; z) &:= [x, [y, z]_{\hbar}]_{\hbar} + [y, [z, x]_{\hbar}]_{\hbar} + [z, [x, y]_{\hbar}]_{\hbar} \\ &= \hat{J}_{\hbar}^1(x; y; z)e_1 + \hat{J}_{\hbar}^2(x; y; z)e_2 + \hat{J}_{\hbar}^3(x; y; z)e_3, \end{aligned}$$

where we again omitted the trivial terms, because  $\hat{J}_{\hbar}^i = 0$  for i > 3. In [9] the quantum Jacobi operators were calculated for all real 3-dimensional Lie algebras. Here we concentrate only on III<sub>a=1</sub><sup>ħ</sup>, VI<sub>a≠1</sub><sup>ħ</sup>, and VII<sub>a</sub><sup>ħ</sup>. Denote

$$(x, y, z) := \begin{pmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ z^1 & z^2 & z^3 \end{pmatrix}, \quad \hat{\xi}^1 := \omega \hat{q} \hat{Q} + (\hat{p} - p_0) \hat{P}, \quad \hat{\xi}^2 := \omega \hat{q} \hat{P} - (\hat{p} + p_0) \hat{Q}.$$

**Table 3.**  $VII_a^{\hbar}, III_{a=1}^{\hbar}, VI_{a \neq 1}^{\hbar}$

Quantum Bianchi type	$\hat{\mu}_{12}^1$	$\hat{\mu}_{12}^2$	$\hat{\mu}_{12}^3$	$\hat{\mu}_{23}^1$	$\hat{\mu}_{23}^2$	$\hat{\mu}_{23}^3$	$\hat{\mu}_{31}^1$	$\hat{\mu}_{31}^2$	$\hat{\mu}_{31}^3$
$VII_a^{\hbar}$	$\frac{a\hat{Q}}{\sqrt{2p_0}}$	$\frac{-a\hat{P}}{\sqrt{2p_0}}$	1	$\frac{\hat{p}-p_0}{-2p_0}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{-a\hat{Q}}{\sqrt{2p_0}}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{a\hat{P}}{\sqrt{2p_0}}$
$III_{a=1}^{\hbar}$	$\frac{\hat{Q}}{\sqrt{2p_0}}$	$\frac{-\hat{P}}{\sqrt{2p_0}}$	-1	$\frac{\hat{p}-p_0}{-2p_0}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{-\hat{Q}}{\sqrt{2p_0}}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{\hat{P}}{\sqrt{2p_0}}$
$VI_{a \neq 1}^{\hbar}$	$\frac{a\hat{Q}}{\sqrt{2p_0}}$	$\frac{-a\hat{P}}{\sqrt{2p_0}}$	-1	$\frac{\hat{p}-p_0}{-2p_0}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{-a\hat{Q}}{\sqrt{2p_0}}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{a\hat{P}}{\sqrt{2p_0}}$

Recall

**Theorem 8.1** ([9]). *The Jacobi operator components of  $VI_{a \neq 1}^{\hbar}$  and  $VII_a^{\hbar}$  read*

$$\hat{J}_h^1(x; y; z) = -\frac{a(x, y, z)}{\sqrt{2p_0^3}} \hat{\xi}^1, \quad \hat{J}_h^2(x; y; z) = -\frac{a(x, y, z)}{\sqrt{2p_0^3}} \hat{\xi}^2, \quad \hat{J}_h^3(x; y; z) = \frac{a^2(x, y, z)}{p_0} [\hat{P}, \hat{Q}].$$

For  $III_{a=1}^{\hbar}$  one has the same formulae with  $a = 1$ .

### 9. SEMICLASSICAL QUANTUM CONDITIONS

**Theorem 9.1** (Poisson brackets of quasi-canonical coordinates). *The quasi-canonical coordinates  $Q$  and  $P$  satisfy the relations*

$$\{P, P\} = 0 = \{Q, Q\}, \quad \{P, Q\} = \varepsilon := \frac{\omega}{2\sqrt{2H}}. \tag{9.1}$$

*Proof.* While the first two relations in (9.1) are evident, we have only to check the third one. Using several times the Leibniz rule for the Poisson brackets, calculate

$$\begin{aligned} 2\omega &= 2\omega\{p, q\} = \{P^2 - Q^2, PQ\} \\ &= \{P^2, PQ\} - \{Q^2, PQ\} \\ &= P\{P^2, Q\} - \{Q^2, P\}Q \\ &= P\{PP, Q\} - \{QQ, P\}Q \\ &= 2(P^2 + Q^2)\{P, Q\} \\ &= 4\sqrt{2H}\{P, Q\}. \end{aligned} \quad \square$$

When performing the quantization of the quasi-canonical variables, we shall use the Schrödinger picture, i.e. the operators  $\hat{q}, \hat{p}, \hat{H}$  and  $\hat{Q}, \hat{P}$  do not depend on time. Denote by  $[\cdot, \cdot]$  the ordinary commutator bracketing. Following the canonical quantization prescription, the quasi-canonical coordinates would satisfy in the semiclassical limit ( $\hbar \rightarrow 0$ ) the constraints

$$\hat{P}^2 + \hat{Q}^2 \approx 2\sqrt{2\hat{H}}, \quad \hat{P}^2 - \hat{Q}^2 \approx 2\hat{p}, \quad \hat{P}\hat{Q} + \hat{Q}\hat{P} \approx 2\omega\hat{q} \tag{9.2}$$

and the *quasi-canonical commutation relations* (quasi-CCR) read as follows:

$$[\hat{P}, \hat{P}] = 0 = [\hat{Q}, \hat{Q}], \quad [\hat{P}, \hat{Q}] = \frac{\hbar}{i} \hat{\varepsilon} := \frac{\hbar}{i} \frac{\omega}{2\sqrt{2\hat{H}}}. \quad (9.3)$$

## 10. SEMICLASSICAL APPROXIMATION OF THE JACOBI OPERATOR

**Theorem 10.1.** *Let constraints (9.2) and (9.3) hold. Then we have*

$$\begin{aligned} \hat{J}_\hbar^1(x; y; z) &\approx \frac{a(x, y, z)}{\sqrt{2p_0^3}} \left[ \hat{P} \left( \sqrt{2E} - \sqrt{2\hat{H}} \right) - \frac{\hbar}{i} \hat{Q} \frac{\hat{\varepsilon}}{2} \right], \\ \hat{J}_\hbar^2(x; y; z) &\approx \frac{a(x, y, z)}{\sqrt{2p_0^3}} \left[ \hat{Q} \left( \sqrt{2E} - \sqrt{2\hat{H}} \right) + \frac{\hbar}{i} \hat{P} \frac{\hat{\varepsilon}}{2} \right], \\ \hat{J}_\hbar^3(x; y; z) &= \frac{\hbar}{i} \frac{a^2(x, y, z)}{p_0} \hat{\varepsilon}. \end{aligned}$$

*Proof.* Using relations (9.2) and (9.3), first calculate

$$\begin{aligned} \hat{\xi}^1 &:= \omega \hat{q} \hat{Q} + (\hat{p} - p_0) \hat{P} \\ &\approx \frac{1}{2} (\hat{P} \hat{Q} + \hat{Q} \hat{P}) \hat{Q} + \frac{1}{2} (\hat{P}^2 - \hat{Q}^2) \hat{P} - p_0 \hat{P} \\ &\approx \frac{1}{2} (\hat{P} \hat{Q}^2 + \hat{Q} \hat{P} \hat{Q} + \hat{P}^3 - \hat{Q}^2 \hat{P}) - p_0 \hat{P} \\ &\approx \frac{1}{2} [\hat{Q} (\hat{P} \hat{Q} - \hat{Q} \hat{P}) + \hat{P} (\hat{Q}^2 + \hat{P}^2)] - p_0 \hat{P} \\ &\approx \frac{1}{2} \hat{Q} [\hat{P}, \hat{Q}] + \frac{1}{2} \hat{P} (\hat{P}^2 + \hat{Q}^2) - p_0 \hat{P} \\ &\approx \frac{\hbar}{i} \hat{Q} \frac{\hat{\varepsilon}}{2} + \hat{P} \sqrt{2\hat{H}} - \sqrt{2E} \hat{P} \\ &\approx \frac{\hbar}{i} \hat{Q} \frac{\hat{\varepsilon}}{2} + \hat{P} \left( \sqrt{2\hat{H}} - \sqrt{2E} \right). \end{aligned}$$

Next calculate

$$\begin{aligned} \hat{\xi}^2 &:= \omega \hat{q} \hat{P} - (\hat{p} + p_0) \hat{Q} \\ &\approx \frac{1}{2} (\hat{P} \hat{Q} + \hat{Q} \hat{P}) \hat{P} - \frac{1}{2} (\hat{P}^2 - \hat{Q}^2) \hat{Q} - p_0 \hat{Q} \\ &\approx \frac{1}{2} (\hat{P} \hat{Q} \hat{P} + \hat{Q} \hat{P}^2 - \hat{P}^2 \hat{Q} + \hat{Q}^3) - p_0 \hat{Q} \\ &\approx \frac{1}{2} [\hat{P} (\hat{Q} \hat{P} - \hat{P} \hat{Q}) + \hat{Q} (\hat{P}^2 + \hat{Q}^2)] - p_0 \hat{Q} \\ &\approx -\frac{1}{2} \hat{P} [\hat{P}, \hat{Q}] + \frac{1}{2} \hat{Q} (\hat{P}^2 + \hat{Q}^2) - p_0 \hat{Q} \\ &\approx -\frac{\hbar}{i} \hat{P} \frac{\hat{\varepsilon}}{2} + \hat{Q} \sqrt{2\hat{H}} - \sqrt{2E} \hat{Q} \\ &\approx -\frac{\hbar}{i} \hat{P} \frac{\hat{\varepsilon}}{2} + \hat{Q} \left( \sqrt{2\hat{H}} - \sqrt{2E} \right). \end{aligned} \quad \square$$

**Corollary 10.2.** Let constraints (9.2), (9.3) and  $\hat{H} = E$  hold. Then we have

$$\begin{aligned}\hat{f}_\hbar^1(x; y; z) &\approx -\frac{\hbar}{i} \frac{a(x, y, z)}{\sqrt{(2p_0)^3}} \frac{\omega}{2\sqrt{2E}} \hat{Q}, \\ \hat{f}_\hbar^2(x; y; z) &\approx +\frac{\hbar}{i} \frac{a(x, y, z)}{\sqrt{(2p_0)^3}} \frac{\omega}{2\sqrt{2E}} \hat{P}, \\ \hat{f}_\hbar^3(x; y; z) &= +\frac{\hbar}{i} \frac{a^2(x, y, z)}{p_0} \frac{\omega}{2\sqrt{2E}}.\end{aligned}$$

**Remark 10.3.** The last Corollary explicitly shows how the fundamental law of the quantum theory  $[\hat{p}, \hat{q}] = \hbar/i$  spoils the Jacobi identity.

## 11. CONCLUDING REMARKS

Following the algebraic point of view concerning mechanics, we considered the following question [4]: how can the time evolution of the linear algebraic operations (multiplications)  $V^{\otimes n} \rightarrow V$  be described? It turned out that quite a natural way to introduce dynamics in algebras is using the *operadic* Lax representations over given dynamical systems.

In this paper the operadic Lax representations for the harmonic oscillator are used to construct the quantum counterparts of some real 3-dimensional Lie algebras. The Jacobi operators of these quantum algebras are studied in semiclassical approximation.

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## Lie algebrate VII<sub>a</sub>, III<sub>a=1</sub>, VI<sub>a≠1</sub> kvantanaloogid üle harmoonilise ostsillaatori poolklassikalises lähenduses

Eugen Paal ja Jüri Virkepu

Harmoonilise ostsillaatori Laxi esituste operaadanalooži kasutades on konstrueeritud mõnede reaalsete kolmemõõtmeliste Lie algebrate kvantpartnerid. On uuritud nende kvantalgebrate Jacobi operaatoreid poolklassikalises lähenduses.