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MATHEMATICS

# Commutativity and ideals in category crossed products

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**Abstract.** In order to simultaneously generalize matrix rings and group graded crossed products, we introduce category crossed products. For such algebras we describe the centre and the commutant of the coefficient ring. We also investigate the connection between on the one hand maximal commutativity of the coefficient ring and on the other hand nonemptiness of intersections of the coefficient ring by nonzero two-sided ideals.

**Key words:** category graded rings, crossed products, ideals.

## 1. INTRODUCTION

Let $R$ be a ring. By this we always mean that $R$ is an additive group equipped with a multiplication which is associative and unital. The identity element of $R$ is denoted $1_R$ and the set of ring endomorphisms of $R$ is denoted $\text{End}(R)$. We always assume that ring homomorphisms respect the multiplicative identities. The centre of $R$ is denoted $Z(R)$ and by the commutant of a subset of $R$ we mean the collection of elements in $R$ that commute with all the elements in the subset.

Suppose that $R_1$ is a subring of $R$, i.e. there is an injective ring homomorphism $R_1 \to R$. Recall that if $R_1$ is commutative, then it is called a maximal commutative subring of $R$ if it coincides with its commutant in $R$. A lot of work has been devoted to investigating the connection between on the one hand maximal commutativity of $R_1$ in $R$ and on the other hand nonemptiness of intersections of $R_1$ with nonzero two-sided ideals of $R$ (see [2,3,5,6,9–11,16]). Recently (see [18–22]) such a connection was established for the commutant $R_1$ of the coefficient ring of crossed products $R$ (see Theorem 1 below). Recall that crossed products are defined by first specifying a crossed system, i.e. a quadruple $\{A, G, \sigma, \alpha\}$ where $A$ is a ring, $G$ is a group (written multiplicatively and with identity element $e$) and $\sigma : G \to \text{End}(A)$ and $\alpha : G \times G \to A$ are maps satisfying the following four conditions:

$$\sigma_e = \text{id}_A, \tag{1}$$

$$\alpha(s,e) = \alpha(e,s) = 1_A, \tag{2}$$

$$\alpha(s,t)\alpha(st,r) = \sigma_s(\alpha(t,r))\alpha(s,tr), \tag{3}$$

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$$\sigma_s(\sigma_t(a))\alpha(s,t) = \alpha(s,t)\sigma_{st}(a) \tag{4}$$

for all $s,t,r \in G$ and all $a \in A$. The crossed product, denoted $A \rtimes_\alpha^\sigma G$, associated to this quadruple, is the collection of formal sums $\sum_{s\in G} a_s u_s$, where $a_s \in A$, $s \in G$, are chosen so that all but finitely many of them are zero. By abuse of notation we write $u_s$ instead of $1_A u_s$ for all $s \in G$. The addition on $A \rtimes_\alpha^\sigma G$ is defined pointwise

$$\sum_{s\in G} a_s u_s + \sum_{s\in G} b_s u_s = \sum_{s\in G} (a_s + b_s) u_s \tag{5}$$

and the multiplication on $A \rtimes_\alpha^\sigma G$ is defined by the bilinear extension of the relation

$$(a_s u_s)(b_t u_t) = a_s \sigma_s(b_t)\alpha(s,t) u_{st} \tag{6}$$

for all $s,t \in G$ and all $a_s, b_t \in A$. By (1) and (2) $u_e$ is a multiplicative identity of $A \rtimes_\alpha^\sigma G$ and by (3) the multiplication on $A \rtimes_\alpha^\sigma G$ is associative. There is also an $A$-bimodule structure on $A \rtimes_\alpha^\sigma G$ defined by the linear extension of the relations $a(bu_s) = (ab)u_s$ and $(au_s)b = (a\sigma_s(b))u_s$ for all $a,b \in A$ and all $s,t \in G$, which, by (4), makes $A \rtimes_\alpha^\sigma G$ an $A$-algebra. In the article [18], the first author and Silvestrov show the following result.

**Theorem 1.** *If $A \rtimes_\alpha^\sigma G$ is a crossed product with $A$ commutative, all $\sigma_s$, $s \in G$, are ring automorphisms and all $\alpha(s,s^{-1})$, $s \in G$, are units in $A$, then every intersection of a nonzero two-sided ideal of $A \rtimes_\alpha^\sigma G$ with the commutant of $A$ in $A \rtimes_\alpha^\sigma G$ is nonzero.*

In [18] the first author and Silvestrov determine the centre of crossed products and in particular when crossed products are commutative; they also give a description of the commutant of $A$ in $A \rtimes_\alpha^\sigma G$. Theorem 1 has been generalized somewhat by relaxing the conditions on $\sigma$ and $\alpha$ (see [20,21]) and by considering general strongly group graded rings (see [22]). For more details concerning group graded rings in general and crossed product algebras in particular, see e.g. [1,7,17].

Many natural examples of rings, such as rings of matrices, crossed product algebras defined by separable extensions and category rings, are not in any natural way graded by groups, but instead by categories (see [12–14] and Remark 1). The purpose of this article is to define a category graded generalization of crossed products and to analyse commutativity questions similar to the ones discussed above for such algebras. In particular, we wish to generalize Theorem 1 from groups to groupoids (see Theorem 2 in Section 4). To be more precise, suppose that $G$ is a category. The family of objects of $G$ is denoted $\mathrm{ob}(G)$; we will often identify an object in $G$ with its associated identity morphism. The family of morphisms in $G$ is denoted $\mathrm{mor}(G)$; by abuse of notation, we will often write $s \in G$ when we mean $s \in \mathrm{mor}(G)$. The domain and codomain of a morphism $s$ in $G$ are denoted $d(s)$ and $c(s)$ respectively. We let $G^{(2)}$ denote the collection of composable pairs of morphisms in $G$, i.e. all $(s,t)$ in $\mathrm{mor}(G) \times \mathrm{mor}(G)$ satisfying $d(s) = c(t)$. Analogously, we let $G^{(3)}$ denote the collection of all composable triples of morphisms in $G$, i.e. all $(s,t,r)$ in $\mathrm{mor}(G) \times \mathrm{mor}(G) \times \mathrm{mor}(G)$ satisfying $(s,t) \in G^{(2)}$ and $(t,r) \in G^{(2)}$. Throughout the article $G$ is assumed to be small, i.e. with the property that $\mathrm{mor}(G)$ is a set. A category is called a groupoid[1] if all its morphisms are invertible. By a crossed system we mean a quadruple $\{A, G, \sigma, \alpha\}$ where $A$ is the direct sum of rings $A_e$, $e \in \mathrm{ob}(G)$, $\sigma_s : A_{d(s)} \to A_{c(s)}$, for $s \in G$, are ring homomorphisms and $\alpha$ is a map from $G^{(2)}$ to the disjoint union of the sets $A_e$, for $e \in \mathrm{ob}(G)$, with $\alpha(s,t) \in A_{c(s)}$, for $(s,t) \in G^{(2)}$, satisfying the following five conditions:

$$\sigma_e = \mathrm{id}_{A_e}, \tag{7}$$

$$\alpha(s,d(s)) = 1_{A_{c(s)}}, \tag{8}$$

$$\alpha(c(t),t) = 1_{A_{c(t)}}, \tag{9}$$

$$\alpha(s,t)\alpha(st,r) = \sigma_s(\alpha(t,r))\alpha(s,tr), \tag{10}$$

[1] The term *groupoid* has various meanings in the literature, e.g. in [8], a set with a binary operation is referred to as a *groupoid*.

$$\sigma_s(\sigma_t(a))\alpha(s,t) = \alpha(s,t)\sigma_{st}(a) \tag{11}$$

for all $e \in \mathrm{ob}(G)$, all $(s,t,r) \in G^{(3)}$ and all $a \in A_{d(t)}$. Let $A \rtimes_\alpha^\sigma G$ denote the collection of formal sums $\sum_{s\in G} a_s u_s$, where $a_s \in A_{c(s)}$, $s \in G$, are chosen so that all but finitely many of them are zero. Define addition on $A \rtimes_\alpha^\sigma G$ by (5) and define multiplication on $A \rtimes_\alpha^\sigma G$ by (6) if $(s,t) \in G^{(2)}$ and $(a_s u_s)(b_t u_t) = 0$ otherwise where $a_s \in A_{c(s)}$ and $b_t \in A_{c(t)}$. By (7), (8), and (9) it follows that $A \rtimes_\alpha^\sigma G$ has a multiplicative identity if and only if $\mathrm{ob}(G)$ is finite; in that case the multiplicative identity is $\sum_{e\in \mathrm{ob}(G)} u_e$. By (10) the multiplication on $A \rtimes_\alpha^\sigma G$ is associative. Define a left $A$-module structure on $A \rtimes_\alpha^\sigma G$ by the bilinear extension of the rule $a_e(b_s u_s) = (a_e b_s)u_s$ if $e = c(s)$ and $a_e(b_s u_s) = 0$ otherwise for all $a_e \in A_e$, $b_s \in A_{c(s)}$, $e \in \mathrm{ob}(G)$, $s \in G$. Analogously, define a right $A$-module structure on $A \rtimes_\alpha^\sigma G$ by the bilinear extension of the rule $(b_s u_s)c_f = (b_s \sigma_s(c_f))u_s$ if $f = d(s)$ and $(b_s u_s)c_f = 0$ otherwise for all $b_s \in A_{c(s)}$, $c_f \in A_f$, $f \in \mathrm{ob}(G)$, $s \in G$. By (11) this $A$-bimodule structure makes $A \rtimes_\alpha^\sigma G$ an $A$-algebra. We will often identify $A$ with $\bigoplus_{e\in \mathrm{ob}(G)} A_e u_e$; this ring will be referred to as the coefficient ring of $A \rtimes_\alpha^\sigma G$. It is clear that $A \rtimes_\alpha^\sigma G$ is a category graded ring in the sense defined in [13] and it is strongly graded if and only if each $\alpha(s,t)$, $(s,t) \in G^{(2)}$, has a left inverse in $A_{c(s)}$. We call $A \rtimes_\alpha^\sigma G$ the category crossed product algebra associated to the crossed system $\{A, G, \sigma, \alpha\}$.

In Section 2, we determine the centre of category crossed products. In particular, we determine when category crossed products are commutative. In Section 3, we describe the commutant of the coefficient ring in category crossed products. In Section 4, we investigate the connection between on the one hand maximal commutativity of the coefficient ring and on the other hand nonemptiness of intersections of the coefficient ring by nonzero two-sided ideals. At the end of each section, we indicate how our results generalize earlier results for other algebraic structures such as group crossed products and matrix rings (see Remarks 1–6 and Remark 8).

## 2. THE CENTRE

For the rest of the article, unless otherwise stated, we suppose that $A \rtimes_\alpha^\sigma G$ is a category crossed product. We say that $\alpha$ is symmetric if $\alpha(s,t) = \alpha(t,s)$ for all $s,t \in G$ with $d(s) = c(s) = d(t) = c(t)$. We say that $A \rtimes_\alpha^\sigma G$ is a monoid (groupoid, group) crossed product if $G$ is a monoid (groupoid, group). We say that $A \rtimes_\alpha^\sigma G$ is a twisted category (monoid, groupoid, group) algebra if each $\sigma_s$, $s \in G$, with $d(s) = c(s)$ equals the identity map on $A_{d(s)} = A_{c(s)}$; in that case the category (monoid, groupoid, group) crossed product is denoted $A \rtimes_\alpha G$. We say that $A \rtimes_\alpha^\sigma G$ is a skew category (monoid, groupoid, group) algebra if $\alpha(s,t) = 1_{A_{c(s)}}$, for $(s,t) \in G^{(2)}$; in that case the category (monoid, groupoid, group) crossed product is denoted $A \rtimes^\sigma G$. If $G$ is a monoid, then we let $A^G$ denote the set of elements in $A$ fixed by all $\sigma_s$, $s \in G$. We say that $G$ is cancellable if any equality of the form $s_1 t_1 = s_2 t_2$, when $(s_i, t_i) \in G^{(2)}$, for $i = 1,2$, implies that $s_1 = s_2$ (or $t_1 = t_2$) whenever $t_1 = t_2$ (or $s_1 = s_2$). For $e, f \in \mathrm{ob}(G)$ we let $G_{f,e}$ denote the collection of $s \in G$ with $c(s) = f$ and $d(s) = e$; we let $G_e$ denote the monoid $G_{e,e}$. We let the restriction of $\alpha$ (or $\sigma$) to $G_e^2$ (or $G_e$) be denoted by $\alpha_e$ (or $\sigma_e$). With this notation all $A_e \rtimes_{\alpha_e}^{\sigma_e} G_e$, for $e \in \mathrm{ob}(G)$, are monoid crossed products.

**Proposition 1.** *The centre of a monoid crossed product $A \rtimes_\alpha^\sigma G$ is the collection of $\sum_{s\in G} a_s u_s$ in $A \rtimes_\alpha^\sigma G$ satisfying the following two conditions*: (i) $a_s \sigma_s(a) = a a_s$, *for $s \in G$ and $a \in A$*; (ii) *for all $t,r \in G$ the following equality holds* $\sum_{\substack{s\in G \\ st=r}} a_s \alpha(s,t) = \sum_{\substack{s\in G \\ ts=r}} \sigma_t(a_s)\alpha(t,s)$.

*Proof.* Let $e$ denote the identity element of $G$. Take $x := \sum_{s\in G} a_s u_s$ in the centre of $A \rtimes_\alpha^\sigma G$. Condition (i) follows from the fact that $x a u_e = a u_e x$ for all $a \in A$. Condition (ii) follows from the fact that $x u_t = u_t x$ for all $t \in G$. Conversely, it is clear that conditions (i) and (ii) are sufficient for $x$ to be in the centre of $A \rtimes_\alpha^\sigma G$. □

**Corollary 1.** *The centre of a twisted monoid ring $A \rtimes_\alpha G$ is the collection of $\sum_{s\in G} a_s u_s$ in $A \rtimes_\alpha G$ satisfying the following two conditions:* (i) *$a_s \in Z(A)$, for $s \in G$*; (ii) *for all $t,r \in G$, the following equality holds* $\sum_{\substack{s\in G \\ st=r}} a_s \alpha(s,t) = \sum_{\substack{s\in G \\ ts=r}} a_s \alpha(t,s)$.

*Proof.* This follows immediately from Proposition 1. □

**Corollary 2.** *If $G$ is an abelian cancellable monoid, $\alpha$ is symmetric and has the property that none of the $\alpha(s,t)$, for $(s,t) \in G^{(2)}$, is a zero-divisor, then the centre of $A \rtimes_\alpha^\sigma G$ is the collection of $\sum_{s\in G} a_s u_s$ in $A \rtimes_\alpha^\sigma G$ satisfying the following two conditions*: (i) $a_s\sigma_s(a) = aa_s$, *for* $s \in G$ *and* $a \in A$; (ii) $a_s \in A^G$, *for* $s \in G$. *In particular, if $A \rtimes^\sigma G$ is a skew monoid ring where $G$ is abelian and cancellable, then the same description of the centre is valid.*

*Proof.* Take $x := \sum_{s\in G} a_s u_s$ in $A \rtimes_\alpha^\sigma G$. Suppose that $x$ belongs to the centre of $A \rtimes_\alpha^\sigma G$. Condition (i) follows from the first part of Proposition 1. Now we show condition (ii). Take $s,t \in G$ and let $r = st$. Since $G$ is commutative and cancellable, we get, by the second part of Proposition 1, that $a_s\alpha(s,t) = \sigma_t(a_s)\alpha(t,s)$. Since $\alpha$ is symmetric and $\alpha(s,t)$ is not a zero-divisor, this implies that $a_s = \sigma_t(a_s)$. Since $s$ and $t$ were arbitrarily chosen from $G$, this implies that $a_s \in A^G$, for $s \in G$. On the other hand, by Proposition 1, it is clear that (i) and (ii) are sufficient conditions for $x$ to be in the centre of $A \rtimes_\alpha^\sigma G$. The second part of the claim is obvious. □

Now we show that the centre of a category crossed product is a particular subring of the direct sum of the centres of the corresponding monoid crossed products.

**Proposition 2.** *The centre of a category crossed product $A \rtimes_\alpha^\sigma G$ equals the collection of $\sum_{e\in \mathrm{ob}(G)} \sum_{s\in G_e} a_s u_s$ in $\sum_{e\in\mathrm{ob}(G)} Z(A_e \rtimes_{\alpha_e}^{\sigma_e} G_e)$ satisfying $\sum_{\substack{s\in G_e\\ rs=g}} \sigma_r(a_s)\alpha(r,s) = \sum_{\substack{t\in G_f\\ tr=g}} a_t\alpha(t,r)$ for all $e, f \in \mathrm{ob}(G)$ with $e \neq f$, and all $r,g \in G_{f,e}$.*

*Proof.* Take $x := \sum_{s\in G} a_s u_s$ in the centre of $A \rtimes_\alpha^\sigma G$. By the equalities $u_e x = x u_e$, for $e \in \mathrm{ob}(G)$, it follows that $a_s = 0$ for all $s \in G$ with $d(s) \neq c(s)$. Therefore we can write $x = \sum_{e\in\mathrm{ob}(G)} \sum_{s\in G_e} a_s u_s$ where $\sum_{s\in G_e} a_s u_s \in Z(A_e \rtimes_{\alpha_e}^{\sigma_e} G_e)$, for $e \in \mathrm{ob}(G)$. The last part of the claim follows from the fact that the equality $u_r\left(\sum_{s\in G_e} a_s u_s\right) = \left(\sum_{s\in G_e} a_s u_s\right) u_r$ holds for all $e, f \in \mathrm{ob}(G)$, all $e \neq f$, and all $r \in G_{f,e}$. □

**Proposition 3.** *Suppose that $A \rtimes_\alpha^\sigma G$ is a category crossed product and consider the following six conditions*: (0) *all $\alpha(s,t)$, for $(s,t) \in G^{(2)}$, are nonzero*; (i) *$A \rtimes_\alpha^\sigma G$ is commutative*; (ii) *$G$ is the disjoint union of the monoids $G_e$, for $e \in \mathrm{ob}(G)$, and they are all abelian*; (iii) *each $A_e \rtimes_{\alpha_e}^{\sigma_e} G_e$, for $e \in \mathrm{ob}(G)$, is a twisted monoid algebra*; (iv) *$A$ is commutative*; (v) *$\alpha$ is symmetric. Then* (a) *conditions* (0) *and* (i) *imply conditions* (ii)–(v); (b) *conditions* (ii)–(v) *imply condition* (i).

*Proof.* (a) Suppose that conditions (0) and (i) hold. By Proposition 2, we get that $G$ is the direct sum of $G_e$, for $e \in \mathrm{ob}(G)$, and that each $A_e \rtimes_{\alpha_e}^{\sigma_e} G_e$, for $e \in \mathrm{ob}(G)$, is commutative. The latter and Proposition 1(i) imply that (iii) holds. Corollary 1 now implies that (iv) holds. For the rest of the proof we can suppose that $G$ is a monoid. Take $s,t \in G$. By the commutativity of $A \rtimes_\alpha^\sigma G$ we get that $\alpha(s,t)u_{st} = u_s u_t = u_t u_s = \alpha(t,s)u_{ts}$ for all $s,t \in G$. Since $\alpha$ is nonzero this implies that $st = ts$ and that $\alpha(s,t) = \alpha(t,s)$ for all $s,t \in G$. Therefore, $G$ is abelian and (v) holds.

Conversely, by Corollary 1 and Corollary 2 we get that conditions (ii)–(iv) are sufficient for commutativity of $A \rtimes_\alpha^\sigma G$. □

**Remark 1.** Proposition 2, Corollary 1, Corollary 2, and Proposition 3 generalize Proposition 3 and Corollaries 1–4 in [18] from groups to categories.

**Remark 2.** Let $A \rtimes G$ be a category algebra where all the rings $A_e$, for $e \in \mathrm{ob}(G)$, coincide with a fixed ring $D$. Then $A \rtimes G$ is the usual category algebra $DG$ of $G$ over $D$. Let $H$ denote the disjoint union of the monoids $G_e$, for $e \in \mathrm{ob}(G)$. By Proposition 1 and Proposition 2 the centre of $DG$ is the collection of $\sum_{s\in H} a_s u_s$, for $a_s \in Z(D)$, and $s \in H$, in the induced category algebra $Z(D)H$ satisfying $\sum_{\substack{s\in H\\ st=r}} a_s = \sum_{\substack{s\in H\\ ts=r}} a_s$ for all $r,t \in G$. Note that if $G$ is a groupoid, then the last condition simplifies to $a_{rt^{-1}} = a_{t^{-1}r}$ for all $r,t \in G$ with $c(r) = c(t)$ and $d(r) = d(t)$. This result specializes to two well-known cases. First of all, if $G$ is a group, then we retrieve the usual description of the centre of a group ring (see e.g. [23]). Secondly, if $G$ is the groupoid with the $n$ first positive integers as objects and as arrows all pairs $(i,j)$, for $1 \leq i,j \leq n$, equipped with the partial binary operation defined by letting $(i,j)(k,l)$ be defined and equal to $(i,l)$ precisely when $j=k$, then $DG$ is the ring of square matrices over $D$ of size $n$ and we retrieve the result that $Z(M_n(D))$ equals the $Z(D)1_n$ where $1_n$ is the unit $n \times n$ matrix.

**Remark 3.** Let $L/K$ be a finite separable (not necessarily normal) field extension. Let $N$ denote a normal closure of $L/K$ and let $\mathrm{Gal}(N/K)$ denote the Galois group of $N/K$. Furthermore, let $L = L_1, L_2, \ldots, L_n$ denote the different conjugate fields of $L$ under the action of $\mathrm{Gal}(N/K)$ and put $F = \bigoplus_{i=1}^n L_i$. If $1 \le i,j \le n$, then let $G_{ij}$ denote the set of field isomorphisms from $L_j$ to $L_i$. If $s \in G_{ij}$, then we indicate this by writing $d(s)=j$ and $c(s)=i$. If we let $G$ be the union of the $G_{ij}$, for $1 \le i,j \le n$, then $G$ is a groupoid. For each $s \in G$, let $\sigma_s = s$. Suppose that $\alpha$ is a map $G^{(2)} \to \bigsqcup_{i=1}^n L_i$ with $\alpha(s,t) \in L_{c(s)}$, for $(s,t) \in G^{(2)}$ satisfying (2), (3) and (4) for all $(s,t,r) \in G^{(3)}$ and all $a \in L_{d(t)}$. The category crossed product $F \rtimes_\alpha^\sigma G$ extends the construction usually defined by Galois field extensions $L/K$. By Proposition 2, the centre of $F \rtimes_\alpha^\sigma G$ is the collection of $\sum_{e \in \mathrm{ob}(G)} a_e u_e$ with $a_e = s(a_f)$ for all $e, f \in \mathrm{ob}(G)$ and all $s \in G$ with $c(s)=e$ and $d(s)=f$. Therefore the centre is a field isomorphic to $L^{G_{1,1}}$ and we retrieve the first part of Theorem 4 in [12].

## 3. THE COMMUTANT OF THE COEFFICIENT RING

**Proposition 4.** *The commutant of $A$ in $A \rtimes_\alpha^\sigma G$ is the collection of $\sum_{s\in G} a_s u_s$ in $A \rtimes_\alpha^\sigma G$ satisfying $a_s = 0$, for $s \in G$, with $d(s) \neq c(s)$, and $a_s \sigma_s(a) = a a_s$, for $s \in G$ with $d(s) = c(s)$ and $a \in A_{d(s)}$.*

*Proof.* The first claim follows from the fact that the equality $(\sum_{s\in G} a_s u_s) u_e = u_e (\sum_{s\in G} a_s u_s)$ holds for all $e \in \mathrm{ob}(G)$. The second claim follows from the fact that the equality $(\sum_{s\in G} a_s u_s) a u_e = a u_e (\sum_{s\in G} a_s u_s)$ holds for all $e \in \mathrm{ob}(G)$ and all $a \in A_e$. □

Recall that the annihilator of an element $r$ in a commutative ring $R$ is the collection, denoted $\mathrm{ann}(r)$, of elements $s$ in $R$ with the property that $rs = 0$.

**Corollary 3.** *Suppose that $A$ is commutative. Then the commutant of $A$ in $A \rtimes_\alpha^\sigma G$ is the collection of $\sum_{s\in G} a_s u_s$ in $A \rtimes_\alpha^\sigma G$ satisfying $a_s = 0$, for $s \in G$ with $d(s) \neq c(s)$, and $\sigma_s(a) - a \in \mathrm{ann}(a_s)$, for $s \in G$ with $d(s) = c(s)$ and $a \in A_{d(s)}$. In particular, $A$ is maximal commutative in $A \rtimes_\alpha^\sigma G$ if and only if for all choices of $e \in \mathrm{ob}(G)$, $s \in G_e \setminus \{e\}$, $a_s \in A_e$, there is a nonzero $a \in A_e$ with the property that $\sigma_s(a) - a \notin \mathrm{ann}(a_s)$.*

*Proof.* This follows immediately from Proposition 4. □

**Corollary 4.** *Suppose that each $A_e$, $e \in \mathrm{ob}(G)$, is an integral domain. Then the commutant of $A$ in $A \rtimes_\alpha^\sigma G$ is the collection of $\sum_{s\in G} a_s u_s$ in $A \rtimes_\alpha^\sigma G$ satisfying $a_s = 0$ whenever $\sigma_s$ is not an identity map. In particular, $A$ is maximal commutative in $A \rtimes_\alpha^\sigma G$ if and only if for all nonidentity $s \in G$, the map $\sigma_s$ is not an identity map.*

*Proof.* This follows immediately from Corollary 3. □

**Proposition 5.** *If $A$ is commutative, $G$ a disjoint union of abelian monoids and $\alpha$ is symmetric, then the commutant of $A$ in $A \rtimes_\alpha^\sigma G$ is the unique maximal commutative subalgebra of $A \rtimes_\alpha^\sigma G$ containing $A$.*

*Proof.* We need to show that the commutant of $A$ in $A \rtimes_\alpha^\sigma G$ is commutative. By the first part of Proposition 4, we can assume that $G$ is an abelian monoid. If we take $\sum_{s\in G} a_s u_s$ and $\sum_{t\in G} b_t u_t$ in the commutant of $A$ in $A \rtimes_\alpha^\sigma G$, then, by the second part of Proposition 4 and the fact that $\alpha$ is symmetric, we get that

$$\begin{aligned}\sum_{s\in G} a_s u_s \sum_{t\in G} b_t u_t &= \sum_{s,t\in G} a_s \sigma_s(b_t) \alpha(s,t) u_{st} = \sum_{s,t\in G} a_s b_t \alpha(s,t) u_{st} \\ &= \sum_{s,t\in G} b_t a_s \alpha(t,s) u_{ts} = \sum_{s,t\in G} b_t \sigma_t(a_s) \alpha(t,s) u_{st} = \sum_{t\in G} b_t u_t \sum_{s\in G} a_s u_s.\end{aligned}$$ □

**Remark 4.** Proposition 4, Corollary 3, Corollary 4, and Proposition 5 together generalize Theorem 1, Corollaries 5–10, and Proposition 4 in [18] from groups to categories.

**Remark 5.** Let $A \rtimes G$ be a category algebra where all the rings $A_e$, $e \in \mathrm{ob}(G)$, coincide with a fixed integral domain $D$. Then $A \rtimes G$ is the usual category algebra $DG$ of $G$ over $D$. By Corollary 4, the commutant of $D$ in $DG$ is $DG$ itself. In particular, $A$ is maximal commutative in $DG$ if and only if $G$ is the disjoint union of $|\,\mathrm{ob}(G)|$ copies of the trivial group.

**Remark 6.** Let $L/K$ be a finite separable (not necessarily normal) field extension. We use the same notation as in Remark 3. By Corollary 4, the commutant of $F$ in $F \rtimes_\alpha^\sigma G$ is the collection of $\sum_{i=1}^n \sum_{s\in G_{ii}} a_s u_s$ satisfying $a_s = 0$ whenever $\sigma_s$ is not an identity map. In particular, $F$ is maximal commutative in $F \rtimes_\alpha^\sigma G$ if all groups $G_{i,i}$, $i = 1,\dots,n$, are nontrivial; this of course happens in the case when $L/K$ is a Galois field extension.

## 4. COMMUTATIVITY AND IDEALS

In this section, we investigate the connection between on the one hand maximal commutativity of the coefficient ring and on the other hand nonemptiness of intersections of the coefficient ring by nonzero two-sided ideals. For the rest of the article, we assume that $\mathrm{ob}(G)$ is finite. Recall (from Section 1) that this is equivalent to the fact that $A \rtimes_\alpha^\sigma G$ has a multiplicative identity; in that case the multiplicative identity is $\sum_{e\in \mathrm{ob}(G)} u_e$.

**Theorem 2.** *If $A \rtimes_\alpha^\sigma G$ is a groupoid crossed product such that for every $s \in G$, $\alpha(s,s^{-1})$ is not a zero-divisor in $A_{c(s)}$, then every intersection of a nonzero two-sided ideal of $A \rtimes_\alpha^\sigma G$ with the commutant of $Z(A)$ in $A \rtimes_\alpha^\sigma G$ is nonzero.*

*Proof.* We show the contrapositive statement. Let $C$ denote the commutant of $Z(A)$ in $A \rtimes_\alpha^\sigma G$ and suppose that $I$ is a two-sided ideal of $A \rtimes_\alpha^\sigma G$ with the property that $I \bigcap C = \{0\}$. We wish to show that $I = \{0\}$. Take $x \in I$. If $x \in C$, then by the assumption $x = 0$. Therefore we now assume that $x = \sum_{s\in G} a_s u_s \in I$, $a_s \in A_{c(s)}$, $s \in G$, and that $x$ is chosen so that $x \notin C$ with the set $S := \{s \in G \mid a_s \neq 0\}$ of least possible cardinality $N$. Seeking a contradiction, suppose that $N$ is positive. First note that there is $e \in \mathrm{ob}(G)$ with $u_e x \in I \setminus C$. In fact, if $u_e x \in C$ for all $e \in \mathrm{ob}(G)$, then $x = 1x = \sum_{e\in \mathrm{ob}(G)} u_e x \in C$, which is a contradiction. By minimality of $N$ we can assume that $c(s) = e$, $s \in S$, for some fixed $e \in \mathrm{ob}(G)$. Take $t \in S$ and consider the element $x' := x u_{t^{-1}} \in I$. Since $\alpha(t,t^{-1})$ is not a zero-divisor we get that $x' \neq 0$. Therefore, since $I \bigcap C = \{0\}$, we get that $x' \in I \setminus C$. Take $a = \sum_{f\in \mathrm{ob}(G)} b_f u_f \in Z(A)$ and note that $Z(A) = \bigoplus_{f\in \mathrm{ob}(G)} Z(A_f)$. Then $I \ni x'' := ax' - x'a = \sum_{s\in S} (b_{c(s)} a_s - a_s \sigma_s(b_{d(s)})) u_s$. In the $A_e$ component of this sum we have $b_e a_e - a_e b_e = 0$ since $b_e \in Z(A_e)$. Thus, the summand vanishes for $s = e$, and hence we get, by the assumption on $N$, that $x'' = 0$. Since $a \in Z(A)$ was arbitrarily chosen, we get that $x' \in C$ which is a contradiction. Therefore $N = 0$ and hence $S = \emptyset$ which in turn implies that $x = 0$. Since $x \in I$ was arbitrarily chosen, we finally get that $I = \{0\}$. □

**Corollary 5.** *If $A \rtimes_\alpha^\sigma G$ is a groupoid crossed product with $A$ maximal commutative and for every $s \in G$, $\alpha(s,s^{-1})$ is not a zero-divisor in $A_{c(s)}$, then every intersection of a nonzero two-sided ideal of $A \rtimes_\alpha^\sigma G$ with $A$ is nonzero.*

*Proof.* This follows immediately from Theorem 2. □

Now we examine conditions under which the converse statement of Corollary 5 is true. To this end, we recall some notions from category theory that we need in the sequel (for the details see e.g. [15]). Let $G$ be a category. A congruence relation $R$ on $G$ is a collection of equivalence relations $R_{a,b}$ on $\hom(a,b)$, $a,b \in \mathrm{ob}(G)$, chosen so that if $(s,s') \in R_{a,b}$ and $(t,t') \in R_{b,c}$, then $(ts,t's') \in R_{a,c}$ for all $a,b,c \in \mathrm{ob}(G)$. Given a congruence relation $R$ on $G$ we can define the corresponding quotient category $G/R$ as the category having as objects the objects of $G$ and as arrows the corresponding equivalence classes of arrows from $G$. In that case there is a full functor $\mathscr{Q}_R : G \to G/R$ which is the identity on objects and sends each morphism of $G$ to its equivalence class in $R$. We will often use the notation $[s] := \mathscr{Q}_R(s)$, $s \in G$. Suppose that $H$ is another category and that $F : G \to H$ is a functor. The kernel of $F$, denoted $\ker(F)$, is the congruence relation on $G$ defined by letting $(s,t) \in \ker(F)_{a,b}$, $a,b \in \mathrm{ob}(G)$, whenever $s,t \in \hom(a,b)$ and $F(s) = F(t)$. In that case there is a unique functor $\mathscr{P}_F : G/\ker(F) \to H$ with the property that $\mathscr{P}_F \mathscr{Q}_{\ker(F)} = F$. Furthermore, if there is a congruence relation $R$ on $G$ contained in $\ker(F)$, then there is a unique functor $\mathscr{N} : G/R \to G/\ker(F)$ with the property that $\mathscr{N}\mathscr{Q}_R = \mathscr{Q}_{\ker(F)}$. In that case there is therefore always a factorization $F = \mathscr{P}_F \mathscr{N}\mathscr{Q}_R$; we will refer to this factorization as the canonical one.

**Proposition 6.** *Let* $\{A,G,\sigma,\alpha\}$ *and* $\{A,H,\tau,\beta\}$ *be crossed systems with* $\mathrm{ob}(G) = \mathrm{ob}(H)$. *Suppose that there is a functor* $F : G \to H$ *satisfying the following three criteria*: (i) $F$ *is the identity map on objects*; (ii) $\tau_{F(s)} = \sigma_s$, *for* $s \in G$; (iii) $\beta(F(s),F(t)) = \alpha(s,t)$, *for* $(s,t) \in G^{(2)}$. *Then there is a unique A-algebra homomorphism* $A \rtimes_\alpha^\sigma G \to A \rtimes_\beta^\tau H$, *denoted* $\widetilde{F}$, *satisfying* $\widetilde{F}(u_s) = u_{F(s)}$, *for* $s \in G$.

*Proof.* Take $x := \sum_{s\in G} a_s u_s$ in $A \rtimes_\alpha^\sigma G$ where $a_s \in A_{c(s)}$, for $s \in G$. By $A$-linearity we get that $\widetilde{F}(x) = \sum_{s\in G} a_s \widetilde{F}(u_s) = \sum_{s\in G} a_s u_{F(s)}$. Therefore $\widetilde{F}$ is unique. It is clear that $\widetilde{F}$ is additive. By (i), $\widetilde{F}$ respects the multiplicative identities. Now we show that $\widetilde{F}$ is multiplicative. Take another $y := \sum_{s\in G} b_s u_s$ in $A \rtimes_\alpha^\sigma G$ where $b_s \in A_{c(s)}$, for $s \in G$. Then, by (ii) and (iii), we get that

$$\begin{aligned}\widetilde{F}(xy) &= \widetilde{F}\left(\sum_{(s,t)\in G^{(2)}} a_s\sigma_s(b_t)\alpha(s,t)u_{st}\right) = \sum_{(s,t)\in G^{(2)}} a_s\sigma_s(b_t)\alpha(s,t)u_{F(st)} \\ &= \sum_{(s,t)\in G^{(2)}} a_s\tau_{F(s)}(b_t)\beta(F(s),F(t))u_{F(s)F(t)} = \widetilde{F}(x)\widetilde{F}(y).\end{aligned}$$ □

**Remark 7.** Suppose that $\{A,G,\sigma,\alpha\}$ is a crossed system. By abuse of notation, we let $A$ denote the category with the rings $A_e$, for $e \in \mathrm{ob}(G)$, as objects and ring homomorphisms $A_e \to A_f$, for $e, f \in \mathrm{ob}(G)$, as morphisms. Define a map $\sigma : G \to A$ on objects by $\sigma(e) = A_e$, for $e \in \mathrm{ob}(G)$, and on arrows by $\sigma(s) = \sigma_s$, for $s \in G$. By Eq. (4) it is clear that $\sigma$ is a functor if the following two conditions are satisfied: (i) for all $(s,t) \in G^{(2)}$, $\alpha(s,t)$ belongs to the centre of $A_{c(s)}$; (ii) for all $(s,t) \in G^{(2)}$, $\alpha(s,t)$ is not a zero-divisor in $A_{c(s)}$.

**Proposition 7.** *Let* $A \rtimes_\alpha^\sigma G$ *be a category crossed product with* $\sigma : G \to A$ *a functor. Suppose that* $R$ *is a congruence relation on* $G$ *with the property that the associated quadruple* $\{A, G/R, \sigma([\cdot]), \alpha([\cdot],[\cdot])\}$ *is a crossed system. If* $I$ *is the two-sided ideal in* $A \rtimes_\alpha^\sigma G$ *generated by an element* $\sum_{s\in G} a_s u_s$, *where* $a_s \in A_{c(s)}$, *for* $s \in G$, *satisfying* $a_s = 0$ *if* $s$ *does not belong to any of the classes* $[e]$, *for* $e \in \mathrm{ob}(G)$, *and* $\sum_{s\in[e]} a_s = 0$, *for* $e \in \mathrm{ob}(G)$, *then* $A \bigcap I = \{0\}$.

*Proof.* By Proposition 6, the functor $\mathscr{Q}_R$ induces an $A$-algebra homomorphism $\widetilde{\mathscr{Q}}_R : A \rtimes_\alpha^\sigma G \to A \rtimes_{\alpha([\cdot],[\cdot])}^{\sigma([\cdot])} G/R$. By the definition of $a_s$, for $s \in G$, we get that

$$\widetilde{\mathscr{Q}}_R\left(\sum_{s\in G} a_s u_s\right) = \widetilde{\mathscr{Q}}_R\left(\sum_{e\in\mathrm{ob}(G)}\sum_{s\in[e]} a_s u_s\right) = \sum_{e\in\mathrm{ob}(G)}\sum_{s\in[e]} a_s u_{[s]} = \sum_{e\in\mathrm{ob}(G)}\left(\sum_{s\in[e]} a_s\right) u_{[e]} = 0.$$

This implies that $\widetilde{\mathscr{Q}}_R(I) = \{0\}$. Since $\widetilde{\mathscr{Q}}_R|_A = \mathrm{id}_A$, we therefore get that $I \bigcap A = (\widetilde{\mathscr{Q}}_R|_A)(A \bigcap I) \subseteq \widetilde{\mathscr{Q}}_R(I) = \{0\}$. □

Let $G$ be a groupoid and suppose that we for each $e \in \mathrm{ob}(G)$ are given a subgroup $N_e$ of $G_e$. We say that $N = \bigcup_{e\in\mathrm{ob}(G)} N_e$ is a normal subgroupoid of $G$ if $sN_{d(s)} = N_{c(s)}s$ for all $s \in G$. The normal subgroupoid $N$ induces a congruence relation $\sim$ on $G$ defined by letting $s \sim t$, for $s,t \in G$, if there is $n$ in $N_{d(t)}$ with $s = nt$. The corresponding quotient category is a groupoid which is denoted $G/N$. For more details, see e.g. [4]; note that our definition of normal subgroupoids is more restrictive than the one used in [4].

**Proposition 8.** *Let $A \rtimes_{\alpha}^{\sigma} G$ be a groupoid crossed product such that for each $(s,t) \in G^{(2)}$, $\alpha(s,t) \in Z(A_{c(s)})$ and $\alpha(s,t)$ is not a zero-divisor in $A_{c(s)}$. Suppose that $N$ is a normal subgroupoid of $G$ with the property that $\sigma_n = \mathrm{id}_{A_{c(n)}}$, for $n \in N$, and $\alpha(s,t) = 1_{A_{c(s)}}$ if $s \in N$ or $t \in N$. If $I$ is the two-sided ideal in $A \rtimes_{\alpha}^{\sigma} G$ generated by an element $\sum_{s \in G} a_s u_s$, with $a_s \in A_{c(s)}$, for $s \in G$, satisfying $a_s = 0$ if $s$ does not belong to any of the sets $N_e$, for $e \in \mathrm{ob}(G)$, and $\sum_{s \in N_e} a_s = 0$, for $e \in \mathrm{ob}(G)$, then $A \bigcap I = \{0\}$.*

*Proof.* By Remark 7, $\sigma$ is a functor $G \to A$ and $\sim\ \subseteq \ker(\sigma)$. Therefore, by the discussion preceding Proposition 6, there is a well-defined functor $\sigma[\cdot] : G/N \to A$. Now we show that the induced map $\alpha([\cdot],[\cdot])$ is well-defined. By Eq. (3) with $s = n \in N_{c(t)}$ we get that $\alpha(n,t)\alpha(nt,r) = \sigma_n(\alpha(t,r))\alpha(n,tr)$. By the assumptions on $\alpha$ and $\sigma$ we get that $\alpha(nt,r) = \alpha(t,r)$. Analogously, by Eq. (3) with $t = n \in N_{d(r)}$, we get that $\alpha(s,t) = \alpha(s,tn)$. Therefore, $\alpha([\cdot],[\cdot])$ is well-defined. The rest of the claim now follows immediately from Proposition 7. □

**Proposition 9.** *Let $A \rtimes^{\sigma} G$ be a skew category algebra. Suppose that $R$ is a congruence relation on $G$ contained in* $\ker(\sigma)$. *If $I$ is the two-sided ideal in $A \rtimes^{\sigma} G$ generated by an element $\sum_{s \in G} a_s u_s$, where $a_s \in A_{c(s)}$, for $s \in G$, satisfying $a_s = 0$ if $s$ does not belong to any of the classes $[e]$, for $e \in \mathrm{ob}(G)$, and $\sum_{s \in [e]} a_s = 0$, for $e \in \mathrm{ob}(G)$, then $A \bigcap I = \{0\}$.*

*Proof.* By Remark 7 and the discussion preceding Proposition 6, there is a well-defined functor $\sigma[\cdot] : G/R \to A$. The claim now follows immediately from Proposition 7. □

**Proposition 10.** *Let $A \rtimes^{\sigma} G$ be a skew groupoid ring with all $A_e$, for $e \in \mathrm{ob}(G)$, equal integral domains and each $G_e$, for $e \in \mathrm{ob}(G)$, an abelian group. If every intersection of a nonzero two-sided ideal of $A \rtimes^{\sigma} G$ and $A$ is nonzero, then $A$ is maximal commutative in $A \rtimes^{\sigma} G$.*

*Proof.* We show the contrapositive statement. Suppose that $A$ is not maximal commutative in $A \rtimes^{\sigma} G$. By the second part of Corollary 4, there is $e \in \mathrm{ob}(G)$ and a nonidentity $s \in G_e$ such that $\sigma_s = \mathrm{id}_{A_e}$. Let $N_e$ denote the cyclic subgroup of $G_e$ generated by $s$. Note that since $G_e$ is abelian, $N_e$ is a normal subgroup of $G_e$. For each $f \in \mathrm{ob}(G)$, define a subgroup $N_f$ of $G_f$ in the following way. If $G_{e,f} \neq \emptyset$, then let $N_f = sN_e s^{-1}$, where $s$ is a morphism in $G_{e,f}$. If, on the other hand, $G_{e,f} = \emptyset$, then let $N_f = \{f\}$. Note that if $s_1, s_2 \in G_{e,f}$, then $s_2^{-1}s_1 \in G_e$ and hence $s_1 N_e s_1^{-1} = s_2 s_2^{-1} s_1 N_e (s_2^{-1}s_1)^{-1} s_2^{-1} = s_2 N_e s_2^{-1}$. Therefore, $N_f$ is well-defined. Now put $N = \bigcup_{f \in \mathrm{ob}(G)} N_f$. It is clear that $N$ is a normal subgroupoid of $G$ and that $\sigma_n = \mathrm{id}_{A_e}$, $n \in N$. Let $I$ be the nonzero two-sided ideal of $A \rtimes^{\sigma} G$ generated by $u_e - u_s$. By Proposition 8 (or Proposition 9) it follows that $A \bigcap I = \{0\}$. □

**Remark 8.** Proposition 2, Corollary 5, and Propositions 7–10 together generalize Theorem 2, Corollary 11, Theorem 3, Corollaries 12–15, and Theorem 4 in [18] from groups to categories.

By combining Theorem 2 and Proposition 10, we get the following result.

**Corollary 6.** *If $A \rtimes^{\sigma} G$ is a skew groupoid ring with all $A_e$, for $e \in \mathrm{ob}(G)$, equal integral domains and each $G_e$, for $e \in \mathrm{ob}(G)$, an abelian group, then $A$ is maximal commutative in $A \rtimes^{\sigma} G$ if and only if every intersection of a nonzero two-sided ideal of $A \rtimes^{\sigma} G$ and $A$ is nonzero.*

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## Kommutatiivsus ja ideaalid kategooriaga gradueeritud ristkorrutistes

Johan Öinert ja Patrik Lundström

On defineeritud kategooriaga gradueeritud ristkorrutise mõiste, mis samaaegselt üldistab nii maatriksringide kui ka rühmaga gradueeritud ristkorrutise mõisteid. Kategooriaga gradueeritud ristkorrutis moodustab algebra üle kordajate ringi. Sellise algebra jaoks kirjeldatakse tsenter ja kordajate ringi kommutant. Samuti on uuritud seost selle algebra kahepoolsete ideaalide ja kordajate ringi ühisosa ning kordajate ringi maksimaalse kommutatiivsuse vahel.