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MATHEMATICS

# A note on Lie superalgebras

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**Abstract.** We treat the possible Lie superalgebras where in addition to Poincaré generators there are $n$ supergenerators. These superalgebras are determined with the help of relativistic wave equations. It is shown that structure constants are connected with the matrices of first-order relativistic wave equations. Some of these Lie superalgebras may be interesting from mathematical point of view.

**Key words:** Lie superalgebras, supersymmetry, relativistic wave equations.

## 1. INTRODUCTION

Let us recall the definition of Lie superalgebra $L$. Let $L$ be a complex (real) graded vector space so that $L = L_0 \oplus L_1$. The subspace $L_0$ is the even with dimension $m$ and $L_1$ is the odd with dimension $n$. Any element $a \in L$ that is either even or odd is said to be homogeneous, and degree (or parity or grading) is defined by

$$\deg a = \begin{cases} 0 & \text{if } a \in L_0, \\ 1 & \text{if } a \in L_1. \end{cases}$$

For all $a, b \in L$ there exists a generalized Lie product (or supercommutator) $[a,b\}$, $[a,b\} \in L$ with the properties:

(1) for all $a,b,c \in L$, and any complex (real) numbers $\alpha$ and $\beta$

$$[\alpha a + \beta b, c\} = \alpha[a,c\} + \beta[b,c\};$$

(2) if $a$ and $b$ are homogeneous elements of $L$ then $[a,b\}$ is also a homogeneous element of $L$, whose degree is $(\deg a + \deg b)(\mathrm{mod}\, 2)$; that is $[a,b\}$ is odd if either $a$ or $b$ is odd, $[a,b\}$ is even if $a$ and $b$ are both even or if $a$ and $b$ are both odd;

(3) for any two homogeneous elements $a$ and $b$ of $L$

$$[b,a\} = -(-1)^{(\deg a)(\deg b)}[a,b\};$$

(4) for any three homogeneous elements $a$, $b$, and $c$ of $L$, the generalized Jacobi identity holds

$$[a,[b,c\}\}(-1)^{(\deg a)(\deg c)} + [b,[c,a\}\}(-1)^{(\deg b)(\deg a)} + [c,[a,b\}\}(-1)^{(\deg c)(\deg b)} = 0.$$

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Then $L$ is said to be a complex (real) Lie superalgebra with even dimension $m$ and odd dimension $n$. It follows in particular that a Lie algebra is a Lie superalgebra which has no odd elements. A Lie superalgebra can be obtained from any associative superalgebra by taking the generalized Lie product (or supercommutator) $[a,b\}$ of the form

$$[a,b\} = ab - (-1)^{(\deg\, a)(\deg\, b)} ba. \tag{1}$$

That is if at least one of the elements $a$ and $b$ is even, then this implies that $[a,b] = ab - ba$, and if $a$ and $b$ are odd elements, then $\{a,b\} = ab + ba$. A way to get a Lie superalgebra is to start with a Lie algebra $L_0$, choose a representation of $L_0$ with a carrier space $L_1$, and set up direct sum $L_0 \oplus L_1$. See for example [1–3].

Next we treat the Poincaré superalgebra (supersymmetry algebra). Supersymmetry or Bose–Fermi symmetry transforms bosonic fields into fermionic ones and vice versa. This symmetry requires the extension of the Poincaré Lie algebra with generators of supersymmetry transformations, that is extension of the Poincaré algebra to the Poincaré superalgebra or supersymmetry algebra [4,5], see for reviews [6–8]. In addition to the Poincaré superalgebra there are the conformal superalgebras, the de Sitter superalgebras, and the anti-de Sitter superalgebras associated with space-time symmetries. These are respectively extensions of the conformal, the de Sitter, and the anti-de Sitter algebras. The even part of the Poincaré superalgebra satisfies the constraint which comes from the Coleman–Mandula no-go theorem [7,9]. Their theorem states that the only symmetry of the scattering matrix ($S$-matrix) that includes Poincaré symmetry is the product of the Poincaré symmetry and an internal symmetry group $G \otimes T$. The basic elements of the odd part $L_1$ form the carrier space of some representation of $L_0$. In particular, they must form the carrier space of some representation of the homogeneous Lorentz algebra. By Haag, Lopuszanski, and Sohnius extension of Coleman–Mandula theorem [10] this representation is equivalent to the direct sum of $N$ copies of the 4-dimensional spinor representation (Majorana spinor) $Q^A_\alpha$, for $A = 1,...,N$ and $\alpha = 1,...,4$. Of course their theorems hold only under a number of certain physical assumptions.

We can write the $N$-extended Poincaré superalgebra [10–12] for $D = 4$ dimensional Minkowski space-time as

$$\begin{aligned}
&[P^\mu, P^\nu] = 0,\\
&[M^{\mu\nu}, P^\rho] = i(\eta^{\nu\rho} P^\mu - \eta^{\mu\rho} P^\nu),\\
&[M^{\mu\nu}, M^{\rho\sigma}] = -i(\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\nu\sigma} M^{\mu\rho}),\\
&[Q^A_\alpha, P^\mu] = 0,\\
&[Q^A_\alpha, M^{\mu\nu}] = \frac{1}{2}(\sigma^{\mu\nu})_{\alpha\beta} Q^A_\beta,\\
&\{Q^A_\alpha, Q^B_\beta\} = -2(\gamma^\mu C)_{\alpha\beta} P_\mu \delta^{AB} + C_{\alpha\beta} U^{AB} + (\gamma_5 C)_{\alpha\beta} V^{AB}.\\
&[T_i, T_j] = i f^k_{ij} T_k, [T_i, M^{\mu\nu}] = [T_i, P^\mu] = 0,\\
&[T_i, Q^A_\alpha] = (\xi_i)^A_B Q^B_\alpha + (\zeta_i)^A_B (\gamma_5)^\beta_\alpha Q^B_\beta,\\
&[U^{AB}, \text{anything}] = [U^{AB}, \text{anything}] = 0.
\end{aligned} \tag{2}$$

Here $\mu, \nu, ... = 0,1,2,3$ are space-time indices, $\alpha, \beta, ... = 1,...,4$ are four-spinor indices. As we see from (2) in addition to Poincaré generators $P^\mu$, $M^{\mu\nu}$ there are more than one anticommuting Majorana spinorial charges (supersymmetry generators) $Q^A_\alpha$, (the indices $A = 1,...,N$ label the representation of the internal symmetry group to which $Q^A_\alpha$ belongs). $T_i, i = 1,...,dimT$ is the generator of the internal symmetry group. The antisymmetric operators $U^{AB} = -U^{BA}$ and $V^{AB} = -V^{BA}$ are central charges, and matrices $\xi, \zeta$ that have to satisfy $(\xi_j + i\zeta_j) = -(\xi_j + i\zeta_j)^\dagger$ $\sigma^{\mu\nu}$ are Lorentz generators for a bispinor, $\gamma^\mu$ are Dirac matrices, and $C$ is charge conjugation matrix.

The central charges can appear only for $N \geq 2$. The $N = 1$ Poincaré superalgebra is known as simple Poincaré superalgebra, there are no central charges. In the absence of central charges the Poincaré superalgebra is invariant under a group $U(N)$ of internal symmetries and in case $N=1$ the $U(1)$ invariance is known as $R$-symmetry. The Poincaré superalgebra and other superalgebras associated with space-time symmetries have great importance in physics, but there are relatively new interesting structures such as 3-algebras [13–15].

## 2. POSSIBLE POINCARÉ SUPERALGEBRAS

In this note we deal with algebras where the number of anticommuting generators is arbitrary and give the general form of the structure constants. These superalgebras are definitely interesting from mathematical point of view. We consider an algebra where the generators $P^\mu$ and $M^{\mu\nu}$of the Poincaré group and $n$ generators $S_\alpha$ satisfy the relations:

$$\begin{aligned}
&[P^\mu,P^\nu]=0,\\
&[M^{\mu\nu},P^\rho]=c_\kappa^{\mu\nu,\rho}P^\kappa,\\
&[M^{\mu\nu},M^{\rho\sigma}]=c_{\kappa\lambda}^{\mu\nu,\rho\sigma}M^{\kappa\lambda},\\
&[S_\alpha,P^\mu]=0,\\
&[S_\alpha,M^{\mu\nu}]=B_{\alpha\beta}^{\mu\nu}S^\beta,\\
&\{S_\alpha,S_\beta\}=A_{\kappa,\alpha\beta}P^\kappa,
\end{aligned} \tag{3}$$

where

$$c_{\kappa\lambda}^{\mu\nu,\rho\sigma}=\frac{1}{2}[\eta^{\mu\sigma}(\eta_\kappa^\nu\eta_\lambda^\rho-\eta_\kappa^\rho\eta_\lambda^\nu)+\eta^{\mu\rho}(\eta_\kappa^\mu\eta_\lambda^\sigma-\eta_\kappa^\sigma\eta_\lambda^\mu)$$
$$-\eta^{\mu\rho}(\eta_\kappa^\nu\eta_\lambda^\sigma-\eta_\kappa^\sigma\eta_\lambda^\nu)-\eta^{\nu\sigma}(\eta_\kappa^\mu\eta_\lambda^\rho-\eta_\kappa^\rho\eta_\lambda^\mu)], \tag{4}$$

$$c_\kappa^{\mu\nu,\rho}=\eta^{\nu\rho}\eta_\kappa^\mu-\eta^{\mu\rho}\eta_\kappa^\nu, \tag{5}$$

and $\eta^{00}=-\eta^{11}=-\eta^{22}=-\eta^{33}=1$. In this paper $\kappa,\lambda,\mu,\nu,\rho,\sigma=0,1,2,3$ and $\alpha,\beta,\gamma,\delta=1,2,...,n$, summation over the repeating indices is presumed. Proceeding from the Jacobi identities, it is easy to demonstrate that $B_{\alpha\beta}^{\mu\nu}$ are the matrix elements of the generators $B^{\mu\nu}$ of some $n$-dimensional representation (reducible or irreducible) of the Lorentz group and $A_{\mu,\alpha\beta}$ are matrix elements of matrices $A_\mu$ which are connected with the $\beta$-matrices of an invariant first-order equation

$$(p_\mu\beta^\mu-m)\psi(p)=0. \tag{6}$$

The idea is to reduce the conditions we get from the Jacobi identities to some well-known commutation relations. Starting with the Jacobi identities it is easy to be convinced that we must look through the identities with generators $S_\alpha$ and $M^{\mu\nu}$. First we start from the identities

$$[S_\alpha,[M^{\mu\nu},M^{\rho\sigma}]]+[M^{\rho\sigma},[S_\alpha,M^{\mu\nu}]]+[M^{\mu\nu},[M^{\rho\sigma},S_\alpha]]=0.$$

Using the relations (3), we obtain

$$c_{\kappa\lambda}^{\mu\nu,\rho\sigma}B_{\alpha\gamma}^{\kappa\lambda}-B_{\alpha\beta}^{\mu\nu}B_{\beta\gamma}^{\rho\sigma}+B_{\alpha\beta}^{\rho\sigma}B_{\beta\gamma}^{\mu\nu}=0.$$

That gives for matrices $B^{\mu\nu}$

$$[B^{\mu\nu},B^{\rho\sigma}]=c_{\kappa\lambda}^{\mu\nu,\rho\sigma}B^{\kappa\lambda}. \tag{7}$$

The last relations are the well-known commutation relations of the Lorentz group generators. Therefore the matrices $B^{\mu\nu}$ are the generators of some arbitrary $n$-dimensional representation of the Lorentz group, and the structure constants $B^{\mu\nu}_{\alpha\beta}$ in (3) are its matrix elements. The number of the anticommuting generators is equal of course to the dimension of the $B^{\mu\nu}$ representation. The second Jacobi identity

$$-\{S_\beta,[M^{\mu\nu},S_\alpha]\}+\{S_\alpha,[S_\beta,M^{\mu\nu}]\}+[M^{\mu\nu},\{S_\alpha,S_\beta\}]=0$$

gives the relations

$$B^{\mu\nu}_{\alpha\gamma}A_{\kappa,\beta\gamma}+B^{\mu\nu}_{\beta\gamma}A_{\kappa,\alpha\gamma}+c^{\mu\nu,\lambda}_{\kappa}A_{\lambda,\alpha\beta}=0.$$

For matrices $B^{\mu\nu}$ and $A_\mu$

$$B^{\mu\nu}A_\kappa+A_\kappa\overline{B}^{\mu\nu}=-c^{\mu\nu,\lambda}_{\kappa}A_\lambda. \tag{8}$$

Here $\overline{B}^{\mu\nu}$ is $B^{\mu\nu}$ transposed and we have used the fact that $\overline{A}_\mu=A_\mu$. Now we suppose that there exists a matrix $C$ which satisfies

$$C\overline{B}^{\mu\nu}C^{-1}=-B^{\mu\nu}. \tag{9}$$

Then we can write (8) as

$$[B^{\mu\nu},A_\kappa C^{-1}]=-c^{\mu\nu,\lambda}_{\kappa}A_\lambda C^{-1}.$$

Denoting

$$\beta_\kappa=A_\kappa C^{-1} \tag{10}$$

we obtain

$$[B^{\mu\nu},\beta_\kappa]=-c^{\mu\nu,\lambda}_{\kappa}\beta_\lambda. \tag{11}$$

The last relations are the invariance conditions for the first-order wave equation (6) corresponding to the $B^{\mu\nu}$ representation. Therefore the matrices $A_\kappa$ are connected with the $\beta$-matrices of an invariant equation. From the relation (10)

$$A_\kappa=\beta_\kappa C \tag{12}$$

and therefore

$$\{S_\alpha,S_\beta\}=(\beta_\kappa C)_{\alpha\beta}P^\kappa.$$

As regards matrix $C$ which satisfies (9), it should be mentioned that such a matrix always exists because it exists in the case of the spinor representations $(1/2,0)$ and $(0,1/2)$. To show this, we write the generators $B^{\mu\nu}$ in the form $B^{0k}=\mp\frac{1}{2}\sigma^k$, $B^{kl}=\frac{i}{2}\varepsilon^{kl}_{m}\sigma^m$ $(k,l,m=1,2,3;\ \varepsilon^{123}=1)$, where the minus sign corresponds to the representation $(1/2,0)$, and the plus sign to $(0,1/2)$ and $\sigma^1,\sigma^2,\sigma^3$ are the Pauli matrices. Now the matrix $C$ must satisfy $C\overline{\sigma^k}C^{-1}=-\sigma^k$. In the case of the usual representation of the Pauli matrices one can take $C=\pm\sigma^2$. As other representations are obtainable from direct products of spinor representations, Matrix $C$ also exists. In the case of Dirac bispinor $(1/2,0)\oplus(0,1/2)$ $C$ is charge conjugation operator. The matrices $\beta_\kappa$ and $C$ must be chosen to satisfy $\overline{\beta_\kappa C}=\beta_\kappa C$.

## 3. SOME CONCLUSIONS

In the previous section it is shown that there exist general Poincaré superalgebras (3) where $B^{\mu\nu}$ are generators of some $n$-dimensional representation of the homogeneous Lorentz group and $A_\mu$ are determined by the $\beta$-matrices of the corresponding first-order wave equation (7). Let us consider some examples.

1. There are Lie superalgebras with simply anticommuting $S_\alpha$ generators independently of the choice of generators, since one can always take $A_\mu=0$.
2. If $B^{\mu\nu}$ are the generators of some irreducible representation of the Lorentz group, we have also $A_\mu=0$, since there are no first-order equations. In the case of first-order equations (6), the $\beta$-representation is always reducible and composed of "linked" irreducible representations.
3. The most important supersymmetry algebra (2) in case $N=1$ corresponds to the choice $B^{\mu\nu}=\frac{1}{2}\sigma^{\mu\nu}$, where $B^{\mu\nu}$ are the Lorentz generators for a Dirac bispinor representation and $A_\mu$ are determined by

the well-known Dirac equation $(p_\mu \gamma^\mu - m)\psi = 0$ for a bispinor $A_\mu = \gamma_\mu C$. The $N$-extended Poincaré superalgebras, which are the basics of modern supersymmetry models in quantum field theory, are built similarly.

4. Usually it is assumed that the generators $S_\alpha$ are fermionic, especially in physical applications, since then we have Fermi–Bose symmetry. However, algebra (3) allows also bosonic generators $S_\alpha$. For example, if we take five generators $S_\alpha$, where four are components of some four-vector $S_\mu$ and the fifth is scalar $S$, the matrices $A_\mu$ are connected with the $s = 0$ Kemmer–Duffin matrices [16]. Now there is no Bose–Fermi symmetry, since Bose and Fermi fields are not mixed in the same multiplet.
5. The next physically interesting Poincaré superalgebra is obtained if we take instead of Dirac bispinor supergenerators $S_\alpha$ 16 vector-bispinor generators $S^\mu_\alpha$. In that case $A_\mu$ are connected with the $\beta$-matrices of the Rarita–Schwinger equation. The Rarita–Schwinger equation is used to describe spin $3/2$, but depending on the choice of free parameters present in the equation it may describe single spin $3/2$, spin $3/2$ and one spin $1/2$, spin $3/2$, and two spins $1/2$. The general form of the corresponding $\beta$-matrices is the following [17]

$$(\beta^\mu)^\rho_\sigma = \gamma^\mu \eta^\rho_\sigma + \left(\frac{a}{\sqrt{3}} - \frac{1}{2}\right) \eta^{\mu\rho} \gamma_\sigma + \left(\frac{b}{\sqrt{3}} - \frac{1}{2}\right) \gamma^\rho \eta^\mu_\sigma + \left(\frac{a+b}{4\sqrt{3}} + \frac{c}{4} + \frac{3}{8}\right) \gamma^\rho \gamma^\mu \gamma_\sigma,$$

where $a, b$, and $c$ are some real free parameters. The choice of parameters determines the mass and spin content of a given equation. There exist the following choices of parameters. In the single spin $3/2$ case $ab = -1/4$, $c = -1/2$, if $ab = c/2$ (except $c = -1/2$) in addition to spin $3/2$ there is also one spin $1/2$ state present. If $4ab \geq -(c - 1/2)^2$ we have one spin $3/2$ and two spin $1/2$ states. In other cases the equation turns out to be unphysical, since the spin $1/2$ states do not have real masses. For that reason Lie superalgebras with vector-bisponor generators may offer numerous possibilities. We hope that they may be interesting also in physical applications.

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## Lie superalgebratest

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On vaadeldud võimalikke Lie superalgebraid, milles lisaks Poincaré generaatoritele on $n$ supergeneraatorit. Sellised superalgebrad on määratud relativistlike lainevõrrandite kaudu. On näidatud, et struktuurikonstandid on seotud esimest järku lainevõrrandite maatriksitega. Mõned sellistest Lie superalgebratest võivad pakkuda matemaatilist huvi.