



Minimal Hilbert series for quadratic algebras and the Anick conjecture

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Abstract. We study the question on whether the famous Golod–Shafarevich estimate, which gives a lower bound for the Hilbert series of a (noncommutative) algebra, is attained. This question was considered by Anick in his 1983 paper ‘Generic algebras and CW-complexes’, Princeton Univ. Press, where he proved that the estimate is attained for the number of quadratic relations $d \leq \frac{n^2}{4}$ and $d \geq \frac{n^2}{2}$, and conjectured that it is the case for any number of quadratic relations. The particular point where the number of relations is equal to $\frac{n(n-1)}{2}$ was addressed by Vershik. He conjectured that a generic algebra with this number of relations is finite dimensional.

We announce here the result that over any infinite field, the Anick conjecture holds for $d \geq \frac{4(n^2+n)}{9}$ and an arbitrary number of generators. We also discuss the result that confirms the Vershik conjecture over any field of characteristic 0, and a series of related asymptotic results.

Key words: quadratic algebras, Golod–Shafarevich theorem, Anick conjecture, Vershik conjecture.

Let $\mathcal{F}(n, \mathbb{K}) = \mathbb{K}\langle x_1, \dots, x_n \rangle$ be a free associative algebra with n generators x_1, \dots, x_n over a field \mathbb{K} . Recall that the free algebra carries the natural degree grading

$$\mathcal{F}(n, \mathbb{K}) = \bigoplus_{k=0}^{\infty} \mathcal{F}_k(n, \mathbb{K}),$$

where

$$\mathcal{F}_k(n, \mathbb{K}) = \text{span}_{\mathbb{K}}\{x_{j_1} \dots x_{j_k} : 1 \leq j_1, \dots, j_k \leq n\}.$$

To define this grading we suppose that generators $x_i, i = 1, \dots, n$ all have degree one. We deal with *quadratic algebras generated in degree one*, that is, algebras given by homogeneous relations of degree 2:

$$R = \mathbb{K}\langle x_1, \dots, x_n \rangle / I, \tag{1}$$

where $I = \text{Id}\{f_1, \dots, f_d\}$ is the ideal generated by elements

$$f_1, \dots, f_d \in \mathcal{F}_2(n, \mathbb{K}) : f_j = \sum_{k,m=1}^n c_{j,k,m} x_k x_m, \quad c_{j,k,m} \in \mathbb{K}. \tag{2}$$

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Since relations (2) are homogeneous, an algebra R inherits grading from the free algebra $\mathcal{F}(n, \mathbb{K})$:

$$R = \bigoplus_{k=0}^{\infty} R_k,$$

where

$$I = \bigoplus_{k=0}^{\infty} I_k, \text{ for } I_k = I \cap \mathcal{F}_k(n, \mathbb{K}) \text{ and } R_k = \mathcal{F}_k(n, \mathbb{K})/I_k.$$

Recall also that the *Hilbert series* of R is a polynomial generating function associated to the sequence of dimensions of graded components $a_q = \dim_{\mathbb{K}} R_q$:

$$H_R(t) = \sum_{q=0}^{\infty} (\dim_{\mathbb{K}} R_q) t^q. \tag{3}$$

It belongs to the ring $\mathbb{Z}[[t]]$ of formal power series on one variable and we consider the following ordering on it. For two power series $a(t) = \sum_{j=0}^{\infty} a_j t^j$ and $b(t) = \sum_{j=0}^{\infty} b_j t^j$ (with real coefficients) we write $a(t) \geq b(t)$ if $a_j \geq b_j$ for any $j \in \mathbb{Z}_+$. For such a power series $a(t) = \sum_{j=0}^{\infty} a_j t^j$, we denote by $|a(t)|$ a series obtained from $a(t)$ by replacing all coefficients starting from the first negative one by zeros.

The famous result due to Golod and Shafarevich [4] gives a lower bound for the Hilbert series of algebras with n generators and d quadratic relations. (Further in the text, whenever we are talking about the number of relations, we mean the number of linearly independent relations.)

Theorem GS. *Let \mathbb{K} be a field, $n \in \mathbb{N}$, $0 \leq d \leq n^2$ and R be a quadratic \mathbb{K} -algebra with n generators and d relations. Then $H_R(t) \geq |(1 - nt + dt^2)^{-1}|$.*

Let us note that although we formulated above the Golod–Shafarevich estimate only for quadratic algebras, it is known for algebras with any finite or infinite number of relations. Namely, it is as follows:

$$\left| \left(1 - nt + \sum_{i=2}^{\infty} r_i t^i \right)^{-1} \right| \leq H_A,$$

where r_i stands for the number of relations of degree i .

This estimate allowed constructing a counterexample for the Kurosh problem on the nilpotency of nil algebra and to the General Burnside problem on the existence of a finitely generated infinite group with all elements being torsion. It is also recognized due to other impacts to p -groups and class field theory [4,5].

It will be convenient for our purposes to state the Golod–Shafarevich theorem also in terms of the numbers

$$h_q(\mathbb{K}, n, d) = \min_{R \in \mathcal{R}_{n,d}} \dim R_q, \tag{4}$$

where $\mathcal{R}_{n,d}$ is the set of all quadratic \mathbb{K} -algebras R with n generators and d relations. For $n \in \mathbb{N}$ and $0 \leq d \leq n^2$, consider the series

$$H_{\mathbb{K},n,d}^{\min}(t) = \sum_{q=0}^{\infty} h_q(\mathbb{K}, n, d) t^q. \tag{5}$$

Then Theorem GS admits the following equivalent form.

Theorem GS'. *Let \mathbb{K} be a field, $n \in \mathbb{N}$ and $0 \leq d \leq n^2$. Then $H_{\mathbb{K},n,d}^{\min}(t) \geq |(1 - nt + dt^2)^{-1}|$.*

Note that a priori it is not clear why the algebra with the series $H_{\mathbb{K},n,d}^{\min}$ should exist in the class $R_{n,d}$. In fact, it is not difficult to show that not only it does exist, but it is in ‘general position’ in one or another sense. Usually we mean by ‘generic quadratic algebra’ generic in the sense of Zariski topology. Namely, we consider an algebra from $R_{n,d}$ as a point in \mathbb{K}^{n^2d} , defined by a tuple of all coefficients of its defining relations. Then we say that a generic quadratic \mathbb{K} -algebra with n relations and d generators has a property \mathcal{P} , if the set $\{c_{j,k,m}\} \in \mathbb{K}^{n^2d}$ of coefficient vectors, for which the corresponding algebra R defined in (1) has property \mathcal{P} , contains a dense Zariski open subset of \mathbb{K}^{n^2d} . The following proposition is a well-known fact, see, for instance [2,3,6].

Proposition 1. *Let \mathbb{K} be an infinite field and $n \in \mathbb{N}$, $0 \leq d \leq n^2$. Then $\dim_{\mathbb{K}} R_q = h_q(\mathbb{K}, n, d)$ for a generic quadratic \mathbb{K} -algebra R with n generators and d relations. In particular, if $H_{\mathbb{K},n,d}^{\min}(t)$ is a polynomial, then $H_R(t) = H_{\mathbb{K},n,d}^{\min}(t)$ for a generic quadratic \mathbb{K} -algebra R with n generators and d relations.*

In case $H_{\mathbb{K},n,d}^{\min}$ is not a polynomial there are more subtleties involved in the question whether a generic algebra has this series. There are arguments (see [7]) showing that this is the case when \mathbb{K} is an uncountable algebraically closed field. In [3] we suggest the use of a slightly modified notion of a ‘generic’ algebra, in the sense of the Lebesgue measure, over \mathbb{R} . Namely, we say that the generic in the Lebesgue sense algebra from $R_{n,d}$ has the property \mathcal{P} if the set of algebras not having \mathcal{P} has Lebesgue measure zero. We show that in this (weaker) sense a generic algebra has the series H^{\min} even if it is infinite.

In his 1983 paper ‘Generic algebras and CW-complexes’ [1], Anick studied the behaviour of Hilbert series of algebras given by relations and formulated the following conjecture.

Conjecture A. *For any infinite field \mathbb{K} , any $n, q \in \mathbb{N}$ and $0 \leq d \leq n^2$, a generic quadratic \mathbb{K} -algebra R with n generators and d relations $\dim R_q$ equals the q th coefficient of the series $|(1 - nt + dt^2)^{-1}|$. Equivalently, $H_{\mathbb{K},n,d}^{\min}(t) = |(1 - nt + dt^2)^{-1}|$.*

In other words, Conjecture A states that the lower estimate of the Hilbert series by Golod and Shafarevich is attained and a generic algebra has the minimal Hilbert series.

Calculation of terms $h_q(\mathbb{K}, n, d)$ for $q = 0, 1, 2$ is trivial for an arbitrary algebra given by n generators and d relations: $h_0(\mathbb{K}, n, d) = 1$, $h_1(\mathbb{K}, n, d) = n$ and $h_2(\mathbb{K}, n, d) = n^2 - d$. Anick proved [1,6] that his conjecture holds also for $q = 3$.

Theorem A. *Let \mathbb{K} be any field, $n \in \mathbb{N}$ and $0 \leq d \leq n^2$. Then*

$$h_3(\mathbb{K}, n, d) = \begin{cases} 0 & \text{if } d \geq \frac{n^2}{2}; \\ n^3 - 2nd & \text{if } d < \frac{n^2}{2}. \end{cases} \tag{6}$$

Since the number in the right-hand side of (6) happens to coincide with the third coefficient in $|(1 - nt + dt^2)^{-1}|$, near t^3 , Theorem A proves Conjecture A in the case $q = 3$ and in the case $d \geq \frac{n^2}{2}$. Conjecture A is also known to be true if $d \leq \frac{n^2}{4}$ [6,7]. The region $\frac{n^2}{4} < d < \frac{n^2}{2}$ remained a white zone so far. Let us note that for $d > \frac{n^2}{4}$, the series $|(1 - nt + dt^2)^{-1}|$ is a polynomial. Thus Conjecture A, if true, implies that a generic quadratic \mathbb{K} -algebra over infinite field \mathbb{K} , with n generators and $d > \frac{n^2}{4}$ relations is finite dimensional.

Vershik [8] formulated a conjecture that addresses a specific point of the ‘difficult interval’ $\frac{n^2}{4} < d < \frac{n^2}{2}$, $d = \frac{n(n-1)}{2}$, which is the number of relations defining the algebra of commutative polynomials or any PBW algebra.

Conjecture V. *Let $n \in \mathbb{N}$, $n \geq 3$. Then a generic quadratic \mathbb{C} -algebra with n generators and $\frac{n(n-1)}{2}$ relations is finite dimensional.*

As it is mentioned in [6], there was an attempt to prove this conjecture in [9], but the argument there was incorrect.

Our goal is to move the frame of the interval $\left(\frac{n^2}{4}, \frac{n^2}{2}\right)$ which remains unknown since Anick’s 1983 paper. Namely, we announce here the following result.

Theorem 2. *For any infinite field, the Golod–Shafarevich estimate is attained for a generic quadratic algebra with n generators and $d \geq \frac{4(n^2+n)}{9}$ quadratic relations.*

Namely, the Hilbert series of the generic algebra is:

$$H(t) = |(1 - nt + dt^2)^{-1}| = 1 + nt + (n^2 - d)t^2 + (n^3 - 2nd)t^3.$$

The point $d = \frac{n(n-1)}{2}$ falls into the interval from Proposition 1, for big enough n , so we automatically get as a consequence an affirmative answer to Vershik’s question for $n \geq 17$. After some additional considerations, we get an affirmative answer to Vershik’s question for each $n \geq 3$ over a field of characteristic 0:

Theorem 3. *Let \mathbb{K} be a field of characteristic 0 and $n \in \mathbb{N}$, $n \geq 3$. Then a generic quadratic \mathbb{K} -algebra R with n generators and $\frac{n(n-1)}{2}$ relations has the Hilbert series and the dimension is given by the following formula*

$$H_R(t) = \begin{cases} 1 + nt + \frac{n(n+1)}{2}t^2 + n^2t^3 & \text{if } n \geq 5; \\ 1 + 4t + 10t^2 + 16t^3 + t^4 & \text{if } n = 4; \\ 1 + 3t + 6t^2 + 9t^3 + 9t^4 & \text{if } n = 3, \end{cases} \quad \dim_{\mathbb{K}} R = \begin{cases} \frac{3n(n+1)+2}{2} & \text{if } n \geq 5; \\ 32 & \text{if } n = 4; \\ 28 & \text{if } n = 3. \end{cases} \quad (7)$$

To illustrate some more explicit results, we present here (without proof) some of their asymptotic versions.

For a field \mathbb{K} , $n, q \in \mathbb{N}$ with $q \geq 2$, we denote

$$d(\mathbb{K}, n, q) = \min\{d \in \mathbb{N} : h_q(\mathbb{K}, n, d) = 0\}. \quad (8)$$

That is, $d(\mathbb{K}, n, q)$ is the minimal d for which there is a quadratic \mathbb{K} -algebra R with n generators and d relations satisfying $R_q = \{0\}$. Obviously, $d(\mathbb{K}, n, 2) = n^2$. Similarly

$$d(\mathbb{K}, n, \infty) = \min\left\{d \in \mathbb{N} : \min_{q \in \mathbb{N}} h_q(\mathbb{K}, n, d) = 0\right\}. \quad (9)$$

That is, $d(\mathbb{K}, n, \infty)$ is the minimal d for which there is a finite dimensional quadratic \mathbb{K} -algebra R with n generators and d relations. In order to formulate our asymptotic results we need the following lemma.

Lemma 4. *Let \mathbb{K} be a field and $q \in \mathbb{N}$, $q \geq 2$ or $q = \infty$. Then the limit $\lim_{n \rightarrow \infty} \frac{d(\mathbb{K}, n, q)}{n^2} = \alpha(\mathbb{K}, q)$ does exist and*

$$\alpha(\mathbb{K}, q) = \lim_{n \rightarrow \infty} \frac{d(\mathbb{K}, n, q)}{n^2} = \inf\left\{\frac{d(\mathbb{K}, n, q)}{n^2} : n \in \mathbb{N}\right\}. \quad (10)$$

Moreover, $\{\alpha(\mathbb{K}, q)\}_{q \geq 3}$ is decreasing, $\alpha(\mathbb{K}, \infty) = \lim_{q \rightarrow \infty} \alpha(\mathbb{K}, q) \geq \frac{1}{4}$ and $\alpha(\mathbb{K}, 3) = \frac{1}{2}$.

Theorem 5. *The equalities $\alpha(\mathbb{K}, 4) = \frac{3-\sqrt{5}}{2}$ and $\alpha(\mathbb{K}, 5) = \frac{1}{3}$ hold for any infinite field. Moreover, $\frac{1}{4} \leq \alpha(\mathbb{K}, \infty) \leq \alpha(\mathbb{K}, 6) \leq \frac{5}{16}$ for any field \mathbb{K} of characteristic 0.*

Corollary 6. *Let \mathbb{K} be an infinite field and $\lim_{n \rightarrow \infty} \frac{d_n}{n^2} > \frac{3-\sqrt{5}}{2}$ with $n, d_n \in \mathbb{N}$ and $d_n \leq n^2$. Then for any sufficiently large n , a generic quadratic \mathbb{K} -algebra with n generators and d_n relations has Hilbert series $1 + nt + (n^2 - d_n)t^2 + \max\{0, (n^3 - 2d_n n^2)\}t^3 = |(1 - nt + d_n t^2)^{-1}|$.*

Corollary 7. Let \mathbb{K} be an infinite field and $\liminf_{n \rightarrow \infty} \frac{d_n}{n^2} > \frac{1}{3}$ with $n, d_n \in \mathbb{N}$ and $d_n \leq n^2$. Then for any sufficiently large n , the Hilbert series of a generic quadratic \mathbb{K} -algebra with n generators and d_n relations is a polynomial of degree at most 4.

Corollary 8. Let \mathbb{K} be a field of characteristic 0 and $\liminf_{n \rightarrow \infty} \frac{d_n}{n^2} > \frac{5}{16}$ with $n, d_n \in \mathbb{N}$ and $d_n \leq n^2$. Then for any sufficiently large n , a generic quadratic \mathbb{K} -algebra with n generators and d_n relations has Hilbert series being a polynomial of degree at most 5 and therefore is finite dimensional.

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Ruutalgebrate minimaalsed Hilberti read ja Anicki hüpotees

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On uuritud probleemi, kas kuulub Golodi-Shafarevichi hinnang, mis annab (mittekommutatiivse) algebra Hilberti rea alumise raja, kehtib. Seda probleemi on 1983. aastal uurinud Anick oma artiklis “Generic algebras and CW-complexes” (Princeton Univ. Press), milles ta tõestas, et see hinnang saavutatakse ruutseoste arvuga $d \leq \frac{n^2}{4}$ ja $d \geq \frac{n^2}{2}$ ning et sama juhtub iga ruutseoste arvuga. Erijuhtu, kui seoste arv on $\frac{n(n-1)}{2}$, käsitles Veršik, kes oletas, et üldiselt on algebra selliste seoste arvuga lõplikumõõtmeline. On esitatud tulemus, mille kohaselt Anicki hüpotees on tõene $d \geq \frac{4(n^2+n)}{9}$ jaoks ja suvalise arvu generaatorite korral. Samuti on käsitletud tulemust, mis kinnitab Veršiki hüpoteesi üle suvalise korpuse karakteristikuga 0 ja vastavat asümptootilist rida.