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MATHEMATICS

On the K-theory of the C^*-algebra associated with a one-sided shift space

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Abstract. One-sided shift spaces are a special kind of non-invertible topological dynamical system with which one can associate a C^*-algebra. We show how to construct the C^*-algebra associated with a one-sided shift space as the Cuntz–Pimsner C^*-algebra of a C^*-correspondence and use this to compute its K-theory.

Key words: operator algebra, symbolic dynamics, C^*-algebras, shift spaces, C^*-correspondences, Cuntz–Pimsner C^*-algebras, K-theory of C^*-algebras.

1. INTRODUCTION

A *one-sided shift space* (also called a one-sided subshift) is a closed subset X of $\mathfrak{a}^{\mathbb{N}}$ (here, and in the rest of the paper, $\mathbb{N}$ denotes the set of non-negative integers), where $\mathfrak{a}$ is a finite set equipped with the discrete topology and $\mathfrak{a}^{\mathbb{N}}$ is equipped with the product topology, such that $\sigma(\mathsf{X}) \subseteq \mathsf{X}$, where σ is the map from $\mathfrak{a}^{\mathbb{N}}$ to itself defined by

$$(\sigma(x_n)_{n\in\mathbb{N}})_k = x_{k+1}$$

for $(x_n)_{n\in\mathbb{N}} \in \mathfrak{a}^{\mathbb{N}}$ and $k \in \mathbb{N}$ (we refer the interested reader to [7] and [8, Section 13.8] for more details). We say that $\mathfrak{a}$ is the *alphabet of* X, and that X is a one-sided shift space over $\mathfrak{a}$. If we in the above instead of $\mathbb{N}$ use $\mathbb{Z}$ (the set of integers), we get what is called a *two-sided shift space* (also called a two-sided subshift, cf. [7] and [8]). Every two-sided shift space Λ natural gives rise to a one-sided shift space

$$\{(x_n)_{n\in\mathbb{N}} \mid (x_n)_{n\in\mathbb{Z}} \in \Lambda\}$$

which we denote by X_Λ. A one-sided shift space X is of this form if and only if $\sigma(\mathsf{X}) = \mathsf{X}$.

In [9] Matsumoto associated with every two-sided shift Λ space a C^*-algebra $\mathscr{O}_\Lambda$. Later an alternative definition of $\mathscr{O}_\Lambda$ occurred in [11,13,3]. Heavily inspired by these constructions, the first named author associated in [1] with every one-sided shift space X a C^*-algebra (see [4] for a discussion of the relationship between this C^*-algebra and the above-mentioned C^*-algebras constructed in [9] and in [11,13,3]). This C^*-algebra was further studied by the authors in [4], where it is denoted by $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ and where it is shown that this algebra can be constructed as one of Exel's crossed products by an endomorphism [5].

We will now for the benefit of the reader give a brief description of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ which occurred in [4]. Let X be a one-sided shift space over the alphabet $\mathfrak{a}$ (i.e., $\mathsf{X} \subseteq \mathfrak{a}^{\mathbb{N}}$) and let $\mathfrak{a}^*$ be the set of finite words in $\mathfrak{a}$.

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We call the number of elements (*letters*) in $u \in \mathfrak{a}$ for the length of u and denote it by $|u|$. For $u, v \in \mathfrak{a}^*$ we denote by $C(u,v)$ the subset $\{vx \in \mathsf{X} \mid ux \in \mathsf{X}\}$ of X consisting of all those elements in X which begin with v and for which we get the sequence by replacing the leading v by u also belonging to X. We then let $\mathscr{D}_\mathsf{X}$ be the C^*-algebra of $l^\infty(\mathsf{X})$ (the C^*-algebra of bounded functions on X) generated by $\{1_{C(u,v)} \mid u, v \in \mathfrak{a}^*\}$, where $1_{C(u,v)}$ denote the characteristic function of $C(u,v)$. According to [4, Theorem 10], $\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ is then the universal C^*-algebra generated by a family of partial isometries $(s_u)_{u\in\mathfrak{a}^*}$ satisfying:

1. $s_u s_v = s_{uv}$ for all $u, v \in \mathfrak{a}^*$,
2. the map $1_{C(u,v)} \mapsto s_v s_u^* s_u s_v^*$, $u, v \in \mathfrak{a}^*$ extends to a $*$-homomorphism from $\mathscr{D}_\mathsf{X}$ to the C^*-algebra generated by $\{s_u \mid u \in \mathfrak{a}^*\}$.

It follows from this universal property that there exists an action γ of $\mathbb{T}$ (the unit circle in the complex plan) on $\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ characterized by $\gamma_z(s_u) = z^{|u|} s_u$ for $z \in \mathbb{T}$ and $u \in \mathfrak{a}^*$. This action is called the *gauge action*.

Among the properties of $\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ studied in [4] is its K-theory. In [4, Theorem 26] a description of $K_0(\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$ and $K_1(\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$ was announced. However, no proof of the result was provided, and consequently the formulation, while being right in spirit, was not correct with respect to the definition of the maps B^l and B used in [4, Theorem 26]. The correct definition of the maps B^l and B and the theorem are as follows.

For $l \in \mathbb{N}$ let $\mathfrak{a}_l^*$ denote the words in $\mathfrak{a}$ of length at most l. Following Matsumoto (cf. [12]), for every $l \in \mathbb{N}$ and every $x \in \mathsf{X}$, we define $\mathscr{P}_l(x)$ by

$$\mathscr{P}_l(x) = \{u \in \mathfrak{a}_l^* \mid ux \in \mathsf{X}\}.$$

We then define an equivalence relation $\sim_l$ on X, called *l-past equivalence*, by

$$x \sim_l y \iff \mathscr{P}_l(x) = \mathscr{P}_l(y).$$

For each $l \in \mathbb{N}$, we let $m(l)$ be the number of l-past equivalence classes (which is finite because $\mathfrak{a}_l^*$ is finite), and we denote the l-past equivalence classes by $\mathscr{E}_1^l, \mathscr{E}_2^l, \ldots, \mathscr{E}_{m(l)}^l$. One can show (cf. [4, p. 291]) that $1_{\mathscr{E}_i^l} \in \mathscr{D}_\mathsf{X}$ for $l \in \mathbb{N}$ and $i \in \{1, 2, \ldots, m(l)\}$. For each $l \in \mathbb{N}$, $j \in \{1, 2, \ldots, m(l)\}$, and $i \in \{1, 2, \ldots, m(l+1)\}$, let

$$I_l(i,j) = \begin{cases} 1 & \text{if } \mathscr{E}_i^{l+1} \subseteq \mathscr{E}_j^l \\ 0 & \text{else.} \end{cases}$$

For $l \in \mathbb{N}$ we denote by $e_1, e_2, \ldots, e_{m(l)}$ the canonical generators of the group $\mathbb{Z}^{m(l)}$. There is then a unique group homomorphism $I_0^l : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+1)}$ which for each $j \in \{1, 2 \ldots, m(l)\}$ maps e_j to $\sum_{i=1}^{m(l+1)} I_l(i,j) e_i$. We denote by $\mathbb{Z}_{\mathsf{X}_0}$ the inductive limit $\varinjlim(\mathbb{Z}^{m(l)}, I_0^l)$.

For a subset $\mathscr{E}$ of X and a $u \in \mathfrak{a}^*$, let $u\mathscr{E} = \{ux \in \mathsf{X} \mid x \in \mathscr{E}\}$. For each $l \in \mathbb{N}$, $j \in \{1, 2, \ldots, m(l)\}$, $i \in \{1, 2, \ldots, m(l+1)\}$ and $a \in \mathfrak{a}$, let

$$A_l(i,j,a) = \begin{cases} 1 & \text{if } \emptyset \neq a\mathscr{E}_i^{l+1} \subseteq \mathscr{E}_j^l \\ 0 & \text{else.} \end{cases}$$

For every $l \in \mathbb{N}$ denote by B^l the linear map from $\mathbb{Z}^{m(l)}$ to $\mathbb{Z}^{m(l+1)}$ given by

$$e_j \mapsto \sum_{i=1}^{m(l+1)} \left(I_l(i,j) - \sum_{a\in\mathfrak{a}} A_l(i,j,a) \right) e_i.$$

One can easily check that the following diagram commutes for every $l \in \mathbb{N}$:

$$\begin{array}{ccc} \mathbb{Z}^{m(l)} & \xrightarrow{B^l} & \mathbb{Z}^{m(l+1)} \\ \downarrow{\scriptstyle I_0^l} & & \downarrow{\scriptstyle I_0^{l+1}} \\ \mathbb{Z}^{m(l+1)} & \xrightarrow{B^{l+1}} & \mathbb{Z}^{m(l+2)}. \end{array}$$

Hence the family $\{B^l\}_{l\in\mathbb{N}}$ induces a linear map B from $\mathbb{Z}_{\mathsf{X}_0}$ to $\mathbb{Z}_{\mathsf{X}_0}$. For $l \in \mathbb{N}$ let ι_l denote the map from $\mathbb{Z}^{m(l)}$ to $\mathbb{Z}_{\mathsf{X}_0}$ given by the universal property of the inductive limit $\varinjlim(\mathbb{Z}^{m(l)}, I_0^l) = \mathbb{Z}_{\mathsf{X}_0}$.

Theorem 1. *Let* X *be a one-sided shift space. Then*

$$K_0(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}) \cong \mathbb{Z}_{\mathsf{X}_0}/B\mathbb{Z}_{\mathsf{X}_0},$$

and

$$K_1(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}) \cong \ker(B).$$

More precisely, the map

$$[1_{\mathscr{E}_i^l}]_0 \mapsto \iota_l(e_i)$$

induces an isomorphism from $K_0(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$ *to* $\mathbb{Z}_{\mathsf{X}_0}/B\mathbb{Z}_{\mathsf{X}_0}$.

It should be noticed that when the shift map σ of X is surjective (i.e., when $\mathsf{X} = \mathsf{X}_\Lambda$ for some two-sided shift space Λ), then $\mathbb{Z}_{\mathsf{X}_1} = \mathbb{Z}_{\mathsf{X}_0}$ (where $\mathbb{Z}_{\mathsf{X}_1}$ is as defined in [4, p. 303]), and thus [4, Theorem 26] is correct in this case. It is, however, not difficult to construct examples of one-sided shift spaces X such that [4, Theorem 26] is not correct for these one-sided shift spaces. One should also notice that if the one-sided shift space under consideration is of the form X_Λ for some two-sided shift space Λ, the description of the K-groups of the C^*-algebra $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ can, as mentioned in [4, p. 301], be obtained from [10] and [13].

The purpose of this paper is to give a complete proof of Theorem 1. There are several ways to prove this theorem. We will do it by constructing, for every one-sided shift space X, the C^*-algebra $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ as the Cuntz–Pimsner algebra of a C^*-correspondence and then apply Theorem 8.6 of [6] (which we, for the benefit of the reader, have restated as Theorem 6 in Section 3). In this connection, it is worth mentioning that the construction of $\mathscr{D} \times_{\alpha,\mathscr{L}} \mathbb{N}$ as a Cuntz–Pimsner algebra given in this paper is related to the construction given in [14, Section 6]. But in [14] the λ-graphs systems that the author considers are essential (otherwise the operator v defined in [14, (6.1) on p. 19] would not be an isometry), so that the construction only works when the one-sided shift space under consideration is of the form X_Λ for some two-sided shift space Λ, and not for all one-sided shift spaces as the construction given in this paper does.

Section 2 of this paper contains the above-mentioned construction of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ as a Cuntz–Pimsner algebra and Section 3 contains the proof of Theorem 1.

2. THE CONSTRUCTION OF $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ AS A CUNTZ–PIMSNER ALGEBRA

In this section we will construct a C^*-correspondence H_{X} for an arbitrary one-sided shift space X, such that the Cuntz–Pimsner algebra $\mathscr{O}_{\mathsf{H}_{\mathsf{X}}}$ of H_{X} is canonical isomorphic to $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$. In [1], $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ (which in that paper is denoted by $\mathscr{O}_{\mathsf{X}}$) is constructed as the Cuntz–Pimsner algebra of a C^*-correspondence. However, for our purpose, it will be more useful to construct $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ as the Cuntz–Pimsner algebra of another C^*-correspondence. It should be noted that the construction which we will describe here has previously appeared in the first named author's Master's thesis.

In our discussion of Cuntz–Pimsner algebras and C^*-correspondence, we will use the notation and terminology of [6]. We will, for the benefit of the reader, here briefly recall the definition of a C^*-correspondence and its corresponding Cuntz–Pimsner C^*-algebra. Let $\mathscr{A}$ be a C^*-algebra. A right *Hilbert* $\mathscr{A}$*-module* H is a Banach space with a right action of the C^*-algebra $\mathscr{A}$ and an $\mathscr{A}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathsf{H}}$ satisfying
1. $\langle\xi,\eta a\rangle_{\mathsf{H}} = \langle\xi,\eta\rangle_{\mathsf{H}} a$,
2. $\langle\xi,\eta\rangle_{\mathsf{H}} = \langle\eta,\xi\rangle_{\mathsf{H}}^*$,
3. $\langle\xi,\xi\rangle_{\mathsf{H}} \geq 0$ and $\|\xi\|_{\mathsf{H}} = \|\langle\xi,\xi\rangle_{\mathsf{H}}\|_{\mathscr{A}}^{1/2}$,
for $\xi,\eta \in \mathsf{H}$ and $a \in \mathscr{A}$, where $\|\cdot\|_{\mathsf{H}}$ is the norm in H and $\|\cdot\|_{\mathscr{A}}$ is the norm in $\mathscr{A}$.

A map $\theta : \mathsf{H} \to \mathsf{H}$ is called *adjointable* if there exists a (necessarily unique) map $\theta^* : \mathsf{H} \to \mathsf{H}$ such that $\langle \theta\xi, \eta \rangle_{\mathsf{H}} = \langle \xi, \theta^*\eta \rangle_{\mathsf{H}}$ for all $\xi, \eta \in \mathsf{H}$. We denote by $\mathscr{L}(\mathsf{H})$ the C^*-algebra of all adjointable operators on H. For $\xi, \eta \in \mathsf{H}$, the operator $\theta_{\xi,\eta} \in \mathscr{L}(\mathsf{H})$ is defined by $\theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta \rangle_{\mathsf{H}}$ for $\zeta \in \mathsf{H}$. We define $\mathscr{K}(\mathsf{H}) \subseteq \mathscr{L}(\mathsf{H})$ by

$$\mathscr{K}(\mathsf{H}) = \overline{\operatorname{span}}\{\theta_{\xi,\eta} \mid \xi, \eta \in \mathsf{H}\},$$

where $\overline{\operatorname{span}}\{\cdots\}$ denotes the closure of the linear span of $\{\cdots\}$. We then have that $\mathscr{K}(\mathsf{H})$ is a closed two-sided ideal in $\mathscr{L}(\mathsf{H})$.

Let $\phi : \mathscr{A} \to \mathscr{L}(\mathsf{H})$ be a $*$-homomorphism. Then $ax := \phi(a)x$ defines a left action of $\mathscr{A}$ on H. A Hilbert $\mathscr{A}$-bimodule equipped with such a left action is what we call a *C^*-correspondence over $\mathscr{A}$*.

A *representation* (π, t) of a C^*-correspondence (H, ϕ) over $\mathscr{A}$ on a C^*-algebra B consists of a linear map $t : \mathsf{H} \to B$ and a $*$-homomorphism $\pi : \mathscr{A} \to B$ such that

$$t(\xi a) = t(\xi)\pi(a),\ t(\xi)^*t(\eta) = \pi(\langle \xi, \eta \rangle_{\mathsf{H}}),\ \text{and}\ t(a\xi) = \pi(a)t(\xi)$$

for $\xi, \eta \in \mathsf{H}$ and $a \in \mathscr{A}$. Given such a representation, there is a $*$-homomorphism $\psi_t : \mathscr{K}(\mathsf{H}) \to B$ which satisfies

$$\psi_t(\theta_{\xi,\eta}) = t(\xi)t(\eta)^*$$

for all $\xi, \eta \in \mathsf{H}$.

We denote by J_{H} the closed two-sided ideal $\phi^{-1}(\mathscr{K}(\mathsf{H})) \cap (\ker\phi)^{\perp}$ in $\mathscr{A}$, where $(\ker\phi)^{\perp} = \{a \in \mathscr{A} \mid ab = 0 \text{ for all } b \in \ker\phi\}$, and we say that a representation (π, t) of (H, ϕ) is *covariant* if

$$\psi_t(\phi(a)) = \pi(a)$$

for all $a \in J_{\mathsf{H}}$. The Cuntz–Pimsner C^*-algebra $\mathscr{O}_{\mathsf{H}}$ of the C^*-correspondence (H, ϕ) is then the universal C^*-algebra generated by a covariant representation of (H, ϕ). It follows from the universal property of $\mathscr{O}_{\mathsf{H}}$ that there is an action γ of $\mathbb{T}$ on $\mathscr{O}_{\mathsf{H}}$ characterized by $\gamma(\pi_{\mathsf{H}}(a)) = \pi_{\mathsf{H}}(a)$ and $\gamma(t_{\mathsf{H}}(\xi)) = zt_{\mathsf{H}}(\xi)$ for $z \in \mathbb{T}$, $a \in \mathscr{A}$, and $\xi \in \mathsf{H}$, where $(\pi_{\mathsf{H}}, t_{\mathsf{H}})$ is the universal covariant representation of (H, ϕ) on $\mathscr{O}_{\mathsf{H}}$. This action is called the *gauge action*.

Let X be a one-sided shift space over the alphabet $\mathfrak{a}$. We will now construct a C^*-correspondence $(\mathsf{H}_{\mathsf{X}}, \phi_{\mathsf{X}})$ such that $\mathscr{O}_{\mathsf{H}_{\mathsf{X}}}$ is isomorphic to $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$. We denote by ε the empty word in $\mathfrak{a}^*$ (the word consisting of no letters). We then have that $C(u, \varepsilon) = \{x \in \mathsf{X} \mid ux \in \mathsf{X}\}$ for $u \in \mathfrak{a}^*$. Let $\mathscr{A}_{\mathsf{X}}$ be the C^*-subalgebra of $\mathscr{D}_{\mathsf{X}}$ (and thus of $l^\infty(\mathsf{X})$) generated by $\{1_{C(a,\varepsilon)} \mid u \in \mathfrak{a}^*\}$, and for each $a \in \mathfrak{a}$ let $\mathscr{A}_a$ be the ideal of $\mathscr{A}_{\mathsf{X}}$ generated by $1_{C(a,\varepsilon)}$. We let H_{X} be the right Hilbert $\mathscr{A}$-module $\bigoplus_{a\in\mathfrak{a}} \mathscr{A}_a$, where the right action is defined by $(\chi_a)_{a\in\mathfrak{a}} f = (\chi_a f)_{a\in\mathfrak{a}}$ for $(\chi_a)_{a\in\mathfrak{a}} \in \mathsf{H}_{\mathsf{X}}$ and $f \in \mathscr{A}_{\mathsf{X}}$, and the inner product is defined by

$$\left\langle (\chi_a)_{a\in\mathfrak{a}} \mid (\eta_a)_{a\in\mathfrak{a}} \right\rangle_{\mathsf{H}_{\mathsf{X}}} = \sum_{a\in\mathfrak{a}} \chi_a^* \eta_a$$

for $(\chi_a)_{a\in\mathfrak{a}}, (\eta_a)_{a\in\mathfrak{a}} \in \mathsf{H}_{\mathsf{X}}$.

To make H_{X} into a C^*-correspondence over $\mathscr{A}_{\mathsf{X}}$, we need to specify a left action of $\mathscr{A}_{\mathsf{X}}$ on H_{X}, i.e., a $*$-homomorphism from $\mathscr{A}_{\mathsf{X}}$ to $\mathscr{L}(\mathsf{H}_{\mathsf{X}})$.

For $a \in \mathfrak{a}$ and a function f from X to $\mathbb{C}$ we let $\lambda_a(f)$ be the function from X to $\mathbb{C}$ defined by

$$\lambda_a(f)(x) = \begin{cases} f(ax) & \text{if } ax \in \mathsf{X}, \\ 0 & \text{if } ax \notin \mathsf{X} \end{cases}$$

for $x \in \mathsf{X}$.

Lemma 2. *For every $a \in \mathfrak{a}$ we have that $\lambda_a(\mathscr{A}_{\mathsf{X}}) \subseteq \mathscr{A}_a$.*

Proof. If $u \in \mathfrak{a}^*$, then $\lambda_a(1_{C(u,\varepsilon)}) = 1_{C(ua,\varepsilon)} \leq 1_{C(a,\varepsilon)}$. Thus $\lambda_a(1_{C(u,\varepsilon)}) \in \mathscr{A}_a$ for every $u \in \mathfrak{a}^*$. It is easy to check that λ_a is a $*$-homomorphism from $l^\infty(\mathsf{X})$ to $l^\infty(\mathsf{X})$. Since $\mathscr{A}_\mathsf{X}$ is the C^*-subalgebra of $l^\infty(\mathsf{X})$ generated by $\{1_{C(u,\varepsilon)} \mid u \in \mathfrak{a}^*\}$, and $\lambda_a(1_{C(u,\varepsilon)}) \in \mathscr{A}_a$ for every $u \in \mathfrak{a}^*$, it follows that $\lambda_a(\mathscr{A}_\mathsf{X}) \subseteq \mathscr{A}_a$. □

We can now define our left action ϕ_X of $\mathscr{A}_\mathsf{X}$ on H_X by letting $\phi_\mathsf{X}(f)(\chi_a)_{a\in\mathfrak{a}} = (\lambda_a(f)\chi_a)_{a\in\mathfrak{a}}$ for $f \in \mathscr{A}_\mathsf{X}$ and $(\chi_a)_{a\in\mathfrak{a}} \in \mathsf{H}_\mathsf{X}$.

Lemma 3. *The map ϕ_X is an injective $*$-homomorphism from $\mathscr{A}_\mathsf{X}$ to $\mathscr{L}(\mathsf{H}_\mathsf{X})$.*

Proof. It is easy to check that ϕ_X is a $*$-homomorphism from $\mathscr{A}_\mathsf{X}$ to $\mathscr{L}(\mathsf{H}_\mathsf{X})$.

Assume that $f \in \mathscr{A}_\mathsf{X}$ and $\phi_\mathsf{X}(f) = 0$. We then have that $\lambda_a(f) = 0$ for all $a \in \mathfrak{a}$. If we for $x \in \mathsf{X}$ let a be the first letter of x, we have that $f(x) = \lambda_a(f)(\sigma(x)) = 0$. Thus $f = 0$, which shows that ϕ_X is injective. □

The following lemma is straightforward to check.

Lemma 4. *We have for $f \in \mathscr{A}_\mathsf{X}$ that $\phi_\mathsf{X}(f) = \sum_{a\in\mathfrak{a}} \theta_{e_a, e_a\lambda_a(f)}$. Thus $\phi_\mathsf{X}(\mathscr{A}_\mathsf{X}) \subseteq \mathscr{K}(\mathsf{H}_\mathsf{X})$.*

Theorem 5. *Let X be a one-sided shift space, and let H_X be the Hilbert $\mathscr{A}_\mathsf{X}$-bimodule defined above. Then their exists a unique $*$-isomorphism from the Cuntz–Pimsner algebra $\mathscr{O}_{\mathsf{H}_\mathsf{X}}$ of H_X to the C^*-algebra $\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ which for every $f \in \mathscr{A}_\mathsf{X}$ maps $\pi_{\mathsf{H}_\mathsf{X}}(f)$ to f and which for every $(\chi_a)_{a\in\mathfrak{a}} \in \mathsf{H}_\mathsf{X}$ maps $t_{\mathsf{H}_\mathsf{X}}((\chi_a)_{a\in\mathfrak{a}})$ to $\sum_{a\in\mathfrak{a}} s_a\chi_a$, where $(\pi_{\mathsf{H}_\mathsf{X}}, t_{\mathsf{H}_\mathsf{X}})$ is the universal covariant representation of H_X on $\mathscr{O}_{\mathsf{H}_\mathsf{X}}$.*

Proof. Since $\mathscr{O}_{\mathsf{H}_\mathsf{X}}$ is generated by $\{\pi_{\mathsf{H}_\mathsf{X}}(f) \mid f \in \mathscr{A}_\mathsf{X}\} \cup \{t_{\mathsf{H}_\mathsf{X}}(\eta) \mid \eta \in \mathsf{H}_\mathsf{X}\}$, there can at most be one $*$-homomorphism from $\mathscr{O}_{\mathsf{H}_\mathsf{X}}$ of H_X to $\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ which for every $f \in \mathscr{A}_\mathsf{X}$ maps $\pi_{\mathsf{H}_\mathsf{X}}(f)$ to f and which for every $(\chi_a)_{a\in\mathfrak{a}} \in \mathsf{H}_\mathsf{X}$ maps $t_{\mathsf{H}_\mathsf{X}}((\chi_a)_{a\in\mathfrak{a}})$ to $\sum_{a\in\mathfrak{a}} s_a\chi_a$.

For $f \in \mathscr{A}_\mathsf{X}$ let $\pi(f) = f$, and for $(\chi_a)_{a\in\mathfrak{a}} \in \mathsf{H}_\mathsf{X}$ let $t((\chi_a)_{a\in\mathfrak{a}}) = \sum_{a\in\mathfrak{a}} s_a\chi_a$. Then π is an injective $*$-homomorphism from $\mathscr{A}_\mathsf{X}$ to $\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ (π is just the inclusion of $\mathscr{A}_\mathsf{X}$ into $\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$), and t is a linear map from H_X to $\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ which satisfies that $t(\chi f) = t(\chi)\pi(f)$ for $\chi \in \mathsf{H}_\mathsf{X}$ and $f \in \mathscr{A}_\mathsf{X}$.

It is not difficult to check (cf. [4, p. 286]) that if $a, b \in \mathfrak{a}$, then $s_a^* s_b = 0$ if $a \neq b$ and $s_a^* s_a = 1_{C(a,\varepsilon)}$, so if $(\chi_a)_{a\in\mathfrak{a}}, (\eta_a)_{a\in\mathfrak{a}} \in \mathsf{H}_\mathsf{X}$, then

$$\begin{aligned} t\big((\chi_a)_{a\in\mathfrak{a}}\big)^* t\big((\eta_a)_{a\in\mathfrak{a}}\big) &= \left(\sum_{a\in\mathfrak{a}} s_a\chi_a\right)^* \left(\sum_{a\in\mathfrak{a}} s_a\eta_a\right) = \sum_{a,b\in\mathfrak{a}} \chi_a^* s_a^* s_b \eta_a \\ &= \sum_{a\in\mathfrak{a}} \chi_a^* \eta_a = \pi\big(\langle(\chi_a)_{a\in\mathfrak{a}} \mid (\eta_a)_{a\in\mathfrak{a}}\rangle\big). \end{aligned}$$

It is not difficult to check (cf. [4, p. 286]) that if $a \in \mathfrak{a}$ and $f \in \mathscr{A}_\mathsf{X}$, then $s_a^* f s_a = \lambda_a(f)$, so if $f \in \mathscr{A}_\mathsf{X}$ and $(\chi_a)_{a\in\mathfrak{a}} \in \mathsf{H}_\mathsf{X}$, then

$$\begin{aligned} ft((\chi_a)_{a\in\mathfrak{a}}) &= f\sum_{a\in\mathfrak{a}} s_a\chi_a = \sum_{a\in\mathfrak{a}} f s_a s_a^* s_a \chi_a = \sum_{a\in\mathfrak{a}} s_a s_a^* f s_a \chi_a \\ &= \sum_{a\in\mathfrak{a}} s_a \lambda_a(f)\chi_a = t(\phi_\mathsf{X}(f)(\chi_a)_{a\in\mathfrak{a}}). \end{aligned}$$

Thus, (π, t) is an injective representation of H_X. Hence there exists a $*$-homomorphism ψ_t from $\mathscr{K}(\mathsf{H}_\mathsf{X})$ to $\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ which for $\chi, \eta \in \mathsf{H}_\mathsf{X}$ maps $\theta_{\chi,\eta}$ to $t(\chi)^* t(\eta)$ (cf. [6, Definition 2.3]). If $f \in \mathscr{A}_\mathsf{X}$, we have

$$\psi_t(\phi_\mathsf{X}(f)) = \sum_{a\in\mathfrak{a}} \theta_{e_a, e_a\lambda_a(f)} = \sum_{a\in\mathfrak{a}} s_a\lambda_a(f)s_a^* = \sum_{a\in\mathfrak{a}} s_a s_a^* f s_a s_a^* = f = \pi(f).$$

Thus the representation (π, t) is covariant. Hence there exists a $*$-homomorphism ζ from $\mathscr{O}_{\mathsf{H}_\mathsf{X}}$ to $\mathscr{D}_\mathsf{X} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ which for every $f \in \mathscr{A}_\mathsf{X}$ maps $\pi_{\mathsf{H}_\mathsf{X}}(f)$ to $\pi(f) = f$ and which for every $(\chi_a)_{a\in\mathfrak{a}} \in \mathsf{H}_\mathsf{X}$ maps $t_{\mathsf{H}_\mathsf{X}}((\chi_a)_{a\in\mathfrak{a}})$ to $t((\chi_a)_{a\in\mathfrak{a}}) = \sum_{a\in\mathfrak{a}} s_a\chi_a$.

The gauge action γ on $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ satisfies that $\gamma_z(\pi(f)) = \gamma_z(f) = f$ and

$$\gamma_z(t(\chi_a)_{a\in\mathfrak{a}}) = \gamma_z\left(\sum_{a\in\mathfrak{a}} s_a\chi_a\right) = z\sum_{a\in\mathfrak{a}} s_a\chi_a = zt(\chi_a)_{a\in\mathfrak{a}}$$

for all $z \in \mathbb{T}$, $f \in \mathscr{A}_{\mathsf{X}}$, and $(\chi_a)_{a\in\mathfrak{a}} \in \mathsf{H}_{\mathsf{X}}$. Thus the representation (π,t) admits a gauge action.

Since the representation (π,t) is injective and admits a gauge action, the $*$-homomorphism ζ is injective according to [6, Theorem 6.4]. Since $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$ is generated by $\{s_a \mid a \in \mathfrak{a}\}$ and $\zeta(t_{\mathsf{H}_{\mathsf{X}}}(e_a)) = t(e_a) = s_a$, it follows that ζ is also surjective. □

3. THE PROOF OF THEOREM 1

In this section we will prove Theorem 1. We will do this by applying Theorem 8.6 of [6] to the C^*-correspondence introduced in the previous section. We will, for the benefit of the reader, first state Theorem 8.6 of [6] and explain the necessary notation.

Let (H,ϕ) be a C^*-correspondence over a C^*-algebra $\mathscr{A}$ and let D_{H} denote the C^*-algebra $\mathscr{K}(\mathsf{H}\oplus\mathscr{A})$. We denote the natural embedding of $\mathscr{A}$ into D_{H} by $\iota_{\mathscr{A}}$ and the natural embedding of $\mathscr{K}(\mathsf{H})$ into D_{H} by $\iota_{\mathscr{K}(\mathsf{H})}$. It is shown in [6, Proposition B.3] that the map $(\iota_{\mathscr{A}})_* : K_*(\mathscr{A}) \to K_*(D_{\mathsf{H}})$ induced by $\iota_{\mathscr{A}}$ is an isomorphism. The map $[\mathsf{H}] : K_*(J_{\mathsf{H}}) \to K_*(\mathscr{A})$ is then defined as $(\iota_{\mathscr{A}})_*^{-1} \circ (\iota_{\mathscr{K}\mathsf{H}})_* \circ (\phi)_*$. Let $\iota_* : K_*(J_{\mathsf{H}}) \to K_*(\mathscr{A})$ denote the map induced by the inclusion ι of J_{H} into $\mathscr{A}$ and let $(\pi_{\mathsf{H}})_* : K_*(\mathscr{A}) \to K_*(\mathscr{O}_{\mathsf{H}})$ denote the map induced by the $*$-homomorphism $\pi_{\mathsf{H}} : \mathscr{A} \to \mathscr{O}_{\mathsf{H}}$.

Theorem 6. ([6, Theorem 8.6]). *Let (H,ϕ) be a C^*-correspondence over a C^*-algebra $\mathscr{A}$. Then we have the following exact sequences*:

$$\begin{array}{ccccc} K_0(J_{\mathsf{H}}) & \xrightarrow{\iota_*-[\mathsf{H}]} & K_0(\mathscr{A}) & \xrightarrow{(\pi_{\mathsf{H}})_*} & K_0(\mathscr{O}_{\mathsf{H}}) \\ \uparrow & & & & \downarrow \\ K_1(\mathscr{O}_{\mathsf{H}}) & \xleftarrow{(\pi_{\mathsf{H}})_*} & K_1(\mathscr{A}) & \xleftarrow{\iota_*-[\mathsf{H}]} & K_1(J_{\mathsf{H}}). \end{array}$$

Let X be a one-sided shift space. We will now use Theorem 6 to compute the K-theory of $\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}$. But first a few lemmas.

Lemma 7. *We have that ideal $J_{\mathsf{H}_{\mathsf{X}}} = \phi_{\mathsf{X}}^{-1}(\mathscr{K}(\mathsf{H}_{\mathsf{X}})) \cap (\ker\phi_{\mathsf{X}})^{\perp}$ of $\mathscr{A}_{\mathsf{X}}$ is all of $\mathscr{A}_{\mathsf{X}}$.*

Proof. This follows directly from Lemmas 3 and 4. □

Lemma 8. *Let $p \in \mathscr{A}_{\mathsf{X}}$ be a projection and let $a \in \mathfrak{a}$. Then we have*

$$(\iota_{\mathscr{A}_{\mathsf{X}}})_*([p1_{C(a,\varepsilon)}]_0) = (\iota_{\mathscr{K}(\mathsf{H}_{\mathsf{X}})})_*([\theta_{e_a,e_ap}]_0).$$

Proof. Let $v = \theta_{(e_a,0),(0,p)} \in \mathscr{K}(D_{\mathsf{H}_{\mathsf{X}}})$. It is easy to check that $vv^* = \iota_{\mathscr{K}(\mathsf{H}_{\mathsf{X}})}(\theta_{e_a,e_ap})$ and $v^*v = \iota_{\mathscr{A}_{\mathsf{X}}}(p1_{C(a,\varepsilon)})$. It follows that $(\iota_{\mathscr{A}_{\mathsf{X}}})_*([p1_{C(a,\varepsilon)}]_0)$ and $(\iota_{\mathscr{K}(\mathsf{H}_{\mathsf{X}})})_*([\theta_{e_a,e_ap}]_0)$ are equivalent in $K_0(D_{\mathsf{H}_{\mathsf{X}}})$. □

Lemma 9. *We have that $K_1(\mathscr{A}_{\mathsf{X}}) = 0$ and that there exists an isomorphism ρ from $\mathbb{Z}_{\mathsf{X}_0}$ to $K_0(\mathscr{A}_{\mathsf{X}})$ which for $l \in \mathbb{N}$ and $i \in \{1,2,\ldots,m(l)\}$ maps $\iota_l(e_i)$ to $[1_{\mathscr{E}_i^l}]_0$.*

We furthermore have for every $l \in \mathbb{N}$ and for every $j \in \{1,2,\ldots,m(l)\}$ that

$$\rho^{-1} \circ [\mathsf{H}_{\mathsf{X}}] \circ \rho(\iota_l(e_j)) = \sum_{a\in\mathfrak{a}} \sum_{i=1}^{m(l+1)} A_l(i,j,a)\iota_{l+1}(e_i). \tag{1}$$

Proof. For each $l \in \mathbb{N}$ let $\mathscr{A}_l$ be the C^*-subalgebra of $\mathscr{A}_{\mathsf{X}}$ generated by $\{1_{C(u,\varepsilon)} \mid u \in \mathfrak{a}_l^*\}$. We then have that $\mathscr{A}_{\mathsf{X}}$ is the closure of $\bigcup_{l\in\mathbb{N}} \mathscr{A}_l$ and that $\mathscr{A}_l$, for each $l \in \mathbb{N}$, is isomorphic to $\mathbb{C}^{m(l)}$ by an isomorphism which for each $i \in \{1,2,\dots,m(l)\}$ maps $1_{\mathscr{E}_i^l}$ to e_i [cf. 4, p. 291]. It follows that $\mathscr{A}_{\mathsf{X}}$ is an AF-algebra, and thus that $K_1(\mathscr{A}_{\mathsf{X}}) = 0$. The existence of ρ also follows from this.

Let $l \in \mathbb{N}$ and $j \in \{1,2,\dots,m(l)\}$. It follows from the definition of $A_l(i,j,a)$ (see [4, p. 301]) that $\lambda_a(1_{\mathscr{E}_j^l}) = \sum_{i=1}^{m(l+1)} A_l(i,j,a) 1_{\mathscr{E}_i^{l+1}}$ for each $a \in \mathfrak{a}$. According to Lemma 4, we have that $\phi_{\mathsf{X}}(1_{\mathscr{E}_j^l}) = \sum_{a\in\mathfrak{a}} \theta_{e_a,e_a\lambda_a(1_{E_j^l})}$, so it follows from Lemma 8 that we have

$$\begin{aligned}
(\iota_{\mathscr{K}(\mathsf{H}_{\mathsf{X}})})_* \circ ((\phi_{\mathsf{X}})|_{J_{\mathsf{H}_{\mathsf{X}}}})_*([1_{\mathscr{E}_j^l}]_0) &= (\iota_{\mathscr{K}(\mathsf{H}_{\mathsf{X}})})_* \left(\left[\sum_{a\in\mathfrak{a}} \theta_{e_a,e_a\lambda_a(1_{\mathscr{E}_j^l})} \right]_0 \right) \\
&= \sum_{a\in\mathfrak{a}} (\iota_{\mathscr{K}(\mathsf{H}_{\mathsf{X}})})_*([\theta_{e_a,e_a\lambda_a(1_{\mathscr{E}_j^l})}]_0) \\
&= \sum_{a\in\mathfrak{a}} (\iota_{\mathscr{A}_{\mathsf{X}}})_*([\lambda_a(1_{\mathscr{E}_j^l})]_0) \\
&= \sum_{a\in\mathfrak{a}} \sum_{i=1}^{m(l+1)} A_l(i,j,a)(\iota_{\mathscr{A}_{\mathsf{X}}})_*([1_{\mathscr{E}_i^{l+1}}]_0),
\end{aligned}$$

from which (1) follows. □

Proof of Theorem 1. It follows from Theorems 5 and 6 and Lemmas 7 and 9 that we have the following exact sequence:

$$0 \longrightarrow K_1(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}) \longrightarrow \mathbb{Z}_{\mathsf{X}_0} \xrightarrow{B} \mathbb{Z}_{\mathsf{X}_0} \xrightarrow{\kappa} K_0(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}) \longrightarrow 0,$$

where κ is the group homomorphism from $\mathbb{Z}_{\mathsf{X}_0}$ to $K_0(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$ which for $l \in \mathbb{N}$ and $i \in \{1,2,\dots,m(l)\}$ maps $\iota_l(e_i)$ to $[1_{\mathscr{E}_i^l}]_0$. It follows that $K_1(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N}) \cong \ker(B)$ and that the map $[1_{\mathscr{E}_i^l}]_0 \mapsto \iota_l(e_i)$ induces an isomorphism from $K_0(\mathscr{D}_{\mathsf{X}} \rtimes_{\alpha,\mathscr{L}} \mathbb{N})$ to $\mathbb{Z}_{\mathsf{X}_0}/B\mathbb{Z}_{\mathsf{X}_0}$. □

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Ühepoolse nihkeruumiga assotsieeritud C^*-algebra K-teooriast

Toke Meier Carlsen ja Sergei Silvestrov

Ühepoolne nihkeruum on mittepööratava topoloogilise dünaamilise süsteemi eriliik, millega assotsieerub C^*-algebra. Artiklis on näidatud, kuidas saab konstrueerida ühepoolse nihkeruumiga assotsieeritud C^*-algebrat kui C^*-vastavuse Cuntzi-Pimsneri algebrat ja kasutada konstrueeritud algebrat K-teooria arvutamiseks.