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MATHEMATICS

Tangent and trinormal spherical images of a time-like curve on the pseudohyperbolic space H_0^3

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Abstract. In this work tangent and trinormal spherical images of a time-like curve lying on the pseudohyperbolic space H_0^3 in Minkowski space-time are investigated. We observe that the mentioned spherical images are space-like curves. Besides, we determine relations between Frenet–Serret invariants of spherical images and the base curve. Additionally, some characterizations of spherical images to be general helices if the base curve is a ccr-curve are presented.

Key words: classical differential geometry, spherical images, time-like curves, general helix, ccr-curves.

1. INTRODUCTION

At each point of a differentiable curve a tetrad of mutually orthogonal unit vectors (called tangent, normal, binormal, and trinormal) was defined and constructed. The rates of changes of these vectors along the curve define the curvatures of the curve in the space E_1^4. Spherical images (indicatrices) are a well-known concept in classical differential geometry of curves [3].

At the beginning of the twentieth century Einstein's theory opened a door to new geometries such as Minkowski space-time, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold. In recent years the theory of degenerate submanifolds has been treated by researchers and some of the classical differential geometry topics have been extended to Lorentzian manifolds. Some authors have aimed to determine Frenet–Serret invariants in higher dimensions. There exists a vast literature on this subject, for instance [2,8–10]. In the light of the available literature, in [8] the author extended spherical images of curves to a four-dimensional Lorentzian space and studied such curves in the case where the base curve is a space-like curve according to the signature $(+,+,+,-)$.

In this work we study spherical images of a time-like curve lying on the pseudohyperbolic space H_0^3 in Minkowski space-time. We investigate relations between Frenet–Serret invariants of spherical images and the base curve. Additionally, some characterizations of spherical images to be general helices if the base curve is a ccr-curve are presented.

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2. PRELIMINARIES

To meet the requirements in the next sections, here the basic elements of the theory of curves in the space E_1^4 are briefly presented. (A more complete elementary treatment can be found in [7].)

The Minkowski space-time E_1^4 is the Euclidean space E^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in E_1^4.

Since g is an indefinite metric, recall that a vector $v \in E_1^4$ can have one of the three causal characters: it can be space-like if $g(v,v) > 0$ or $v = 0$, time-like if $g(v,v) < 0$, and null (light-like) if $g(v,v) = 0$ and $v \neq 0$. Similary, an arbitrary curve $\alpha = \alpha(s)$ in E_1^4 can be locally space-like, time-like, or null (light-like), if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like, or null. Also, recall that the norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$. Therefore, v is a unit vector if $g(v,v) = \pm 1$. Next, vectors v, w in E_1^4 are said to be orthogonal if $g(v,w) = 0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$. Let a and b be two space-like vectors in E_1^4; then there is a unique real number $0 \leq \delta \leq \pi$, called the angle between a and b, such that $g(a,b) = \|a\| . \|b\| \cos\delta$. Let $\vartheta = \vartheta(s)$ be a curve in E_1^4. If the tangent vector field of this curve forms a constant angle with a constant vector field U, this curve is called a general helix.

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving Frenet–Serret frame along the curve $\alpha(s)$ in the space E_1^4. Then T, N, B_1, B_2 are, respectively, the tangent, the principal normal, the binormal (the first binormal), and the trinormal (the second binormal) vector fields. A space-like or time-like curve $\alpha(s)$ is said to be parametrized by arclength function s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Let $\alpha(s)$ be a time-like curve in the space-time E_1^4, parametrized by arclength function s. In [7] the following Frenet–Serret equations are given:

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where T, N, B_1, and B_2 are mutually orthogonal vectors satisfying the equations

$$g(T,T) = -1, g(N,N) = g(B_1,B_1) = g(B_2,B_2) = 1,$$

and where κ, τ, and σ are the first, second, and third curvatures of the curve α, respectively.

In the same space, in [1], the authors express a characterization of space-like curves lying on H_0^3 by the following theorem:

Theorem 2.1. *Let $\alpha(s)$ be a unit speed space-like curve in E_1^4, with space-like N, B_1 and curvatures $\kappa \neq 0, \tau \neq 0, \sigma \neq 0$ for each $s \in I \subset R$. Then α lies on a pseudohyperbolic space if and only if*

$$\begin{gathered} \frac{\sigma}{\tau}\frac{d\rho}{ds} = \frac{d}{ds}\left[\frac{1}{\sigma}\left(\rho\tau + \frac{d}{ds}\left(\frac{1}{\tau}\frac{d\rho}{ds}\right)\right)\right], \\ \left\{\frac{1}{\sigma}\left[\rho\tau + \frac{d}{ds}\left(\frac{1}{\tau}\frac{d\rho}{ds}\right)\right]\right\}^2 > \rho^2 + \left(\frac{1}{\tau}\frac{d\rho}{ds}\right)^2, \end{gathered} \tag{2.1}$$

where $\rho = \dfrac{1}{\kappa}$.

Recall that if a regular curve has constant Frenet–Serret curvature ratios (i.e., $\frac{\tau}{\kappa}$ and $\frac{\sigma}{\tau}$ are constants), it is called a *ccr-curve* [4,5]. In the same space the authors of [9] defined a vector product and gave a method to determine the Frenet–Serret invariants for an arbitrary curve by the following definition and the theorem.

Definition 2.2. *Let $a=(a_1,a_2,a_3,a_4)$, $b=(b_1,b_2,b_3,b_4)$, and $c=(c_1,c_2,c_3,c_4)$ be vectors in E_1^4. The vector product in the Minkowski space-time E_1^4 is defined by the determinant*

$$a\wedge b\wedge c=-\begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \tag{2.2}$$

where e_1,e_2,e_3, and e_4 are mutually orthogonal vectors (coordinate direction vectors) satisfying the equations

$$e_1\wedge e_2\wedge e_3=e_4,\ e_2\wedge e_3\wedge e_4=e_1,\ e_3\wedge e_4\wedge e_1=e_2,\ e_4\wedge e_1\wedge e_2=-e_3.$$

Theorem 2.3. *Let $\alpha=\alpha(t)$ be an arbitrary space-like curve in the Minkowski space-time E_1^4. The Frenet–Serret apparatus of α can be written as follows:*

$$T=\frac{\alpha'}{\|\alpha'\|}, \tag{2.3}$$

$$N=\frac{\|\alpha'\|^2\alpha''-g(\alpha',\alpha'')\alpha'}{\left\|\|\alpha'\|^2\alpha''-g(\alpha',\alpha'')\alpha'\right\|}, \tag{2.4}$$

$$B_1=\mu N\wedge T\wedge B_2, \tag{2.5}$$

$$B_2=\mu\frac{T\wedge N\wedge\alpha'''}{\|T\wedge N\wedge\alpha'''\|}, \tag{2.6}$$

$$\kappa=\frac{\left\|\|\alpha'\|^2\alpha''-g(\alpha',\alpha'')\alpha'\right\|}{\|\alpha'\|^4}, \tag{2.7}$$

$$\tau=\frac{\|T\wedge N\wedge\alpha'''\|\,\|\alpha'\|}{\left\|\|\alpha'\|^2\alpha''-g(\alpha',\alpha'')\alpha'\right\|}, \tag{2.8}$$

and

$$\sigma=\frac{g(\alpha^{(IV)},B_2)}{\|T\wedge N\wedge\alpha'''\|\,\|\alpha'\|}, \tag{2.9}$$

where μ is taken -1 or $+1$ to make $+1$ the determinant of the $[T,N,B_1,B_2]$ matrix.

Here, we shall use Frenet–Serret invariants of a time-like curve. Therefore, our calculations do not contain null vectors.

3. TANGENT SPHERICAL IMAGE OF A TIME-LIKE CURVE LYING ON H_0^3

Following the paper [8], we first adapt the tangent spherical image definition to time-like curves of Minkowski space-time. Moreover, we give the definition of the trinormal spherical image for time-like curves at the beginning of Section 4.

Definition 3.1. *Let $\beta=\beta(s)$ be a unit speed time-like curve in Minkowski space-time. If we translate the tangent vector to the centre O of the pseudohyperbolic space H_0^3, we obtain a curve $\delta=\delta(s_\delta)$. This curve is called the tangent spherical image or tangent indicatrix of the curve β in E_1^4.*

Theorem 3.2. *Let $\beta = \beta(s)$ be a unit speed time-like curve and $\delta = \delta(s_\delta)$ be its tangent spherical image. Then*
(i) $\delta = \delta(s_\delta)$ *is a space-like curve*;
(ii) *the Frenet–Serret apparatus of* δ, $\{T_\delta, N_\delta, B_{1\delta}, B_{2\delta}, \kappa_\delta, \tau_\delta, \sigma_\delta\}$ *can be formed by the apparatus of* β, $\{T, N, B_1, B_2, \kappa, \tau, \sigma\}$.

Proof. Let $\beta = \beta(s)$ be a unit speed time-like curve and $\delta = \delta(s_\delta)$ be its tangent spherical image. One can write

$$\delta = T(s). \tag{3.1}$$

By differentiation with respect to s we get

$$\delta' = \dot{\delta}\frac{ds_\delta}{ds} = \kappa N. \tag{3.2}$$

Here we shall denote differentiation according to s by a dash, and differentiation according to s_δ by a dot. So, we immediately arrive at

$$T_\delta = N \tag{3.3}$$

and

$$\|\delta'\| = \frac{ds_\delta}{ds} = \kappa. \tag{3.4}$$

By the inner product $g(\delta', \delta') = \kappa^2 > 0$, thus the tangent spherical image $\delta(s_\delta)$ is a space-like curve. Considering the previous method, we form the following differentiations with respect to s:

$$\begin{cases} \delta'' = \kappa^2 T + \kappa' N + \kappa\tau B_1, \\ \delta''' = 3\kappa\kappa' T + (\kappa'' + \kappa^3 - \kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B_1 + \kappa\tau\sigma B_2, \\ \delta^{(IV)} = \lambda_1 T + \lambda_2 N + \lambda_3 B_1 + \lambda_4 B_2. \end{cases} \tag{3.5}$$

By equation (2.4) we form

$$\|\delta'\|^2 \delta'' - g(\delta', \delta'')\delta' = \kappa^3(\kappa T + \tau B_1). \tag{3.6}$$

Then one can easily have the principal normal vector

$$N_\delta = \frac{\kappa T + \tau B_1}{\sqrt{|\tau^2 - \kappa^2|}}, \tag{3.7}$$

and the first curvature

$$\kappa_\delta = \frac{\sqrt{|\tau^2 - \kappa^2|}}{\kappa}, \tag{3.8}$$

where $\kappa \neq \tau$. Now let us form the vector $T_\delta \wedge N_\delta \wedge \delta'''$

$$T_\delta \wedge N_\delta \wedge \delta''' = -\frac{1}{\chi}\begin{vmatrix} -T & N & B_1 & B_2 \\ 0 & 1 & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ l_1 & l_2 & l_3 & l_4 \end{vmatrix}, \tag{3.9}$$

where l_i are the components of the differentiable vector function of δ''' and $\chi = \sqrt{|\tau^2 - \kappa^2|}$. This product yields

$$T_\delta \wedge N_\delta \wedge \delta''' = \frac{\kappa\tau}{\chi}\left\{\tau\sigma T + \kappa\sigma B_1 + \tau\left(\frac{\kappa}{\tau}\right)' B_2\right\}. \tag{3.10}$$

Thus, we express the trinormal (second binormal) vector field of the curve $\delta(s_\delta)$ as

$$B_{2\delta} = \mu \frac{\tau\sigma T + \kappa\sigma B_1 + \tau\left(\frac{\kappa}{\tau}\right)' B_2}{\sqrt{\left|\left[\tau\left(\frac{\kappa}{\tau}\right)'\right]^2 + \sigma^2(\kappa^2 - \tau^2)\right|}}. \tag{3.11}$$

Taking the norm of both sides of (3.10), we have the second curvature

$$\tau_\delta = \frac{\tau\sqrt{\left|\left[\tau\left(\frac{\kappa}{\tau}\right)'\right]^2 + \sigma^2(\kappa^2 - \tau^2)\right|}}{\kappa(\tau^2 - \kappa^2)}. \tag{3.12}$$

To determine the binormal vector field, we express

$$N_\delta \wedge T_\delta \wedge B_{2\delta} = -\frac{1}{\chi\vartheta}\begin{vmatrix} -T & N & B_1 & B_2 \\ \kappa & 0 & \tau & 0 \\ 0 & 1 & 0 & 0 \\ \tau\sigma & 0 & \kappa\sigma & \tau\left(\frac{\kappa}{\tau}\right)' \end{vmatrix}, \tag{3.13}$$

where $\vartheta = \sqrt{\left|\left[\tau\left(\frac{\kappa}{\tau}\right)'\right]^2 + \sigma^2(\kappa^2 - \tau^2)\right|}$. So, we get

$$B_{1\delta} = \mu \frac{1}{\chi\vartheta}\left\{-\tau^2\left(\frac{\kappa}{\tau}\right)' T - \kappa\tau\left(\frac{\kappa}{\tau}\right)' B_1 + \sigma(\kappa^2 - \tau^2)B_2\right\}. \tag{3.14}$$

Finally, using (2.9) and the obtained equations, we have the third curvature

$$\sigma_\delta = \frac{-\lambda_1\tau\sigma + \lambda_3\kappa\sigma + \lambda_4\tau\left(\frac{\kappa}{\tau}\right)'}{\kappa\left\{\left[\tau\left(\frac{\kappa}{\tau}\right)'\right]^2 + \sigma^2(\kappa^2 - \tau^2)\right\}}. \tag{3.15}$$

□

Corollary 3.3. *$\{T_\delta, N_\delta, B_{1\delta}, B_{2\delta}\}$ is an orthonormal frame of Minkowski space-time.*

Considering the above theorem, we also give:

Theorem 3.4. *Let $\beta = \beta(s)$ be a time-like unit speed curve and $\delta(s_\delta)$ be its tangent spherical image. If β is a ccr-curve or a helix (i.e. W-curve), then δ is also a helix.*

Proof. Let $\beta = \beta(s)$ be a time-like unit speed ccr-curve. Then we know $\frac{\kappa}{\tau} = c_1 = \text{constant}$ and $\frac{\tau}{\sigma} = c_2 = \text{constant}$. Since, in terms of the above theorem, we have, respectively,

$$\kappa_\delta = \sqrt{\left|1 - \frac{1}{c_1^2}\right|} = \text{constant} \tag{3.16}$$

and

$$\tau_\delta = -\frac{c_2}{c_1\sqrt{c_1^2 - 1}} = \text{constant}, \tag{3.17}$$

$\delta(s_\delta)$ is a spherical curve; so we may substitute κ_δ and τ_δ to the first part of formula (2.1). In this way we easily have

$$\sigma_\delta = \frac{c_2}{c_1^2} = \text{constant}. \tag{3.18}$$

Therefore, $\delta(s_\delta)$ is also a helix. The case of β being a helix can be immediately seen from the above equations. $\square$

By this theorem we present a characterization of the tangent spherical image with respect to constant Frenet–Serret curvature ratios (or helices). We observe that the mentioned indicatrix can be a helix, so, one can ask whether this tangent spherical image is a general helix or not. Therefore, we investigate it by the following statements.

Let $\beta = \beta(s)$ be a unit speed time-like curve and $\delta = \delta(s_\delta)$ be its space-like tangent spherical image. If $\delta = \delta(s_\delta)$ is a general helix, then, for a constant space-like vector U, we may express

$$g(T_\delta, U) = \cos\theta, \tag{3.19}$$

where θ is a constant angle. Equation (3.19) is also congruent to

$$g(N, U) = \cos\theta. \tag{3.20}$$

One can form a constant vector U according to $\{T, N, B_1, B_2\}$ as follows:

$$U = \varepsilon_1 T + \varepsilon_2 N + \varepsilon_3 B_1 + \varepsilon_4 B_2. \tag{3.21}$$

Differentiating (3.21) with respect to s, we have the following system of ordinary differential equations:

$$\begin{cases} \varepsilon_1' + \varepsilon_2\kappa = 0, \\ \varepsilon_1\kappa - \varepsilon_3\tau = 0, \\ \varepsilon_3' + \varepsilon_2\tau - \varepsilon_4\sigma = 0, \\ \varepsilon_4' + \varepsilon_3\sigma = 0. \end{cases} \tag{3.22}$$

We know that $\varepsilon_2 = c \neq 0$ is a constant. Using this system, we have two differential equations according to ε_3:

$$\begin{cases} \varepsilon_3'\left(\dfrac{\tau}{\kappa}\right) + \varepsilon_3\left(\dfrac{\tau}{\kappa}\right)' + c\kappa = 0, \\ \left(\dfrac{\varepsilon_3'}{\sigma}\right)' + c\left(\dfrac{\tau}{\kappa}\right)' + \varepsilon_3\sigma = 0. \end{cases} \tag{3.23}$$

The solution of these two differential equations with variable coefficients is not easy. Besides, a general solution of it has not yet been found. Due to this, we shall prove the following theorem which contains a special solution of system (3.23).

Theorem 3.5. *Let $\beta = \beta(s)$ be a unit speed time-like ccr-curve and $\delta = \delta(s_\delta)$ be its space-like tangent spherical image. If δ is a general helix, then there exists a relation among Frenet–Serret curvatures of β*

$$cc_1\int_0^s \kappa ds + \alpha_1\cos\int_0^s \sigma ds + \alpha_2\sin\int_0^s \sigma ds = 0, \tag{3.24}$$

where $c, c_1 = \frac{\kappa}{\tau}, \alpha_1$, and α_2 are constants.

Proof. Let us suppose $\beta=\beta(s)$ is a unit speed time-like ccr-curve and $\delta=\delta(s_\delta)$ is its space-like tangent spherical image; then relations (3.23) hold. We know that a ccr-curve has constant curvature ratios such that $\frac{\kappa}{\tau}=c_1$ and $\frac{\tau}{\sigma}=c_2$ are constants. Thus the differential equations (3.23) transform to

$$\begin{cases} \varepsilon_3'+cc_1\kappa=0,\\ \left(\dfrac{\varepsilon_3'}{\sigma}\right)'+\varepsilon_3\sigma=0. \end{cases} \tag{3.25}$$

By the first equation we have

$$\varepsilon_3=-cc_1\int_0^s \kappa ds. \tag{3.26}$$

Using an exchange variable $t=\int_0^s \sigma ds$ in $(3.25)_2$, we obtain

$$\frac{d^2\varepsilon_3}{dt^2}+\varepsilon_3=0. \tag{3.27}$$

The solution of it and (3.26) give us

$$cc_1\int_0^s \kappa ds+\alpha_1\cos\int_0^s \sigma ds+\alpha_2\sin\int_0^s \sigma ds=0, \tag{3.28}$$

for the real numbers c, c_1, α_1, and α_2, as desired. □

Corollary 3.6. *The fixed direction (constant vector U) can be composed by the components*

$$\begin{cases} \varepsilon_1=-c\displaystyle\int_0^s \kappa ds,\\ \varepsilon_2=c,\\ \varepsilon_3=-cc_1\displaystyle\int_0^s \kappa ds,\\ \varepsilon_4=cc_2(1-c_1^2). \end{cases} \tag{3.29}$$

4. TRINORMAL SPHERICAL IMAGE OF A TIME-LIKE CURVE LYING ON S_1^3

Definition 4.1. *Let $\beta=\beta(s)$ be a unit speed time-like curve in Minkowski space-time. If we translate the space-like trinormal vector fields to the centre O of the pseudohyperbolic space H_0^3, we obtain a curve $\varphi=\varphi(s_\varphi)$. This curve is called the trinormal spherical image or trinormal indicatrix of the curve β in E_1^4.*

We follow the same procedure to prove the following theorems. Owing to this, we express them without proofs.

Theorem 4.2. *Let $\beta=\beta(s)$ be a unit speed time-like curve and $\varphi=\varphi(s_\varphi)$ be its trinormal spherical image. Then*

(i) *$\varphi=\varphi(s_\varphi)$ is a space-like curve;*

(ii) *the Frenet–Serret apparatus of φ can be determined by the apparatus of $\beta=\beta(s)$ by the following formulas:*

$$T_\varphi=B_1, \tag{4.1}$$

$$N_\varphi = \frac{\tau N + \sigma B_2}{\sqrt{\tau^2+\sigma^2}}, \tag{4.2}$$

$$B_{1\varphi} = \mu \frac{1}{\phi\eta}\left\{\kappa\tau[\sigma^2-\tau^2]T - \sigma(\tau\sigma)'N + \tau(\tau\sigma)'B_2\right\}, \tag{4.3}$$

$$B_{2\varphi} = \mu \frac{(\tau\sigma)'T - \kappa\tau\sigma N - \kappa\tau^2 B_2}{\sqrt{\left|(\kappa\tau\sigma)^2+(\kappa\tau^2)^2-[(\tau\sigma)']^2\right|}}, \tag{4.4}$$

$$\kappa_\varphi = \frac{\sqrt{\tau^2+\sigma^2}}{\sigma}, \tag{4.5}$$

$$\tau_\varphi = \frac{\sqrt{\left|(\kappa\tau\sigma)^2+(\kappa\tau^2)^2-[(\tau\sigma)']^2\right|}}{\sigma(\tau^2+\sigma^2)}, \tag{4.6}$$

$$\sigma_\varphi = \frac{-\omega_1(\tau\sigma)' - \omega_2\kappa\tau\sigma - \omega_4\kappa\tau^2}{\sigma\sqrt{\left|(\kappa\tau\sigma)^2+(\kappa\tau^2)^2-[(\tau\sigma)']^2\right|}}, \tag{4.7}$$

where $\eta = \sqrt{\left|(\kappa\tau\sigma)^2+(\kappa\tau^2)^2-[(\tau\sigma)']^2\right|}$, $\phi = \sqrt{\tau^2+\sigma^2}$, *and* ω_i *are the components of the differentiable function* $\varphi^{(IV)}$ *with respect to* s;

(iii) *if* β *is a helix, then* φ *is also a helix.*

Theorem 4.3. *Let* $\beta = \beta(s)$ *be a unit speed time-like ccr-curve and* $\varphi = \varphi(s_\varphi)$ *be its space-like trinormal spherical image. If* φ *is a general helix, then*

(i) *there exists the relation*

$$\frac{c}{c_2}\int_0^s \sigma ds + \psi_1 \cosh \int_0^s \kappa ds + \psi_2 \sinh \int_0^s \kappa ds = 0, \tag{4.8}$$

where $c, c_2 = \frac{\tau}{\sigma}, \psi_1$ *and* ψ_2 *are constants*;

(ii) *the fixed direction of the axis of the general helix is*

$$U = \frac{2c}{c_1}T - \left(\frac{c}{c_2}\int_0^s \sigma ds\right)N + cB_1 - \left(c\int_0^s \sigma ds\right)B_2. \tag{4.9}$$

5. CONCLUSION AND FURTHER REMARKS

In this work we extended the spherical image concept to the time-like curves of Minkowski space-time. We investigated tangent and trinormal spherical images of a time-like curve and observed that such spherical curves are space-like curves. Thereafter, we determined relations between Frenet–Serret invariants of spherical images and the base curve. In the light of the obtained results, we also express some open problems for further studies.

The involute of a given curve $\alpha = \alpha(s)$, which is a well-known concept in classical differential geometry, is defined by

$$\gamma = \alpha + \lambda T_\alpha, \tag{5.1}$$

where λ is $(c - s_\alpha)$ for constant c [6]. Recently, involutes of a space-like curve in Minkowski space-time were studied by [6]. Similary, if we express the involute of the tangent spherical image, we get

$$\gamma = \delta + \lambda T_\delta \tag{5.2}$$

or, in other words,

$$\gamma = T + \lambda N. \tag{5.3}$$

This equation belongs to the spherical image's involute with respect to the base curve. Besides, in an analogous way we may express the same case for the trinormal spherical image by

$$\zeta = \varphi + \lambda T_\varphi \tag{5.4}$$

or

$$\zeta = B_2 + \lambda B_1. \tag{5.5}$$

So, one can determine the relations between Frenet–Serret invariants of the involute of the tangent spherical image and the Frenet–Serret apparatus of the base curve.

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Pseudohüperboolse ruumi H_0^3 ajasarnase joone puutuja ja trinormaali sfäärilised kujutised

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Kasutades Frenet-Serret' valemeid, on uuritud ruumi H_0^3 ajasarnaste joonte sfäärilisi kujutisi. On tõestatud, et need kujutised osutuvad ruumisarnasteks joonteks.