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MATHEMATICS

On $|A|_k$ summability factors of infinite series

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Abstract. In an earlier paper (Rhoades, B. E. and Savaş, E. Some necessary conditions for absolute matrix summability factors. *Indian J. Pure Appl. Math.*, 2002, **33**(7), 1003–1009) the authors obtained necessary conditions for the series $\sum a_n$ to be absolutely summable of order k by a triangular matrix. In this paper we present sufficient conditions for absolute matrix summability factors. As a corollary we obtain a result of N. Singh (On $|\overline{N}, p_n|$ summability factors of infinite series. *Indian J. Math.*, 1968, **10**, 19–24).

Key words: absolute summability, summability factors.

Let A be a lower triangular matrix, $\{s_n\}$ any sequence. Then

$$A_n := \sum_{\nu=0}^{n} a_{n\nu} s_\nu .$$

A series $\sum a_n$, with partial sums s_n, is said to be summable $|A|_k, k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |A_n - A_{n-1}|^k < \infty .$$

We may associate with A two lower triangular matrices $\overline{A}$ and $\hat{A}$ as follows:

$$\bar{a}_{n\nu} = \sum_{r=\nu}^{n} a_{nr}, \qquad n, \nu = 0, 1, 2, \ldots,$$

and

$$\hat{a}_{n\nu} = \bar{a}_{n\nu} - \bar{a}_{n-1,\nu}, \qquad n = 1, 2, 3, \ldots .$$

In our previous work on absolute summability [1,2] we have assumed that the triangular matrix A had row sums one. This condition rules out the consideration of factorable matrices that are not weighted mean matrices. A lower triangular matrix A is said to be factorable if the nonzero terms a_{nk} can be written as $a_n b_k$ for $0 \leq k \leq n$. If A is a factorable matrix with row sums one, then it is a weighted mean matrix.

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We shall first establish a general theorem for triangular matrices, which also applies to factorable matrices which need not be weighted mean matrices, and then we shall specialize this result to triangular matrices with row sums one.

A series $\sum a_n$ with partial sums s_n is said to be bounded $|A|_k, k \geq 1$, if $\sum_{\nu=1}^{m} a_{m\nu}|s_\nu|^k = O(1)$ as $m \to \infty$.

Theorem 1. *Let A be a lower triangular matrix satisfying*

(i) $\displaystyle\sum_{\nu=1}^{n-1} |\Delta_\nu \hat{a}_{n\nu}| = O(|a_{nn}|), n \geq 2,$

(ii) $\displaystyle\sum_{n=\nu+1}^{m+1} |\Delta_\nu \hat{a}_{n\nu}| = O(|a_{\nu\nu}|), m \geq \nu,$

(iii) $n|a_{nn}| = O(1),$

(iv) $|a_{\nu r} - a_{\nu+1,r}| = O(|a_{\nu+1,\nu+1}a_{\nu r}|), \qquad 0 \leq r \leq \nu,$

(v) $\displaystyle\sum_{\nu=1}^{n-1} |a_{\nu\nu}\hat{a}_{n,\nu+1}| = O(|a_{nn}|), n \geq 2,$ *and*

(vi) $\displaystyle\sum_{n=\nu+1}^{m+1} |\hat{a}_{n,\nu+1}| = O(1), \quad m \geq \nu\ldots.$

If $\sum a_n$ is bounded $|A|_k$ and $\{\lambda_n\}$ is a bounded nonzero sequence satisfying

(vii) $\displaystyle\sum_{n=1}^{m} |a_{nn}||\lambda_n|^k = O(1),$ *and*

(viii) $|\Delta|\lambda_n|^k| = O(|a_{nn}||\lambda_n|^k),$

then the series $\sum a_n\lambda_n$ is summable $|A|_k, k > 1$.

Proof. Let (y_n) be the nth term of the A-transform of $\sum_{i=0}^{n} \lambda_i a_i$. Then

$$\begin{aligned} y_n &= \sum_{i=0}^{n} a_{ni}s_i = \sum_{i=0}^{n} a_{ni} \sum_{\nu=0}^{i} \lambda_\nu a_\nu \\ &= \sum_{\nu=0}^{n} \lambda_\nu a_\nu \sum_{i=\nu}^{n} a_{ni} = \sum_{\nu=0}^{n} \bar{a}_{n\nu}\lambda_\nu a_\nu \end{aligned}$$

and, for $n > 0$,

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n} (\bar{a}_{n\nu} - \bar{a}_{n-1,\nu})\lambda_\nu a_\nu = \sum_{\nu=0}^{n} \hat{a}_{n\nu}\lambda_\nu a_\nu.$$

Using Abel's transformation, we have, for $n > 1$,

$$\begin{aligned} Y_n &:= \sum_{\nu=1}^{n-1} (\Delta_\nu \hat{a}_{n\nu})\lambda_\nu s_\nu + \sum_{\nu=1}^{n-1} \hat{a}_{n,\nu+1}(\Delta\lambda_\nu)s_\nu + a_{nn}\lambda_n s_n \\ &= T_{n1} + T_{n2} + T_{n3}, \quad \text{say.} \end{aligned}$$

Since Y_1 is bounded, in order to prove our theorem, it is sufficient, by Minkowski's inequality, to show that

$$\sum_{n=2}^{\infty} n^{k-1}|T_{nr}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3.$$

Using Hölder's inequality and (i), (iii), and (ii),

$$\begin{aligned}
I_1 &:= \sum_{n=2}^{m+1} n^{k-1}|T_{n1}|^k = \sum_{n=2}^{m} n^{k-1}\Big|\sum_{\nu=1}^{n-1}\Delta_\nu \hat{a}_{n\nu}\lambda_\nu s_\nu\Big|^k \\
&= O(1)\sum_{n=2}^{m+1} n^{k-1}\Big(\sum_{\nu=2}^{n-1}|\Delta_\nu \hat{a}_{n\nu}||\lambda_\nu||s_\nu|\Big)^k \\
&= O(1)\sum_{n=2}^{m+1} n^{k-1}\Big(\sum_{\nu=1}^{n-1}|\Delta_\nu \hat{a}_{n\nu}||\lambda_\nu|^k|s_\nu|^k\Big) \\
&\qquad \times \Big(\sum_{\nu=1}^{n-1}|\Delta_\nu \hat{a}_{n\nu}|\Big)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}(n|a_{nn}|^{k-1}\sum_{\nu=1}^{n-1}|\Delta_\nu \hat{a}_{n,\nu}||\lambda_\nu|^k|s_\nu|^k \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_\nu|^k|s_\nu|^k\sum_{n=\nu+1}^{m+1}|\Delta_\nu \hat{a}_{n\nu}| \\
&= O(1)\sum_{\nu=1}^{m}|a_{\nu\nu}||\lambda_\nu|^k|s_\nu|^k.
\end{aligned}$$

Using the boundedness of $\sum a_n$ and $\{\lambda_n\}$, (iv), (viii), and (vii),

$$\begin{aligned}
I_1 &= O(1)\sum_{\nu=1}^{m}|\lambda_\nu|^k\Big[\sum_{i=0}^{\nu}|a_{\nu i}||s_i|^k - \sum_{i=0}^{\nu-1}|a_{\nu i}||s_i|^k\Big] \\
&= O(1)\Big[\sum_{\nu=1}^{m}|\lambda_\nu|^k\sum_{i=0}^{\nu}|a_{\nu i}||s_i|^k - \sum_{\nu=0}^{m-1}|\lambda_{\nu+1}|^k\sum_{i=0}^{\nu}|a_{\nu+1,i}||s_i|^k\Big] \\
&\le O(1)\Big[|\lambda_m|^k\sum_{i=0}^{m}|a_{mi}||s_i|^k \\
&\qquad + \sum_{\nu=1}^{m-1}\Big(|\lambda_\nu|^k\sum_{i=0}^{\nu}|a_{\nu i}||s_i|^k - |\lambda_{\nu+1}|^k\sum_{i=0}^{\nu}|a_{\nu+1,i}||s_i|^k\Big)\Big] \\
&\le O(1)+O(1)\Big[\sum_{\nu=1}^{m-1}\Big(|\lambda_\nu|^k - |\lambda_{\nu+1}|^k\Big)\sum_{i=0}^{\nu}|a_{\nu i}||s_i|^k \\
&\qquad + \sum_{\nu=1}^{m-1}|\lambda_{\nu+1}|^k\sum_{i=0}^{\nu}|a_{\nu i}-a_{\nu+1,i}||s_i|^k\Big] \\
&= O(1)+O(1)\sum_{\nu=1}^{m-1}|\Delta(|\lambda_\nu|^k)| + O(1)\sum_{\nu=1}^{m-1}|\lambda_{\nu+1}|^k|a_{\nu+1,\nu+1}|\sum_{i=0}^{\nu}|a_{\nu i}||s_i|^k \\
&= O(1)+O(1)\sum_{\nu=1}^{m-1}|a_{\nu\nu}||\lambda_\nu|^k + O(1)\sum_{\nu=1}^{m-1}|a_{\nu+1,\nu+1}||\lambda_{\nu+1}|^k \\
&= O(1).
\end{aligned}$$

Using (viii), Hölder's inequality, (v), (iii), and (vi),

$$
\begin{aligned}
I_2 := \sum_{n=2}^{m+1} n^{k-1}|T_{n2}|^k &= \sum_{n=2}^{m+1} n^{k-1}\Big|\sum_{\nu=1}^{n-1} \hat{a}_{n,\nu+1}(\Delta\lambda_\nu)s_\nu\Big|^k \\
&\leq \sum_{n=2}^{m+1} n^{k-1}\Big[\sum_{\nu=1}^{n-1} |\hat{a}_{n,\nu+1}||\Delta\lambda_\nu||s_\nu|\Big]^k \\
&= O(1)\sum_{n=2}^{m+1} n^{k-1}\Big[\sum_{\nu=1}^{n-1} |\lambda_\nu|^k|s_\nu|^k|a_{\nu\nu}\hat{a}_{n,\nu+1}|\Big] \times \Big[\sum_{\nu=1}^{n-1} |a_{\nu\nu}\hat{a}_{n,\nu+1}|\Big]^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} (n|a_{nn}|)^{k-1}\sum_{\nu=1}^{n-1} |\hat{a}_{n,\nu+1}a_{\nu\nu}||\lambda_\nu|^k|s_\nu|^k \\
&= O(1)\sum_{\nu=1}^{m} a_{\nu\nu}|\lambda_\nu|^k|s_\nu|^k\sum_{n=i+1}^{m+1} |\hat{a}_{n,\nu+1}| \\
&= O(1)\sum_{\nu=1}^{m} a_{\nu\nu}|\lambda_\nu|^k|s_\nu|^k = O(1),
\end{aligned}
$$

as in the proof of I_1.

Finally, using (iii),

$$
\begin{aligned}
\sum_{n=1}^{m} n^{k-1}|T_{n3}|^k &= \sum_{n=1}^{m} n^{k-1}\Big|a_{nn}\lambda_n s_n\Big|^k \\
&= O(1)\sum_{n=1}^{m} n^{k-1}|a_{nn}|^k|\lambda_n|^k|s_n|^k \\
&= O(1)\sum_{n=1}^{m} (n|a_{nn}|)^{k-1}a_{nn}|\lambda_n|^{k-1}|\lambda_n||s_n|^k \\
&= O(1)\sum_{n=1}^{m} |a_{nn}\lambda_n||s_n|^k \\
&= O(1),
\end{aligned}
$$

as in the proof of I_1. □

Theorem 2. *Let A be a lower triangular matrix with nonnegative entries satisfying*
(ix) $\bar{a}_{n0} = 1, n = 0, 1, 2, \ldots,$
(x) $a_{n-1,\nu} \geq a_{n\nu}$ *for* $n \geq \nu + 1$,
and conditions (iii)–(v) *of Theorem* 1.

If $\sum a_n$ *is bounded* $|A|_k$ *and* $\{\lambda_n\}$ *is a bounded nonzero sequence satisfying conditions* (vii) *and* (viii) *of Theorem* 1, *then the series* $\sum a_n\lambda_n$ *is summable* $|A|_k, k > 1$.

Proof. Upon examining the conditions of Theorem 1 it is clear that one needs to show that conditions (ix) and (x) imply that $\hat{a}_{n,\nu+1} \geq 0$ and that conditions (i), (ii), and (vi) of Theorem 1 hold.

Using the definitions of $\hat{a}_{n\nu}$ and $\bar{a}_{n\nu}$, and (ix) and (x),

$$
\begin{aligned}
\Delta_\nu\hat{a}_{n\nu} &= \hat{a}_{n\nu} - \hat{a}_{n,\nu+1} \\
&= \bar{a}_{n\nu} - \bar{a}_{n-1,\nu} - \bar{a}_{n,\nu+1} + \bar{a}_{n-1,\nu+1} \\
&= a_{n\nu} - a_{n-1,\nu} \leq 0.
\end{aligned}
$$

Therefore

$$\sum_{\nu=1}^{n-1} |\Delta_\nu \hat{a}_{n\nu}| = \sum_{\nu=1}^{n-1} (a_{n-1,\nu} - a_{n\nu})$$
$$= 1 - 1 + a_{n0} + a_{nn} \le a_{nn},$$

and condition (i) of Theorem 1 is true.

Also,

$$\sum_{n=\nu+1}^{m+1} |\Delta_\nu \hat{a}_{n\nu}| = \sum_{n=\nu+1}^{m+1} (a_{n-1,\nu} - a_{\nu\nu}) = a_{\nu\nu} - a_{m+1,\nu} \le a_{\nu\nu},$$

and condition (ii) of Theorem 1 is true.

Finally,

$$\sum_{n=\nu+1}^{m+1} |\hat{a}_{n,\nu+1}| = \sum_{n=\nu+1}^{m+1} \sum_{i=0}^{\nu} (a_{n-1,i} - a_{ni})$$
$$= \sum_{i=0}^{\nu} \sum_{n=\nu+1}^{m+1} (a_{n-1,i} - a_{ni})$$
$$= \sum_{i=0}^{\nu} (a_{\nu i} - a_{m+1,i}) \le \sum_{i=0}^{\nu} a_{\nu i} = 1,$$

and condition (vi) of Theorem 1 is satisfied. □

If one is dealing with absolute summability of order 1, then conditions (iii) and (iv) of Theorem 1 are not needed.

Theorem 3. *Let A be a lower triangular matrix satisfying conditions* (ii), (iv), *and* (vi) *of Theorem* 1. *If $\sum a_n$ is bounded $|A|$ and $\{\lambda_n\}$ is a bounded nonzero sequence satisfying conditions* (vii) *and* (viii) *of Theorem* 1 *(with $k = 1$), then the series $\sum a_n \lambda_n$ is summable $|A|$.*

Proof. This can be proved by using the techniques similar to that of Theorem 1. So we omit it. □

Theorem 4. *Let A be a lower triangular matrix with nonnegative entries satisfying conditions* (ix) *and* (x) *of Theorem* 2 *and condition* (iv) *of Theorem* 1. *If $\sum a_n$ is bounded $|A|$ and $\{\lambda_n\}$ is a bounded nonzero sequence satisfying conditions* (vii) *and* (viii) *of Theorem* 1, *then the the series $\sum a_n \lambda_n$ is summable $|A|$.*

Proof. As in the proof of Theorem 2, conditions (ix) and (x) of Theorem 2 imply conditions (i) and (ii) of Theorem 1. □

A weighted mean matrix is a lower triangular matrix with entries $a_{nk} = p_k/P_n$, where $\{p_k\}$ is a nonnegative sequence with $p_0 > 0$ and $P_n := \sum_{k=0}^n p_k$. A weighted mean matrix is denoted by $(\overline{N}, p_n)$.

Corollary 1. *Let $\{p_n\}$ be a positive sequence such that $P_n := \sum_{k=0}^n p_k \to \infty$, and satisfies*

(xi) $np_n = O(P_n)$.

If $\sum a_n \lambda_n$ is bounded $|\overline{N}, p_n|_k$ and $\{\lambda_n\}$ is a bounded nonzero sequence satisfying

(xii) $\displaystyle\sum_{n=1}^{\infty} \frac{p_n}{P_n} |\lambda_n|^k = O(1),$ *and*

(xiii) $|\Delta\lambda_n| = O\left(\dfrac{p_n}{P_n}|\lambda_n|\right),$

then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k, k \ge 1$.

Proof. Conditions (i), (iv), and (v) of Theorem 1 are automatically satisfied for any weighted mean method. Conditions (iii), (vii), and (viii) of Theorem 1 become, respectively, conditions (xi), (xii), and (xiii) of Corollary 1. □

Corollary 2. *If $\sum a_n$ is bounded $|\overline{N}, p|$ and $\{\lambda_n\}$ is a bounded nonzero sequence satisfying*

(a) $\displaystyle\sum_{n=1}^{m} \frac{p_n}{P_n}|\lambda_n| = O(1)$, *and*

(b) $\displaystyle\frac{P_n}{p_n}|\Delta\lambda_n| = O(|\lambda_n|)$,

then $\sum a_n\lambda_n$ is summable $|\overline{N}, p|$.

Proof. A weighted mean matrix automatically satisfies conditions (i)–(iii) of Theorem 1. Conditions (vii) and (viii) of Theorem 1 reduce to conditions (a) and (b) of Corollary 2, respectively.

Corollary 2 is a result of [3]. □

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Lõpmatute ridade $|A|_k$-summeeruvusteguritest

B. E. Rhoades ja Ekrem Savaş

Olgu A kolmnurkne maatriks ja $k \geq 1$. Artiklis on defineeritud rea $|A|_k$-summeeruvuse ja $|A|_k$-tõkestatuse mõisted. On leitud piisavad tingimused selleks, et arvud λ_n oleksid maatriksi A k-järku absoluutse summeeruvuse tegurid ehk rida $\sum_n a_n\lambda_n$, kus (λ_n) on tõkestatud jada, oleks $|A|_k$-summeeruv, kui rida $\sum_n a_n$ on $|A|_k$-tõkestatud. Saadud tulemus üldistab N. Singhi tulemust Rieszi kaalutud keskmiste menetluse $(\overline{N}, p_n)$ absoluutse summeeruvuse tegurite kohta [3].