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MECHANICS

Asymptotics and stabilization for dynamic models of nonlinear beams

Dedicated to Jüri Engelbrecht with friendship and admiration

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Abstract. We prove that the von Kármán model for vibrating beams can be obtained as a singular limit of a modified Mindlin–Timoshenko system when the modulus of elasticity in shear k tends to infinity, provided a regularizing term through a fourth-order dispersive operator is added. We also show that the energy of solutions for this modified Mindlin–Timoshenko system decays exponentially, uniformly with respect to the parameter k, when suitable damping terms are added. As $k \to \infty$, one deduces the uniform exponential decay of the energy of the von Kármán model.

Key words: vibrating beams, Mindlin–Timoshenko system, von Kármán system, singular limit, uniform stabilization.

The Mindlin–Timoshenko system of equations is a mathematical model for describing the transverse vibrations of beams. It is more accurate than the Euler–Bernoulli theory, since it also takes transverse shear effects into account. It is used, for example, to model aircraft wings (see, for instance, [3]).

For a beam of length $L > 0$, this one-dimensional nonlinear system reads as

$$\begin{aligned}
&\frac{\rho h^3}{12}\phi_{tt} - D\phi_{xx} + k(\phi + \psi_x) = 0 && \text{in} \quad Q,\\
&\rho h \psi_{tt} - k(\phi + \psi_x)_x - Eh\left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x = 0 && \text{in} \quad Q,\\
&\rho h \eta_{tt} - Eh\left(\eta_x + \frac{1}{2}\psi_x^2\right)_x = 0 && \text{in} \quad Q,
\end{aligned} \tag{1}$$

where $Q = (0,L) \times (0,T)$, with $(0,L)$ being the segment occupied by the beam, and T a given positive time.

In system (1), subscripts mean partial derivatives. The unknowns $\phi = \phi(x,t)$, $\psi = \psi(x,t)$, and $\eta = \eta(x,t)$ represent, respectively, the angle of rotation, the vertical displacement, and the longitudinal displacement at time t of the cross section located x units from the end-point $x = 0$. The constant $h > 0$ represents the thickness of the beam which, in this model, is considered to be small and uniform with respect to x. The constant ρ is the mass density per unit volume of the beam and $D = Eh^3/12\left(1-\mu^2\right)$ is called the modulus of flexural rigidity, where E is Young's modulus and μ is Poisson's ratio, $0 < \mu < 1/2$. The

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parameter k is the so-called modulus of elasticity in shear. It is given by the expression $k = \hat{k}Eh/2(1+\mu)$, where $\hat{k}$ is a shear correction coefficient. For more details concerning the Mindlin–Timoshenko hypotheses and governing equations see Lagnese and Lions [8].

A large amount of literature is available on this model, addressing problems of the existence, uniqueness, and asymptotic behaviour in time when some damping effects are considered, as well as some other important properties (see [6,8] and references therein). These models are also relevant when analysing the effect of the microstructure on the dispersive properties of materials (see [4]).

When one assumes the linear filament of the beam to remain orthogonal to the deformed middle surface, the transverse shear effects are neglected, and one obtains the so-called von Kármán system (see [8]):

$$\begin{aligned} &\rho h \psi_{tt} - \frac{\rho h^3}{12}\psi_{xxtt} + D\psi_{xxxx} - Eh\left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x = 0 \quad \text{in} \quad Q, \\ &\rho h \eta_{tt} - Eh\left(\eta_x + \frac{1}{2}\psi_x^2\right)_x = 0 \quad \text{in} \quad Q. \end{aligned} \tag{2}$$

Note that neglecting the shear effects of the beam is formally equivalent to considering the modulus $k \to \infty$ in system (1), since k is inversely proportional to the shear angle. There is also an extensive literature about system (2) (see [5–8,10] and references therein).

In the context of systems of elasticity, it is well known that reduced models are often singular limits of more complete ones. The issue of the dependence of the decay rate or, more generally, the stabilizability properties with respect to this singular parameter is also an interesting subject that has been the object of intensive research. In the present paper this issue is addressed in the context of nonlinear models for vibrating beams. Our interest is to analyse the asymptotic limit of the nonlinear Mindlin–Timoshenko system (1), when the shear modulus k tends to infinity. This problem was mentioned in [8], where it was conjectured that *system* (1) *approaches*, *in some sense*, *the von Kármán system* (2) *as* $k \to \infty$.

In order to achieve our results, we need to perturb system (1) by a regularizing term. More precisely, instead of system (1) we consider the modified system

$$\begin{aligned} &\frac{\rho h^3}{12}\phi_{tt} - D\phi_{xx} + k(\phi + \psi_x) = 0 \quad \text{in} \quad Q, \\ &\rho h \psi_{tt} - k(\phi + \psi_x)_x - Eh\left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x + \frac{\sigma}{k}\psi_{xxxx} = 0 \quad \text{in} \quad Q, \\ &\rho h \eta_{tt} - Eh\left(\eta_x + \frac{1}{2}\psi_x^2\right)_x = 0 \quad \text{in} \quad Q, \end{aligned} \tag{3}$$

with $\sigma > 0$ being a constant characterizing physical properties of the beam, and show the following:

(i) system (3) converges to system (2) as $k \to \infty$;

(ii) by adding appropriate damping terms, there is a uniform (with respect to k) rate of decay for the total energy of the solutions of (3) as $t \to \infty$.

As a consequence of this analysis, we obtain a decay rate for the total energy of the solutions of the von Kármán system (as $t \to \infty$) as a singular limit of the uniform (with respect to k) decay rate of the energy of the Mindlin–Timoshenko system. Note, however, that our results need the fourth-order regularizing term in the component ψ for compactness purposes. Reproducing the results of this paper without that regularizing term is an interesting open problem. These results are proved under suitable boundary and initial conditions that we shall make precise below.

The connections between these two systems and the asymptotic limit problem of passing to the limit as k tends to infinity have been recently investigated in a number of different situations. For the linear case, that is, in the absence of the term $\left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x$, and without the equation for η, it was proved in [8] (see also [1]) that the linear Mindlin–Timoshenko system

$$\begin{aligned} &\frac{\rho h^3}{12}\phi_{tt} - D\phi_{xx} + k(\phi + \psi_x) = 0, \\ &\rho h \psi_{tt} - k(\phi + \psi_x)_x = 0, \end{aligned} \tag{4}$$

approaches, as $k \to \infty$, the Kirchhoff equation

$$\rho h \psi_{tt} - \frac{\rho h^3}{12}\psi_{xxtt} + D\psi_{xxxx} = 0, \tag{5}$$

under various boundary conditions. Controllability properties of these systems have also been studied in [1]. The problem of singular perturbations and stabilization related to the nonlinear von Kármán model has also been intensively investigated. We refer, for instance, to [9,11–13], where these issues are addressed under various boundary conditions.

We consider system (3) under Dirichlet boundary conditions for ϕ, clamped ends for ψ, and Neumann boundary conditions on η:

$$\phi(0,\cdot) = \phi(L,\cdot) = \psi(0,\cdot) = \psi(L,\cdot) = \psi_x(0,\cdot) = \psi_x(L,\cdot) = \eta_x(0,\cdot) = \eta_x(L,\cdot) = 0 \quad \text{on} \quad (0,T), \tag{6}$$

and the initial data

$$\begin{aligned} &\{\phi(\cdot,0), \psi(\cdot,0), \eta(\cdot,0)\} = \{\phi_0, \psi_0, \eta_0\} \quad \text{in} \quad (0,L), \\ &\{\phi_t(\cdot,0), \psi_t(\cdot,0), \eta_t(\cdot,0)\} = \{\phi_1, \psi_1, \eta_1\} \quad \text{in} \quad (0,L). \end{aligned} \tag{7}$$

These models can be proved to be well-behaved in a suitable functional framework. We consider the Hilbert space $\mathscr{X}$:

$$\mathscr{X} = H_0^1(0,L) \times L^2(0,L) \times H_0^2(0,L) \times L^2(0,L) \times V \times H,$$

where $V = H^1(0,L) \cap H$ and $H = \left\{v \in L^2(0,L);\ \int_0^L v(x)\,dx = 0\right\}$, equipped with the norm

$$\|\{u_1,u_2,v_1,v_2,w_1,w_2\}\|_{\mathscr{X}}^2 = D|u_{1x}|^2 + \frac{\rho h^3}{12}|u_2|^2 + k|u_1 + v_{1x}|^2 + \frac{\sigma}{k}|v_{1xx}|^2 + \rho h|v_2|^2 + Eh|w_{1x}|^2 + \rho h|w_2|^2,$$

where $|\cdot|$ denotes the norm in $L^2(0,L)$.

Theorem 1. *If* $(\phi_0,\phi_1,\psi_0,\psi_1,\eta_0,\eta_1) \in \mathscr{X}$, *then problem* (3), (6), (7) *has a unique weak solution in the class*

$$\{\phi,\psi,\eta\} \in C^0\left([0,\infty); H_0^1(0,L) \times H_0^2(0,L) \times V\right) \cap C^1\left([0,\infty); \left[L^2(0,L)\right]^2 \times H\right). \tag{8}$$

Moreover, the energy $E_k(t)$, *given by*

$$\begin{aligned} E_k(t) = &\frac{1}{2}\Bigg(\frac{\rho h^3}{12}|\phi_t(t)|^2 + \rho h|\psi_t(t)|^2 + \rho h|\eta_t(t)|^2 + D|\phi_x(t)|^2 \\ &+ k|\phi(t) + \psi_x(t)|^2 + Eh\left|\eta_x(t) + \frac{1}{2}[\psi_x(t)]^2\right|^2 + \frac{\sigma}{k}|\psi_{xx}(t)|^2\Bigg), \end{aligned} \tag{9}$$

satisfies

$$E_k(t) = E_k(0), \quad \forall t \geq 0. \tag{10}$$

Theorem 2. *Let* $\{\phi^k, \psi^k, \eta^k\}$ *be the unique solution of* $(3), (6), (7)$ *with data* $\{\phi_0, \phi_1, \psi_0, \psi_1, \eta_0, \eta_1\} \in \mathscr{X}$ *satisfying*

$$\phi_0 + \psi_{0x} = 0 \quad \textit{in} \quad (0,L). \tag{11}$$

Then, letting $k \to \infty$, *one gets*

$$\{\phi^k, \psi^k, \eta^k\} \to \{-\psi_x, \psi, \eta\} \textit{ weak* in } L^\infty\left(0,T;\left[H_0^1(0,L)\right]^2 \times V\right),$$

where $\{\psi, \eta\}$ *solves the von Kármán system*

$$\begin{array}{ll} \rho h \psi_{tt} - \frac{\rho h^3}{12}\psi_{xxtt} + D\psi_{xxxx} - Eh\left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x = 0 & \textit{in} \quad Q, \\ \rho h \eta_{tt} - Eh\left(\eta_x + \frac{1}{2}\psi_x^2\right)_x = 0 & \textit{in} \quad Q, \\ \psi(0,\cdot) = \psi(L,\cdot) = \psi_x(0,\cdot) = \psi_x(L,\cdot) = \eta_x(0,\cdot) = \eta_x(L,\cdot) = 0 & \textit{on} \quad (0,T), \\ \psi(\cdot,0) = \psi_0, \quad \left(\psi - \frac{h^2}{12}\psi_{xx}\right)_t(0) = \psi_1 + \frac{h^2}{12}\phi_{1x} & \textit{in} \quad (0,L), \\ \eta(\cdot,0) = \eta_0, \quad \eta_t(\cdot,0) = \eta_1 & \textit{in} \quad (0,L). \end{array} \tag{12}$$

One also has that the exponential decay for the energy

$$E(t) = \frac{1}{2}\left(\rho h[|\psi_t(t)|^2 + |\eta_t(t)|^2] + \frac{\rho h^3}{12}|\psi_{xt}(t)|^2 + D|\psi_{xx}(t)|^2 + Eh\left|\eta_x(t) + \frac{1}{2}[\psi_x(t)]^2\right|^2\right),$$

associated to the solution of the von Kármán system

$$\begin{array}{ll} \rho h \psi_{tt} - \frac{\rho h^3}{12}\psi_{xxtt} + D\psi_{xxxx} - Eh\left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x + \beta\psi_t - \alpha\psi_{xxt} = 0 & \text{in} \quad Q, \\ \rho h \eta_{tt} - Eh\left(\eta_x + \frac{1}{2}\psi_x^2\right)_x + \gamma\eta_t = 0 & \text{in} \quad Q, \\ \psi(0,\cdot) = \psi(L,\cdot) = \psi_x(0,\cdot) = \psi_x(L,\cdot) = \eta_x(0,\cdot) = \eta_x(L,\cdot) = 0 & \text{on} \quad (0,T), \\ \{\psi(\cdot,0), \psi_t(\cdot,0), \eta(\cdot,0), \eta_t(\cdot,0)\} = \{\psi_0, \psi_1, \eta_0, \eta_1\} & \text{in} \quad (0,L), \end{array} \tag{13}$$

can be obtained as a limit (as $k \to \infty$) of the uniform stabilization of the perturbed Mindlin–Timoshenko system

$$\begin{array}{ll} \frac{\rho h^3}{12}\phi_{tt} - D\phi_{xx} + k(\phi + \psi_x) + \alpha\phi_t = 0 & \text{in} \quad Q, \\ \rho h \psi_{tt} - k(\phi + \psi_x)_x - Eh\left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x + \frac{\sigma}{k}\psi_{xxxx} + \beta\psi_t = 0 & \text{in} \quad Q, \\ \rho h \eta_{tt} - Eh\left(\eta_x + \frac{1}{2}\psi_x^2\right)_x + \gamma\eta_t = 0 & \text{in} \quad Q, \\ \phi(0,\cdot) = \phi(L,\cdot) = \psi(0,\cdot) = \psi(L,\cdot) = \eta_x(0,\cdot) = \eta_x(L,\cdot) = 0 & \text{on} \quad (0,T), \\ \{\phi(\cdot,0), \psi(\cdot,0), \eta(\cdot,0)\} = \{\phi_0, \psi_0, \eta_0\} & \text{in} \quad (0,L), \\ \{\phi_t(\cdot,0), \psi_t(\cdot,0), \eta_t(\cdot,0)\} = \{\phi_1, \psi_1, \eta_1\} & \text{in} \quad (0,L), \end{array} \tag{14}$$

where α, β, and γ are positive constants.

System (13) can be obtained as a limit, when $k \to \infty$, of system (14).

Notice that the energy of system (14) is given by (9) and it is dissipated according to the law

$$\frac{dE_k(t)}{dt} = -\left(\alpha\left|\phi_t(t)\right|^2 + \beta\left|\psi_t(t)\right|^2 + \gamma\left|\eta_t(t)\right|^2\right). \tag{15}$$

This energy decays exponentially (as $t \to \infty$) uniformly with respect to k:

Theorem 3. *Let* $\{\phi, \psi, \eta\}$ *be the global solution of system* (14) *for data* $\{\phi_0, \phi_1, \psi_0, \psi_1, \eta_0, \eta_1\} \in \mathscr{X}$ *satisfying* (11)*. Then there exists a constant* $\omega > 0$ *such that*

$$E_k(t) \leq 4E_k(0)\, e^{-\frac{\omega}{2}t}, \ \forall t \geq 0. \tag{16}$$

As a consequence of inequality (16), letting $k \to \infty$, one recovers the exponential decay of the energy $E(t)$, associated to system (13). This is in agreement with the results obtained in [9].

The interested reader is referred to [2] for further details on this topic and, in particular, for the proofs of the results presented here.

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Mittelineaarsete talade dünaamiliste mudelite asümptootika ja stabiliseerimine

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Talade piki- ja põikivõnkumised kujutavad olulist probleemi mitte ainult teoreetilisest, vaid ka praktilisest aspektist. Seni lahendamata küsimuseks on olnud erinevate matemaatiliste mudelite seostatus ja sellest tulenevalt ka nende rakenduspiirkondade määramine. Artiklis on vaadeldud Mindlini-Timoshenko mudeli (nihkedeformatsioonid on arvesse võetud) ja von Kármáni mudeli seostatust. On tuletatud Mindlini-Timoshenko mudeli asümptootika, mis nihkemooduli kasvades läheneb von Kármáni mudelile. On tõestatud teoreemid, mille tingimuste täitmine nõuab sumbuvusteguri arvestamist ja mis kindlustab siis asümptootika stabiilse käitumise singulaarse piirjuhtumini, s.o von Kármáni mudelini.