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MECHANICS

Interaction of solitary pulses in active dispersive–dissipative media

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Abstract. We examine weak interaction and formation of bound states of pulses for the generalized Kuramoto–Sivashinsky (gKS) equation, which is one of the simplest prototypes describing active media with energy supply, dissipation, dispersion, and nonlinearity. We derive a system of ordinary differential equations describing the leading-order dynamics of the pulses of the gKS equation and prove a criterion for the existence of a countable infinite or finite number of bound states. Our theory is corroborated by computations of the full equation.

Key words: dissipative solitons, solitary-pulse interaction, bound states, generalized Kuramoto–Sivashinsky equation.

1. INTRODUCTION

In the present paper, we consider the generalized Kuramoto–Sivashinsky (gKS) equation, also known as the Kuramoto–Sivashinsky–Korteweg–de Vries equation and the Benney equation (Feng et al., 2002),

$$\begin{gathered} h_t + hh_x + h_{xx} + \delta h_{xxx} + h_{xxxx} = 0, \\ (x,t) \in (-\infty,\infty) \times [0,\infty). \end{gathered} \tag{1}$$

This equation is one of the simplest prototypes modelling a nonlinear active medium with energy supply, energy dissipation, and dispersion and whose dynamics is dominated by localized nonlinear pulses. The gKS equation has been derived in many physical contexts, including plasma waves with dispersion due to finite ion banana width (Cohen et al., 1976), liquid films sheared by a turbulent gas (Jurman and McCready, 1989), falling liquid films in the presence of a viscous stress at the free surface (Oron and Edwards, 1993), liquid films flowing down a uniformly heated wall (Kalliadasis et al., 2003). In the context of the falling liquid film problem, x denotes a streamwise coordinate, t denotes time, and h denotes the scaled local film amplitude (i.e. the deviation of the film surface from the flat-film solution). We have recently derived this equation for a viscous thin film coating a vertical fibre to obtain a theoretical insight into the interaction of the droplike pulses and formation of bound states observed in that system (Duprat et al., 2009). The gKS model is obtained through a weakly nonlinear expansion for the Navier–Stokes equations and the corresponding boundary conditions at the fibre and the free surface written in cylindrical coordinates. A self-consistent derivation is possible only when the radius, R, is assumed to be large compared to the undisturbed film thickness, H_0, the Weber number, We, is assumed to be large, and the Reynolds number, Re, is taken to be small, more precisely,

$$R/H_0 = O(\varepsilon^{-1}), \quad We = O(\varepsilon^{-2}), \quad Re = O(\varepsilon), \tag{2}$$

where, $\varepsilon \ll 1$ is the so-called long-wave or film parameter and is typically defined as the ratio of H_0 to a lengthscale over which streamwise variations occur. By taking the amplitude deviation from H_0 to be of $O(\varepsilon^2)$ we then derive equation (1), where the dispersion parameter, δ, is defined by $\delta = 6/(We\,Re\,A^{1/2})$ and where $A = 8/(5We) + H_0^2/R^2$.

It is well known that for small δ the gKS equation exhibits complicated chaotic dynamics in both space and time. However, a sufficiently large δ arrests the spatio-temporal chaos such that the solution evolves into a regular array of pulses that interact indefinitely

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with each other through their tails (Kawahara, 1983). In this case, it is feasible to consider the solution as a superposition of such pulses and to develop a weak interaction theory. The idea of weak interaction theory for solitary pulses in other systems has been implemented, for example, by Ei (2002) and Sandstede (2002). As far as the gKS equation is concerned, previous efforts to develop weak interaction approaches include Elphick et al. (1991), Ei and Ohta (1994), and Chang and Demekhin (2002). However, all previous studies for the gKS equation appear to be either incomplete or sometimes overlook important details and subtleties. For instance, the spectrum of the adjoint operator of the equation linearized around a pulse has not been analysed carefully in this case. Our aim is to obtain a clear and complete understanding of the pulse interaction problem for the gKS equation and to scrutinize our results by detailed comparisons with computations.

The rest of the paper is organized as follows. In Section 2 we develop a weak interaction theory of solitary pulses for the gKS equation. In Section 3 we analyse bound states of the pulses and compare our theoretical results with computations. Section 4 is devoted to the discussion of theoretical results and conclusions.

2. PULSE-INTERACTION THEORY FOR THE gKS EQUATION

In a frame moving with the velocity c_δ of a solitary pulse, the gKS equation (1) takes the form

$$h_t - c_\delta h_x + hh_x + h_{xx} + \delta h_{xxx} + h_{xxxx} = 0. \qquad (3)$$

Let $h_0 = h_0(x)$ be a stationary pulse. It satisfies the steady version of (3). It can be shown that $h_0(x)$ tends to zero exponentially and monotonically as $x \to -\infty$ and it tends to zero exponentially either in an oscillatory manner or monotonically as $x \to \infty$ depending on whether δ is below or above a threshold value $\delta^* \approx 3.71$ (Kawahara and Toh, 1988). More specifically,

$$\begin{aligned} h_0(x) &\sim C_1 \mathrm{e}^{\lambda_1 x} \ \text{ as } \ x \to -\infty, \\ h_0(x) &\sim \mathrm{Re}(C_2 \mathrm{e}^{\lambda_2 x}) \ \text{ as } \ x \to \infty, \end{aligned} \qquad (4)$$

where C_1 is a real constant and C_2 is, in general, a complex one. Here, λ_1 and λ_2 are nonzero roots of the characteristic equation of the linearized stationary equation:

$$\lambda^3 + \delta\lambda^2 + \lambda - c_\delta = 0. \qquad (5)$$

As was shown by Kawahara and Toh (1988), for any value of δ there is one root that is real and positive, which we denote by λ_1. If δ is below the threshold value δ^*, there is a pair of complex conjugate roots, λ_2 and $\bar{\lambda}_2$, with negative real parts. Otherwise, if δ is above δ^*, there are two real roots with negative real parts. In this case, we denote the root with the larger real part by λ_2.

We assume that the solution, h, is described as a superposition of n quasi-stationary pulses $h_1, \ldots, h_n$ located at $x_1(t), \ldots, x_n(t)$, respectively; namely,

$$h_i(x,t) = h_0(x - x_i(t)), \qquad i = 1, \ldots, n, \qquad (6)$$

and a small overlap (or correction) function $\hat{h}$. Thus, we use the ansatz

$$h = \sum_{i=1}^{n} h_i + \hat{h} \qquad (7)$$

for the solution. Our aim is to derive a system of equations governing the locations of the pulses. We consider weak interaction assuming that the pulses are sufficiently separated and, therefore, that for each pulse it is sufficient to take into account its interaction with only immediate neighbours. More precisely, we assume that $l_i \equiv x_{i+1} - x_i = \log\alpha + O(1)$ for $i = 1, \ldots, n-1$, where $\alpha \ll 1$, and that the velocities of the pulses, x_i', $i = 1, \ldots, n$, and the overlap function, $\hat{h}$, are $O(\alpha)$. The linearized equation for the overlap function, $\hat{h}$, in the vicinity of the ith pulse takes the form

$$\hat{h}_t - x_1' h_{1x} = \mathscr{L}_1 \hat{h} - (h_1 h_2)_x \qquad (8)$$

for $i = 1$,

$$\hat{h}_t - x_i' h_{ix} = \mathscr{L}_i \hat{h} - (h_{i-1} h_i)_x - (h_i h_{i+1})_x \qquad (9)$$

for $2 \le i \le n$, and

$$\hat{h}_t - x_n' h_{nx} = \mathscr{L}_n \hat{h} - (h_{n-1} h_n)_x \qquad (10)$$

for $i = n$, where $\mathscr{L}_i$'s are linear operators defined by

$$\mathscr{L}_i f = c_\delta f_x - f_{xx} - \delta f_{xxx} - f_{xxxx} - (h_i f)_x \qquad (11)$$

for $i = 1, \ldots, n$. The formal adjoint operators, $\mathscr{L}_i^*$'s, with respect to the usual inner product in $L^2_{\mathbb{C}}$ are given by

$$\mathscr{L}_i^* f = -c_\delta f_x - f_{xx} + \delta f_{xxx} - f_{xxxx} + h_i f_x. \qquad (12)$$

We note that weak interaction theories with some rigorous results have been formulated for other systems that presume a stable primary soliton (which is not the case in the present study, unless the analysis is done in an exponentially weighted space, as will be discussed later on) in Ei (2002) and Sandstede (2002), for instance. Renormalization group techniques have been used to capture the leading-order pulse motion for well-separated pulses, for example, for nonlinear Schrödinger equation, e.g. Promislow (2002). In general, the ansatz (7) can be rigorously justified for certain equations by proving the existence of a centre manifold that is formed by pulse packets. For the existence of centre manifolds, semigroups generated by linearized operators (which are sectorial) should have exponential trichotomies.

Our next goal is to project the dynamics in the vicinity of the ith pulse onto the null space of $\mathscr{L}_i$ spanned by the translational mode h_{ix}. It can be shown that the null space of $\mathscr{L}_i^*$ is spanned by a constant and a function, which we denote by Ψ^i, tending exponentially to different constants as $x \to \pm\infty$. It can be easily seen that $\Psi^i(x) = \Psi^0(x - x_i)$, where Ψ^0 is the function tending to different constants as $x \to \pm\infty$ and belonging to the null space of $\mathscr{L}_0^*$, where $\mathscr{L}_0^*$ is defined by (12) with h_i replaced with h_0. Therefore, zero is not in the point spectrum of $\mathscr{L}_i^*$ on an infinite interval, and the projection onto the null space of $\mathscr{L}_i$ cannot be made in a straightforward way. Nevertheless, projections can be made rigorously by choosing an appropriate weighted space, namely,

$$L_a^2 = \{f : \mathrm{e}^{ax} f \in L_{\mathbb{C}}^2\}, \qquad (13)$$

where a is a positive sufficiently small number with the inner product $\langle f, g\rangle_a = \langle \mathrm{e}^{ax} f, \mathrm{e}^{ax} g\rangle$, where $\langle \cdot, \cdot\rangle$ denotes the usual inner product in $L_{\mathbb{C}}^2$. As noticed by Pego and Weinstein (1992), studying the spectrum of $\mathscr{L}_i$ in L_a^2 is equivalent to studying the spectrum of the operator defined by $\mathscr{L}_i^a f = \mathrm{e}^{ax}\mathscr{L}_i(\mathrm{e}^{-ax} f)$ on $L_{\mathbb{C}}^2$. With such a construction, zero is an isolated eigenvalue of both $\mathscr{L}_i^a$ and $\mathscr{L}_i^{a*}$ of both algebraic and geometric multiplicity unity and projections can be made in a straightforward way. Assuming that "$\hat{h}$ is free of translational modes", i.e. that it is in the null spaces of the projections, we arrive at the following system describing the dynamics of the locations of the pulses:

$$x_1' = S_1(x_2 - x_1), \qquad (14)$$

$$x_i' = S_2(x_i - x_{i-1}) + S_1(x_{i+1} - x_i), \quad 1 < i < n, \qquad (15)$$

$$x_n' = S_2(x_n - x_{n-1}), \qquad (16)$$

where

$$S_{1,2}(l) \equiv -\int_{-\infty}^{\infty} h_0(x + l/2) h_0(x - l/2) \Psi_x^0(x \pm l/2)\,\mathrm{d}x, \qquad (17)$$

where the subscript 1 or 2 on the left-hand side corresponds to the plus or minus sign, respectively, on the right-hand side.

3. BOUND-STATES THEORY AND COMPARISON WITH COMPUTATIONS

We note that (14)–(16) can be transformed to a system for the separation distances l_i's, and by studying its fixed points we can obtain the bound states of the pulses, i.e. the states when the pulses travel together with the same velocity.

For instance, for a bound state of two pulses we must have:

$$S_1(l_1) = S_2(l_1). \qquad (18)$$

The graphs of S_1 and S_2 are shown in Fig. 1(a) for $\delta = 0.4$. The abscissas of the intersection points indicate the separation distances for which bound states can be formed, and the ordinates indicate the corresponding velocities of the bound states relative to c_δ. It is also interesting to note that the ordinates of the intersection points are always negative, i.e. the velocity of a two-pulse bound state is always less than that of an individual pulse. Another interesting observation is that for $\delta = 0.4$ we, apparently, get a countable infinite number of bound states. This observation can be proved analytically by examining the behaviour of S_1 and S_2 as $l \to \infty$; namely, it can be shown that if δ is less than a threshold value $\tilde{\delta} \approx 0.85$, then there is a countable infinite number of intersections of S_1 and S_2. Otherwise, there is a finite number of intersections of S_1 and S_2. More specifically, it can be shown that $S_1(l) \sim D_1 \mathrm{e}^{-\lambda_1 l}$, $S_2(l) \sim \mathrm{Re}(D_2 \mathrm{e}^{\lambda_2 l})$ as $l \to \infty$, where D_1 is a real constant and D_2 is, in general, a complex number. Recall that λ_1 is the real positive root of (5) and λ_2 is the root of (5) with the maximum negative real part. Therefore, if $\lambda_1 + \mathrm{Re}\,\lambda_2 > 0$, there exists a countable infinite number of two-pulse bound states. Otherwise, if $\lambda_1 + \mathrm{Re}\,\lambda_2 < 0$, there exists a finite number of two-pulse bound states (or no bound states at all). The calculations for the gKS equation show that $\lambda_1 + \mathrm{Re}\,\lambda_2 < 0$ iff $\delta < \tilde{\delta} \approx 0.85$.

To validate the bound-states theory, we first compared our theoretical predictions with the numerically found bound states. To find two-pulse bound states numerically, we use a superposition of two individual pulses as an initial guess for our numerical method based on a pseudospectral representation of the derivatives and Newton iterations. Our numerical results show an excellent agreement with the theory. The numerical scheme converges only when the pulse separation distance for the initial condition was sufficiently close to a theoretically predicted bound-state separation distance, giving a perfect match between the theory and computations, especially for well separated pulses. Some results are presented in Fig. 2(a) and (b) for $\delta = 0.5$. We can see that for $l_1 \approx 9.4$ the theoretical and numerical results are graphically indistinguishable. For $l_1 \approx 6.7$, there is a reasonably small discrepancy between the theory and numerics.

Further, to validate the pulse-interaction theory, we solved the system (14)–(16) for $\delta = 0.4$ numerically for two pulses and compared the results with the numerical solutions of the full equation (3) when the initial condition was a superposition of two pulses with the same separation distance as that for (14)–(16). We found very good agreement between these results. (To solve (3) numerically, we implemented a pseudospectral numerical method with a linear propagator so that the linear part of the spatial operator is done exactly in the Fourier space and the stiffness is removed, e.g. Trefethen, 2000.)

In Fig. 1(b) we present a typical solution of the system (14)–(16) where the space-time trajectories of 24 pulses are shown. We note that the results are shown in the frame moving with the velocity of a solitary pulse. Here $\delta = 0.4$. We can observe both attractions and

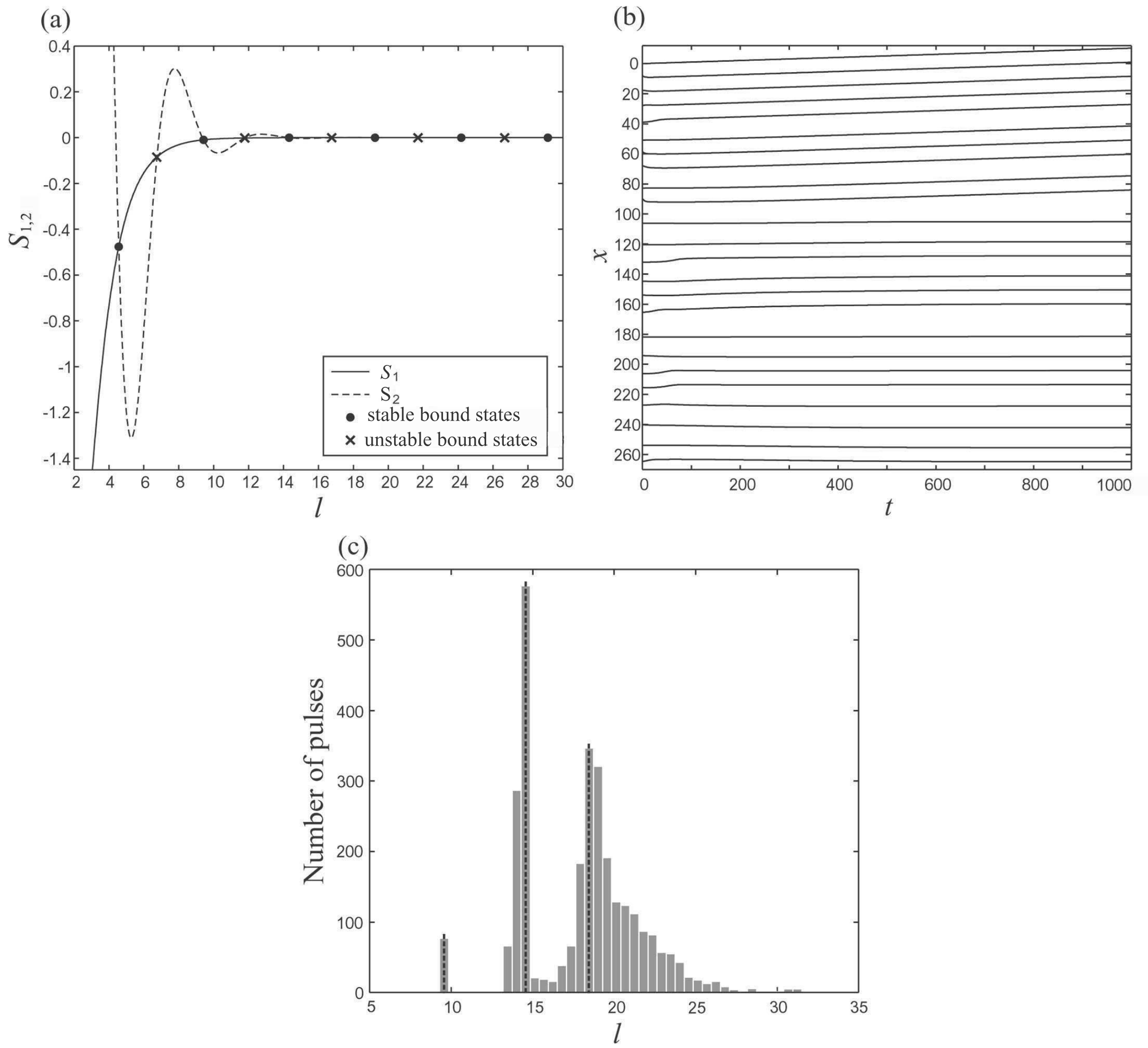

Fig. 1. (a) Dependence of S_1 and S_2 on the separation distance between two pulses (solid and dashed lines, respectively) for $\delta = 0.4$. Circles and crosses correspond to stable and unstable two-pulse bound states, respectively. (b) Evolution of pulses of the gKS equation for $\delta = 0.4$ obtained by solving the system (14)–(16). Attractions and repulsions can be observed as well as the formation of bound states. (c) The histogram obtained on the statistics on 3000 pulse separation distances at $t = 1000$.

repulsions and the formation of two- and three-pulse bound states. Figure 1(c) shows a histogram of the pulse separation distance obtained on the statistics on 3000 pulse separation distances at $t = 1000$. The initial distribution of the separation distances was taken to be normal with mean 18 and standard deviation 3. We can observe three clear peaks. We note that the peaks are formed at around 9.5, 14, and 18.5, in very good agreement with the stable two-pulse bound state distances shown in Fig. 1(a).

4. DISCUSSION

We have developed a weak interaction theory of the pulses of the gKS equation by representing its solution as a superposition of such pulses and a small overlap function and by deriving a system of linearized equations for the overlap function in the vicinity of each pulse. We found that zero is an eigenvalue of the linearized operators of geometric and algebraic multiplicity unity which is embedded into the essential spectrum. This

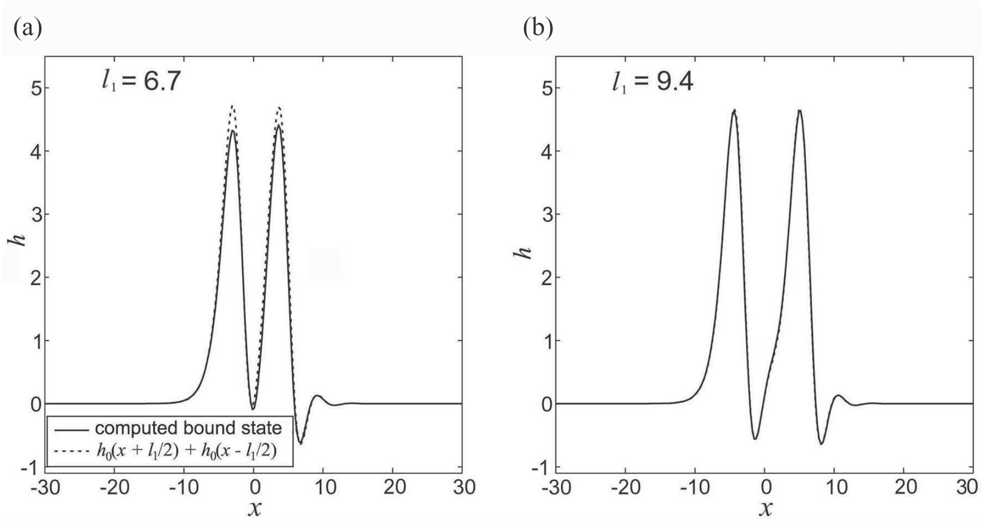

Fig. 2. Comparison of numerically computed bound states (solid lines) with superpositions of two pulses (dashed lines) with the theoretically predicted separation distances $l_1 \approx 6.7$ and 9.4 (panels (a) and (b), respectively). The results are shown for $\delta = 0.5$.

eigenvalue is associated with the translational invariance of the equation. However, zero is not an eigenvalue for the corresponding adjoint operators. The null spaces of the adjoint operators are spanned by a constant and a function having a jump at infinity. Despite this, we showed that projections onto the null spaces, spanned by translational modes, can be made rigorous in an appropriate weighted space and the derivation of a dynamic system describing the interaction of the pulses due to translational modes can be rigorously justified. This system can be written in terms of the separation distances between consecutive pulses. By analysing its fixed points, we obtained bound states consisting of a number of pulses. In particular, we analysed in detail the bound states of two pulses and provided a criterion for the existence of a countable infinite or finite number of bound states, depending on the strength of the dispersive term in the equation. Interestingly, this criterion exactly coincides with Shilnikov's criterion on the existence of subsidiary homoclinic orbits (Glendinning and Sparrow, 1984). Our approach, however, in addition to providing an existence result for the bound states, also gives the description of the dynamics of the pulses. Besides, it can be extended to higher dimensions. This extension is left as a topic for future research. Our interaction theory and the resulting bound states were corroborated by computational experiments. In particular, we found that our theory was capable of predicting pulse-separation distances of true bound states, and that interaction of the pulses was well described by our simplified model.

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Üksikimpulsside interaktsioon aktiivsetes dispergeeruvates ja dissipeeruvates keskkondades

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Üldistatud Kuramoto-Sivashinsky (gKS) võrrandist lähtudes on vaadeldud impulsside nõrka interaktsiooni ja omavahel seotud olekute formeerumist. Võrrand on üks lihtsamatest prototüüpidest, mis kirjeldab energia varustuse, dissipatsiooni ja mittelineaarsusega aktiivset keskkonda.

On tuletatud harilike diferentsiaalvõrrandite süsteem, mis kirjeldab gKS-võrrandi impulsside dünaamika põhiosa, ja tõestatud loenduvate lõpmatute ning lõplike arvudega seotud olekute eksisteerimise kriteerium. Arvutused täisvõrrandil tõestavad meie teooria õigsust.