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MATHEMATICS

Some new scales of characterization of Hardy's inequality

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Abstract. Let $1 < p \leq q < \infty$. Inspired by some recent results concerning Hardy-type inequalities where the equivalence of four scales of integral conditions was proved, we use related ideas to find ten new equivalence scales of integral conditions. By applying this result to the original Hardy-type inequality, we obtain a new proof of a number of characterizations of the Hardy inequality and also some new weight characterizations.

Key words: Hardy operator, Hardy's inequality.

1. INTRODUCTION

We consider the general one-dimensional Hardy inequality

$$\left(\int_0^b \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_0^b f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{1}$$

with a fixed b, $0 < b \leq \infty$, for measurable functions $f \geq 0$, weights u and v and for the parameters p, q satisfying

$$1 < p \leq q < \infty.$$

Inequality (1) is usually characterized by the (so-called Muckenhoupt) condition

$$A_M := \sup_{0<x<b} A_M(x) < \infty, \tag{2}$$

where

$$A_M(x) := \left(\int_x^b u(t)dt\right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(t)dt\right)^{\frac{1}{p'}}. \tag{3}$$

Here and in the sequel $p' = p/(p-1)$. Further, let us denote

$$U(x) := \int_x^b u(t)dt, \qquad V(x) := \int_0^x v^{1-p'}(t)dt, \tag{4}$$

and assume that $U(x) < \infty$, $V(x) < \infty$ for every $x \in (0,b)$.

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The index M in $A_M := \sup_{0<x<b} U^{\frac{1}{q}}(x)V^{\frac{1}{p'}}(x)$ is B. Muckenhoupt, who gave in 1972 a nice proof of the fact that $A_M < \infty$ is necessary and sufficient for (1) to hold (for the case $p = q$, see [7]).

Besides the condition $A_M < \infty$, some other equivalent conditions have been derived during the next decades, e.g. the conditions $A_T < \infty$ or $A_T^* < \infty$, where

$$\begin{aligned} A_T &:= \inf_{h>0} \sup_{0<x<b} \left(\frac{1}{h(x)} \int_0^x u(t)(h(t)+V(t))^{\frac{q}{p'}+1} dt \right)^{\frac{1}{q}}; \\ A_T^* &:= \inf_{h>0} \sup_{0<x<b} \left(\frac{1}{h(x)} \int_x^b v^{1-p'}(t)(h(t)+U(t))^{\frac{p'}{q}+1} dt \right)^{\frac{1}{p'}} \end{aligned} \tag{5}$$

(see Tomaselli [10] for $p = q$, Gurka [3] for $p < q$), and the conditions $A_{PS} < \infty$ or $A_{PS}^* < \infty$, where

$$\begin{aligned} A_{PS} &:= \sup_{0<x<b} \left(\int_0^x u(t)V^q(t)dt \right)^{\frac{1}{q}} V^{\frac{-1}{p}}(x); \\ A_{PS}^* &:= \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t)U^{p'}(t)dt \right)^{\frac{1}{p'}} U^{\frac{-1}{q'}}(x) \end{aligned} \tag{6}$$

(see Persson and Stepanov [9]).

The first scale of conditions was derived in [5]. It reads $A_W(r) < \infty$ (with $1 < r < p$) or $A_W^*(r) < \infty$ (with $1 < r < q'$), where

$$\begin{aligned} A_W(r) &:= \sup_{0<x<b} \left(\int_x^b u(t)V^{\frac{q(p-r)}{p}}(t)dt \right)^{\frac{1}{q}} V^{\frac{r-1}{p}}(x), \quad 1 < r < p; \\ A_W^*(r) &:= \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t)U^{\frac{p'(q'-r)}{q'}}(t)dt \right)^{\frac{1}{p'}} U^{\frac{r-1}{q'}}(x), \quad 1 < r < q'. \end{aligned} \tag{7}$$

A detailed account of the history of the topic can be found in the book [6]; see also [4] and [8].

In 2004 four new scales of equivalent conditions were derived. In [1] the equivalence of four scales of rather general integral conditions was proved and this result was used to characterize inequality (1) by some scales (with the usual Muckenhoupt condition as a special case). The proof was carried out by first proving an equivalence theorem of independent interest.

Here, we will extend the existing list of (equivalent) scales.

The paper is organized as follows. In Section 2 we again formulate an equivalence theorem of independent interest, which extends the equivalence theorem from [1] (Theorem 1) and in Section 3 we use this equivalence theorem to describe some new scales of weight characterization of the Hardy inequality. The main result is formulated in Theorem 2, which includes the results mentioned in (2), (5), (6), and (7) (see Remark 3), but gives also ten new weight characterizations. In Section 4 we give some outlines of the proof of the equivalence theorem (Theorem 1), emphasizing the main ideas and showing the connection with the other cases. In fact, in the proof ten different equivalent conditions are considered; five of them are proved in detail and for the remaining cases hints are given for every step. A complete proof containing all details can be found in the research note [2].

2. THE EQUIVALENCE THEOREM

Theorem 1. *For $-\infty \le a < b \le \infty$, α, β, and s positive numbers and f, g, h measurable functions positive a.e. in (a,b), let us denote*

$$F(x) := \int_x^b f(t)dt, \qquad G(x) := \int_a^x g(t)dt, \tag{8}$$

suppose that $F(x) < \infty$, $G(x) < \infty$ *for every* $x \in (a,b)$. *Furthermore denote*

$$\begin{aligned}
B_1(x;\alpha,\beta) &:= F^{\alpha}(x)G^{\beta}(x);\\
B_2(x;\alpha,\beta,s) &:= \left(\int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)dt\right)^{\alpha} G^s(x);\\
B_3(x;\alpha,\beta,s) &:= \left(\int_a^x g(t)F^{\frac{\alpha-s}{\beta}}(t)dt\right)^{\beta} F^s(x);\\
B_4(x;\alpha,\beta,s) &:= \left(\int_a^x f(t)G^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\alpha} G^{-s}(x);\\
B_5(x;\alpha,\beta,s) &:= \left(\int_x^b g(t)F^{\frac{\alpha+s}{\beta}}(t)dt\right)^{\beta} F^{-s}(x);\\
B_6(x;\alpha,\beta,s) &:= \left(\int_x^b f(t)G^{\frac{\beta}{\alpha+s}}(t)dt\right)^{\alpha+s} F^{-s}(x);\\
B_7(x;\alpha,\beta,s) &:= \left(\int_a^x g(t)F^{\frac{\alpha}{\beta+s}}(t)dt\right)^{\beta+s} G^{-s}(x);\\
B_8(x;\alpha,\beta,s) &:= \left(\int_a^x f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(x), \quad \alpha > s;\\
B_9(x;\alpha,\beta,s) &:= \left(\int_x^b f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(x), \quad \alpha < s;\\
B_{10}(x;\alpha,\beta,s) &:= \left(\int_x^b g(t)F^{\frac{\alpha}{\beta-s}}(t)dt\right)^{\beta-s} G^s(x), \quad \beta > s;\\
B_{11}(x;\alpha,\beta,s) &:= \left(\int_a^x g(t)F^{\frac{\alpha}{\beta-s}}(t)dt\right)^{\beta-s} G^s(x), \quad \beta < s;\\
B_{12}(x;\alpha,\beta,s;h) &:= \left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{\alpha} (h(x)+G(x))^s, \quad \beta < s;\\
B_{13}(x;\alpha,\beta,s;h) &:= \left(\int_a^x g(t)h^{\frac{\alpha-s}{\beta}}(t)dt\right)^{\beta} (h(x)+F(x))^s, \quad \alpha < s;\\
B_{14}(x;\alpha,\beta,s;h) &:= \left(\int_a^x f(t)(h(t)+G(t))^{\frac{\beta+s}{\alpha}}dt\right)^{\alpha} h^{-s}(x);\\
B_{15}(x;\alpha,\beta,s;h) &:= \left(\int_x^b g(t)(h(t)+F(t))^{\frac{\alpha+s}{\beta}}dt\right)^{\beta} h^{-s}(x).
\end{aligned} \tag{9}$$

The numbers $B_1(\alpha,\beta) := \sup_{a<x<b} B_1(x;\alpha,\beta)$, $B_i(\alpha,\beta,s) := \sup_{a<x<b} B_i(x;\alpha,\beta,s)$ $(i = 2,3,\ldots,11)$ *and* $B_i(\alpha,\beta,s) := \inf_{h\geq 0} \sup_{a<x<b} B_i(x;\alpha,\beta,s;h)$ $(i = 12,13,14,15)$ *are mutually equivalent. More precisely, there are positive constants* c_i *and* d_i *so that*

$$c_i B_i(\alpha,\beta,s) \leq B_1(\alpha,\beta) \leq d_i B_i(\alpha,\beta,s),\ i = 2,3,\ldots,15. \tag{10}$$

The constants are independent of f *and* g *and can depend on* α,β, *and* s.

3. THE MAIN RESULT

Theorem 2. *Let* $1 < p \le q < \infty$, $0 < s < \infty$, *and define, for the weight functions* u, v, *the functions* U *and* V *by* (4), *and the functions* $A_i(s)$, $i = 1, 2, \ldots, 14$, *as follows*:

$$
\begin{aligned}
A_1(s) &:= \sup_{0<x<b} \left(\int_x^b u(t) V^{q(\frac{1}{p'}-s)}(t)dt \right)^{\frac{1}{q}} V^s(x); \\
A_2(s) &:= \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t) U^{p'(\frac{1}{q}-s)}(t)dt \right)^{\frac{1}{p'}} U^s(x); \\
A_3(s) &:= \sup_{0<x<b} \left(\int_0^x u(t) V^{q(\frac{1}{p'}+s)}(t)dt \right)^{\frac{1}{q}} V^{-s}(x); \\
A_4(s) &:= \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t) U^{p'(\frac{1}{q}+s)}(t)dt \right)^{\frac{1}{p'}} U^{-s}(x); \\
A_5(s) &:= \sup_{0<x<b} \left(\int_x^b u(t) V^{\frac{q}{p'(1+sq)}}(t)dt \right)^{\frac{1+sq}{q}} U^{-s}(x); \\
A_6(s) &:= \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t) U^{\frac{p'}{q(1+sp')}}(t)dt \right)^{\frac{1+sp'}{p'}} V^{-s}(x); \\
A_7(s) &:= \sup_{0<x<b} \left(\int_0^x u(t) V^{\frac{q}{p'(1-sq)}}(t)dt \right)^{\frac{1-sq}{q}} U^{s}(x), \quad qs < 1; \\
A_8(s) &:= \sup_{0<x<b} \left(\int_x^b u(t) V^{\frac{q}{p'(1-sq)}}(t)dt \right)^{\frac{1-sq}{q}} U^{s}(x), \quad qs > 1; \\
A_9(s) &:= \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t) U^{\frac{p'}{q(1-sp')}}(t)dt \right)^{\frac{1-sp'}{p'}} V^{s}(x), \quad p's < 1; \\
A_{10}(s) &:= \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t) U^{\frac{p'}{q(1-sp')}}(t)dt \right)^{\frac{1-sp'}{p'}} V^{s}(x), \quad p's > 1; \\
A_{11}(s) &:= \inf_{h>0} \sup_{0<x<b} \left(\int_x^b u(t) h(t)^{q(\frac{1}{p'}-s)} dt \right)^{\frac{1}{q}} (h(x)+V(x))^s, \quad p's > 1; \\
A_{12}(s) &:= \inf_{h>0} \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t) h(t)^{p'(\frac{1}{q}-s)} dt \right)^{\frac{1}{p'}} (h(x)+U(x))^s, \quad qs > 1; \\
A_{13}(s) &:= \inf_{h>0} \sup_{0<x<b} \left(\int_0^x u(t) (h(t)+V(t))^{q(\frac{1}{p'}+s)} dt \right)^{\frac{1}{q}} h^{-s}(x); \\
A_{14}(s) &:= \inf_{h>0} \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t) (h(t)+U(t))^{p'(\frac{1}{q}+s)}(t) \right)^{\frac{1}{p'}} h^{-s}(x).
\end{aligned}
\tag{11}
$$

Then Hardy inequality (1) *holds for all measurable functions* $f \ge 0$ *if and only if any of the quantities* $A_i(s)$, $i = 1, 2, 3, \ldots, 14$, *is finite for some* $0 < s < \infty$. *Moreover, for the best constant* C *in* (1) *we have* $C \approx A_i(s)$, $i = 1, 2, 3, \ldots, 14$. *The constants in the equivalence relations can depend on s.*

Remark 3. The conditions in (2), (5), (6), and (7) can be described in the following way:

$$\begin{aligned}
A_M &= A_1\left(\frac{1}{p'}\right), \\
A_{PS} &= A_3\left(\frac{1}{p}\right), \\
A_W(r) &= A_1\left(\frac{r-1}{p}\right) \text{ with } 1<r<p, \\
A_{PS}^* &= A_4\left(\frac{1}{q'}\right), \\
A_W^*(r) &= A_2\left(\frac{r-1}{q'}\right) \text{ with } 1<r<q', \\
A_T &= A_{13}\left(\frac{1}{q}\right), \\
A_T^* &= A_{14}\left(\frac{1}{p'}\right).
\end{aligned}$$

Hence, Theorem 2 generalizes the corresponding results in [1,5] and also all previous results of this type.

Proof of Theorem 2. In (8) we put $a=0, f(x)=u(x), g(x)=v^{1-p'}(x)$, so that $F(x)=U(x)$, $G(x)=V(x)$, and choose

$$\alpha=\frac{1}{q},\ \ \beta=\frac{1}{p'}.$$

Then the assertion follows from the fact that

$$A_i(s)=B_{i+1}\left(\frac{1}{q},\frac{1}{p'},s\right),\ \ i=1,2,\dots,14,$$

are all equivalent with A_1 from (2) according to Theorem 1 and the finiteness of A_1 is necessary and sufficient for inequality (1) to hold. Moreover, since for the least constant C in (1) we have $C\approx A_1$, it is clear that $C\approx A_i(s)$ and the proof is complete.

Remark 4. The proof of Theorem 1 gives us also the opportunity to estimate e.g. the quantities A_M, $A_W(r)$, $A_W^*(r)$, A_{PS}, A_{PS}^*, A_T and A_T^* in terms of each other.

4. THE PROOF OF THE EQUIVALENCE THEOREM

In the proof, which is rather technical, we use, among other tools, the fact that the function F from (8) is decreasing and the function of G from (8) is increasing, and that

$$f(x)dx=-dF(x),\ \ g(x)dx=dG(x)$$

so that

$$\int_x^b f(t)F^\lambda(t)dt=\frac{1}{\lambda+1}F^{\lambda+1}(x);\ \ \int_a^x g(t)G^\kappa(t)dt=\frac{1}{\kappa+1}G^{\kappa+1}(x). \tag{12}$$

Moreover, the equivalences

$$B_i(\alpha,\beta,s)\approx B_1(\alpha,\beta),\quad i=2,3,4,5, \tag{13}$$

have been proved in Theorem 2.1 in [1], so that it remains to prove the other 10 equivalences.

1. $B_1(\alpha,\beta) \approx B_6(\alpha,\beta,s)$.

(i) $B_1(\alpha,\beta) \lesssim B_6(\alpha,\beta,s)$:

$$\begin{aligned}
B_1(x;\alpha,\beta) &= F^{\alpha}(x)G^{\beta}(x) = F^{\alpha+s}(x)F^{-s}(x)G^{\beta}(x)\\
&= \left(\int_x^b f(t)dt\right)^{\alpha+s} G^{\beta}(x)F^{-s}(x)\\
&= \left(\int_x^b f(t)G^{\frac{\beta}{\alpha+s}}(x)dt\right)^{\alpha+s} F^{-s}(x)\\
&\le \left(\int_x^b f(t)G^{\frac{\beta}{\alpha+s}}(t)dt\right)^{\alpha+s} F^{-s}(x) = B_6(x;\alpha,\beta,s)
\end{aligned}$$

(we have used the fact that G is increasing). Now we take the suprema for $x \in (a,b)$ and have that $B_1(\alpha,\beta) \le B_6(\alpha,\beta,s)$, i.e. $d_6 = 1$ in (10).

(ii) $B_6(\alpha,\beta,s) \lesssim B_1(\alpha,\beta)$:

$$\begin{aligned}
B_6(x;\alpha,\beta,s) &= \left(\int_x^b f(t)G^{\frac{\beta}{\alpha+s}}(t)dt\right)^{\alpha+s} F^{-s}(x)\\
&= \left(\int_x^b f(t)B_1^{\frac{1}{\alpha+s}}(t,\alpha,\beta)F^{-\frac{\alpha}{\alpha+s}}(t)G^{-\frac{\beta}{\alpha+s}}(t)G^{\frac{\beta}{\alpha+s}}(t)dt\right)^{\alpha+s} F^{-s}(x)\\
&\le B_1(\alpha,\beta)\left(\int_x^b f(t)F^{-\frac{\alpha}{\alpha+s}}(t)dt\right)^{\alpha+s} F^{-s}(x)\\
&= B_1(\alpha,\beta)\left(-\frac{\alpha+s}{s}F^{\frac{s}{\alpha+s}}\Big|_x^b\right)^{\alpha+s} F^{-s}(x)\\
&= \left(\frac{\alpha+s}{s}\right)^{\alpha+s} B_1(\alpha,\beta)F^{s}(x)F^{-s}(x) = \left(\frac{\alpha+s}{s}\right)^{\alpha+s} B_1(\alpha,\beta)
\end{aligned}$$

(we have used the fact that $B_1(t,\alpha,\beta) \le B_1(\alpha,\beta)$, and formula (12) for F with $\lambda = -\frac{\alpha}{\alpha+s}$). Taking the supremum on the left-hand side, we have that $B_6(\alpha,\beta,s) \le \frac{1}{c_6}B_1(\alpha,\beta)$ with $c_6 = \left(\frac{s}{\alpha+s}\right)^{\alpha+s}$.

2. $B_1(\alpha,\beta) \approx B_7(\alpha,\beta,s)$; $\alpha > s$ – the procedure is as in 1.

3. $B_1(\alpha,\beta) \approx B_8(\alpha,\beta,s)$; $\alpha > s$.

(i) $B_1(\alpha,\beta) \lesssim B_8(\alpha,\beta,s)$: Fix $x \in (a,b)$ and define $y = y(x) \in (x,b)$ so that

$$\int_x^y f(t)dt = \int_y^b f(t)dt. \tag{14}$$

Then

$$\begin{aligned}
F^{\alpha}(x) &= \left(\int_x^b f(t)dt\right)^{\alpha} = \left(\int_x^y f(t)dt + \int_y^b f(t)dt\right)^{\alpha} = 2^{\alpha}\left(\int_y^b f(t)dt\right)^{\alpha}\\
&= 2^{\alpha}\left(\int_x^y f(t)dt\right)^{\alpha-s}\left(\int_y^b f(t)dt\right)^{\alpha} = 2^{\alpha}\left(\int_y^b f(t)dt\right)^{\alpha-s} F^{s}(y)
\end{aligned}$$

and

$$\begin{aligned}
B_1(x;\alpha,\beta) &= 2^\alpha \left(\int_x^y f(t)dt\right)^{\alpha-s} F^s(y)G^\beta(x)\\
&= 2^\alpha \left(\int_x^y f(t)G^{\frac{\beta}{\alpha-s}}(x)dt\right)^{\alpha-s} F^s(y)\\
&\le 2^\alpha \left(\int_x^y f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(y)\\
&\le 2^\alpha \left(\int_a^y f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(y) = 2^\alpha B_8(y;\alpha,\beta,s)
\end{aligned}$$

(we have used the fact that G is increasing). Taking the supremum with respect to y (right) and x (left), we have that $B_1(\alpha,\beta) \le 2^\alpha B_8(\alpha,\beta,s)$, i.e. $d_8 = 2^\alpha$ in (10).

(ii) $B_8(\alpha,\beta,s) \lesssim B_1(\alpha,\beta)$:

$$\begin{aligned}
B_8(x;\alpha,\beta,s) &= \left(\int_a^x f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(x)\\
&= \left(\int_a^x f(t)B_1^{\frac{\alpha}{\alpha-s}}(t;\alpha,\beta)F^{-\frac{\alpha}{\alpha-s}}(t)G^{-\frac{\beta}{\alpha-s}}(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(x)\\
&\le B_1(\alpha,\beta)\left(\int_a^x f(t)F^{-\frac{\alpha}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(x).
\end{aligned}$$

Now (see (12)) $\int_a^x f(t)F^{-\frac{\alpha}{\alpha-s}}(t)dt = \frac{\alpha-s}{s}\left(F^{-\frac{s}{\alpha-s}}(x) - F^{-\frac{s}{\alpha-s}}(a)\right) \le \frac{\alpha-s}{s}F^{-\frac{s}{\alpha-s}}(x)$ (even if $F(a)=\infty$, since $-\frac{s}{\alpha-s} < 0$).

Hence

$$\begin{aligned}
B_8(x;\alpha,\beta,s) &\le B_1(\alpha,\beta)\left(\frac{\alpha-s}{s}\right)^{\alpha-s}\left(F^{-\frac{s}{\alpha-s}}(x)\right)^{\alpha-s} F^s(x)\\
&= \left(\frac{\alpha-s}{s}\right)^{\alpha-s} B_1(\alpha,\beta),
\end{aligned}$$

and taking the supremum, we have

$$B_8(\alpha,\beta,s) \le \frac{1}{c_8} B_1(\alpha,\beta,) \quad \text{with} \quad c_8 = \left(\frac{s}{\alpha-s}\right)^{\alpha-s}.$$

4. $B_1(\alpha,\beta) \approx B_9(\alpha,\beta,s)$; $\alpha < s$.

(i) $B_1(\alpha,\beta) \approx B_9(\alpha,\beta,s)$:

$$\begin{aligned}
B_1(x;\alpha,\beta) = F^s(x)F^{\alpha-s}(x)G^\beta(x) &= \left(\int_x^b f(t)G^{\frac{\beta}{\alpha-s}}(x)dt\right)^{\alpha-s} F^s(x)\\
&\le \left(\int_x^b f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(x) = B_9(x;\alpha,\beta,s)
\end{aligned}$$

(we used the fact that G is increasing, and *twice* the fact that $\alpha - s < 0$). Taking the supremum, we get $B_1(\alpha,\beta) \le B_9(\alpha,\beta,s)$, i.e. $d_9 = 1$ in (10).

(ii) $B_9(\alpha,\beta,s) \lesssim B_1(\alpha,\beta)$: Fix $x \in (a,b)$ and define $y \in (x,b)$ so that (14) holds. Then $F(x) = 2F(y)$ and

$$\begin{aligned}
B_9(x;\alpha,\beta,s) &= \left(\int_x^b f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(x) \\
&\le \left(\int_x^y f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(x) \\
&\le \left(\int_x^y f(t)G^{\frac{\beta}{\alpha-s}}(y)dt\right)^{\alpha-s} F^s(x) \\
&= \left(\int_x^y f(t)dt\right)^{\alpha-s} G^\beta(y)F^s(x) = \left(\int_y^b f(t)dt\right)^{\alpha-s} G^\beta(y)F^s(x) \\
&= F^{\alpha-s}(y)G^\beta(y)2^sF^s(y) = 2^sB_1(y;\alpha,\beta)
\end{aligned}$$

(we used the fact that G is increasing and that $\alpha - s < 0$). Taking the suprema, we have that $B_9(\alpha,\beta,s) \le 2^sB_1(\alpha,\beta)$, i.e. $c_9 = 2^{-s}$ in (10).

5. $B_1(\alpha,\beta) \approx B_{10}(\alpha,\beta,s)$; $\beta > s$ – the procedure is as in 3.

6. $B_1(\alpha,\beta) \approx B_{11}(\alpha,\beta,s)$; $\beta < s$ – the procedure is as in 4.

7. $B_1(\alpha,\beta) \approx B_{12}(\alpha,\beta,s)$; $\beta < s$.

(i) $B_1(\alpha,\beta) \lesssim B_{12}(\alpha,\beta,s)$: Assume that $B_{12}(\alpha,\beta,s) < \infty$ and denote it for simplicity by B_{12}. Since $\inf_{h>0}\sup_x \left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^\alpha (h(x)+G(x))^s = B_{12}$, there exists a positive function h such that

$$\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^\alpha (h(x)+G(x))^s \le 2B_{12}, \quad x \in (0,b)$$

and consequently

$$\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt \le (2B_{12})^{\frac{1}{\alpha}} h^{-\frac{s}{\alpha}}(x), \tag{15}$$

$$\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt \le (2B_{12})^{\frac{1}{\alpha}} G^{-\frac{s}{\alpha}}(x). \tag{16}$$

From (15) we obtain, raising both sides to the power $\frac{s-\beta}{s} > 0$, multiplying by $f(x)$, and integrating from y to b, that

$$\int_y^b f(x)\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{\frac{s-\beta}{s}} dx \le (2B_{12})^{\frac{s-\beta}{s\alpha}} \int_y^b f(x)h^{\frac{\beta-s}{\alpha}}(x)dx. \tag{17}$$

Now we use the equivalence relation

$$B_5(1,1,1) \approx B_5\left(1,1,\frac{\beta}{s}\right),$$

which holds, since both terms are equivalent to $B_1(1,1)$ (see (13)). This relation reads

$$\sup_x \left(\int_x^b g(t)F^2(t)dt\right) F^{-1}(x) \approx \sup_x \left(\int_x^b g(t)F^{1+\frac{\beta}{s}}(t)dt\right) F^{-\frac{\beta}{s}}(x).$$

We use this relation with $\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt$ for $F(x)$ and with $f(x)\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{-\frac{\beta}{s}-1}$ for $g(x)$. Then we have

$$\begin{aligned}
&\sup_x \left(\int_x^b f(t)dt\right)\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{-\frac{\beta}{s}} \\
&\quad\approx \sup_x \left(\int_x^b f(y)\left(\int_y^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{\frac{s-\beta}{s}} dy\right)\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{-1} \\
&\quad\lesssim B_{12}^{\frac{s-\beta}{s\alpha}},
\end{aligned}$$

where the last inequality follows from (17). Therefore we get

$$\sup_x \left(\int_x^b f(t)dt\right)^{\alpha}\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{-\frac{\alpha\beta}{s}} \lesssim B_{12}^{\frac{s-\beta}{s}}. \tag{18}$$

Taking into account that due to (16)

$$G^{\beta}(x) \le (2B_{12})^{\frac{\beta}{s}}\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{-\frac{\alpha\beta}{s}},$$

we get from (18) that

$$\begin{aligned}
\sup_x F^{\alpha}(x)G^{\beta}(x) &\lesssim B_{12}^{\frac{\beta}{s}} \sup_x F^{\alpha}(x)\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{-\frac{\alpha\beta}{s}} \\
&\lesssim B_{12}^{\frac{\beta}{s}}B_{12}^{1-\frac{\beta}{s}} = B_{12}.
\end{aligned}$$

Therefore, we have that

$$B_1(\alpha,\beta) \lesssim B_{12}(\alpha,\beta,s).$$

(ii) $B_{12}(\alpha,\beta,s) \lesssim B_1(\alpha,\beta)$: Since for $h(x) = G(x)$,

$$\begin{aligned}
B_{12}(x;\alpha,\beta,s,G) &= 2^s\left(\int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)dt\right)^{\alpha} G^s(x) \\
&= 2^s B_2(x;\alpha,\beta,s) \le 2^s B_2(\alpha,\beta,s) \lesssim B_1(\alpha,\beta)
\end{aligned}$$

(see (13) for $i = 2$), we immediately obtain that $B_{12}(\alpha,\beta,s) \lesssim B_1(\alpha,\beta)$.

8. $B_1(\alpha,\beta) \approx B_{13}(\alpha,\beta,s)$; $\beta < s$ – the procedure is as in 7.

9. $B_1(\alpha,\beta) \approx B_{14}(\alpha,\beta,s)$.

(i) $B_1(\alpha,\beta) \lesssim B_{14}(\alpha,\beta,s)$: Assume that $B_{14}(\alpha,\beta,s) := B_{14}$ is finite. Due to the definition of B_{14}, there exists a positive function h such that

$$\left(\int_a^x f(t)(h(t)+G(t))^{\frac{\beta+s}{\alpha}}dt\right)^{\alpha} h^s(x) \le 2B_{14},$$

and consequently

$$\int_a^x f(t)h^{\frac{\beta+s}{\alpha}}(t)dt \le (2B_{14})^{\frac{1}{\alpha}} h^{\frac{s}{\alpha}}(x), \tag{19}$$

$$\int_a^x f(t)G^{\frac{\beta+s}{\alpha}}(t)dt \le (2B_{14})^{\frac{1}{\alpha}} h^{\frac{s}{\alpha}}(x). \tag{20}$$

From (19) we obtain, raising both sides to the power $\frac{\beta+s}{s}$, multiplying by $f(x)$, and integrating from a to y, that

$$\int_a^y f(x)\left(\int_a^x f(t)h^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\frac{\beta+s}{s}} dx \lesssim (2B_{14})^{\frac{\beta+s}{s\alpha}} \int_a^y f(x)h^{\frac{\beta+s}{\alpha}}(x)dx. \tag{21}$$

Now we use the equivalence relation

$$B_1\left(1,\frac{\beta}{s}\right) \approx B_4\left(1,\frac{\beta}{s},1\right),$$

which makes sense due to (13) and which reads

$$\sup_x F(x)G^{\frac{\beta}{s}}(x) \approx \sup_x \left(\int_a^x f(t)G^{\frac{\beta}{s}+1}(t)dt\right)G^{-1}(x).$$

Putting here $\int_a^x f(t)h^{\frac{\beta+s}{\alpha}}(t)dt$ for $G(x)$, we have that

$$\begin{aligned}
&\sup_x \left(\int_x^b f(y)dy\right)\left(\int_a^x f(t)h^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\frac{\beta}{s}} \\
&\approx \sup_x \left(\int_a^x f(y)\left(\int_a^y f(t)h^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\frac{\beta}{s}+1} dy\right)\left(\int_a^x f(t)h^{\frac{\beta+s}{\alpha}}(t)dt\right)^{-1} \\
&\lesssim B_{14}^{\frac{s+\beta}{s\alpha}},
\end{aligned} \tag{22}$$

where the last inequality follows from (21). From (20) we obtain that

$$\int_a^x f(y)\left(\int_a^y f(t)G^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\frac{\beta+s}{s}} dy \lesssim B_{14}^{\frac{\beta+s}{\alpha s}} \int_a^x f(y)h^{\frac{\beta+s}{\alpha}}(y)dy$$

and consequently

$$F^s(x)\left(\int_a^x f(y)\left(\int_a^y f(t)G^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\frac{\beta+s}{s}} dy\right)^{\beta} \lesssim B_{14}^{\beta\frac{\beta+s}{\alpha s}} F^s(x)\left(\int_a^x f(y)h^{\frac{\beta+s}{\alpha}}(y)dy\right)^{\beta}.$$

Hence, using (22), we have that

$$\begin{aligned}
&\sup_x F^s(x)\left(\int_a^x f(y)\left(\int_a^y f(t)G^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\frac{\beta+s}{s}} dy\right)^{\beta} \\
&\approx B_{14}^{\beta\frac{\beta+s}{\alpha s}} \sup_x \left(\int_a^x f(y)dy\right)^s \left(\int_a^x f(y)h^{\frac{\beta+s}{\alpha}}(y)dy\right)^{\beta} \\
&\lesssim B_{14}^{\beta\frac{\beta+s}{\alpha s}} B_{14}^{s\frac{\beta+s}{\alpha s}} = B_{14}^{\frac{(\beta+s)^2}{\alpha s}}.
\end{aligned} \tag{23}$$

Now we use the equivalence relation

$$B_1\left(\beta+s,\frac{\beta(\beta+s)}{s}\right)\approx B_8\left(\beta+s,\frac{\beta(\beta+s)}{s},s\right),$$

which makes sense due to step 3 and which reads

$$\sup_x F^{\beta+s}(x)G^{\beta\frac{\beta+s}{s}}(x)\approx\sup_x\left(\int_a^x f(t)G^{\frac{\beta+s}{s}}(t)dt\right)^{\beta}F^s(x).$$

Putting here $f(t)G^{\frac{\beta+s}{\alpha}}(t)$ for $g(t)$, we have that

$$\begin{aligned}
&\sup_x F^{\beta+s}(x)\left(\int_a^x f(t)G^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\frac{\beta(\beta+s)}{s}}\\
&\qquad\approx\sup_x F^s(x)\left(\int_a^x f(y)\left(\int_a^y f(t)G^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\frac{\beta+s}{s}}dy\right)^{\beta}\\
&\qquad\lesssim B_{14}^{\frac{(\beta+s)^2}{\alpha s}}
\end{aligned}\tag{24}$$

due to (23). Then, using the equivalence relation

$$B_1\left(\frac{(\beta+s)^2}{s},\frac{\beta(\beta+s)^2}{\alpha s}\right)\approx B_8\left(\frac{(\beta+s)^2}{s},\frac{\beta(\beta+s)^2}{\alpha s},\beta+s\right),$$

which makes sense again due to step 3 and which reads

$$\sup_x F^{\frac{(\beta+s)^2}{s}}(x)G^{\frac{\beta(\beta+s)^2}{\alpha s}}(x)\approx\sup_x\left(\int_a^x f(t)G^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\frac{\beta(\beta+s)}{s}}F^{\beta+s}(x),$$

we obtain due to (24) that

$$\sup_x F^{\frac{(\beta+s)^2}{s}}(x)G^{\frac{\beta(\beta+s)^2}{\alpha s}}(x)\lesssim B_{14}^{\frac{(\beta+s)^2}{\alpha s}},$$

i.e.,

$$B_1(\alpha,\beta)=\sup_x F^{\alpha}(x)G^{\beta}(x)\lesssim B_{14}=B_{14}(\alpha,\beta,s).$$

(ii) $B_{14}(\alpha,\beta,s)\lesssim B_1(\alpha,\beta)$: Since for $h(x)=G(x)$,

$$\begin{aligned}
B_{14}(x;\alpha,\beta,s,G)&=2^{\beta+s}\left(\int_a^x f(t)G^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\alpha}G^{-s}(x)\\
&=2^{\beta+s}B_4(x;\alpha,\beta,s)\le 2^{\beta+s}B_4(\alpha,\beta,s)\lesssim B_1(\alpha,\beta)
\end{aligned}$$

due to (13), we have the result immediately.

10. $B_1(\alpha,\beta)\approx B_{15}(\alpha,\beta,s)$ – the procedure is again similar to that of 9.

Hence, the proof is complete.

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REFERENCES

1. Gogatishvili, A., Kufner, A., Persson, L. E., and Wedestig, A. An equivalence theorem for some scales of integral conditions related to Hardy's inequality with applications. *Real Anal. Exchange*, 2004, **29**, 867–880.
2. Gogatishvili, A., Kufner, A., and Persson, L. E. *An Equivalence Theorem with Application to Hardy's Inequality*. Research Report 8, Department of Mathematics, Luleå University of Technology, 2007.
3. Gurka, P. Generalized Hardy's inequality. *Časopis Pěst. Mat.*, 1984, **109**, 194–203.
4. Kufner, A. and Persson, L. E. *Weighted Inequalities of Hardy Type*. World Scientific Publishing Co, Singapore, 2003.
5. Kufner, A., Persson, L. E., and Wedestig, A. A study of some constants characterizing the weighted Hardy inequality. *Banach Center Publ.*, Orlicz Centenary Volume, 2004, 64, 135–146.
6. Kufner, A., Maligranda, L., and Persson, L. E. *The Hardy Inequality. About Its History and Some Related Results*. Vydavatelský servis Publishing House, Pilsen, 2007.
7. Muckenhoupt, B. Hardy's inequality with weights. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, I. *Studia Math.*, 1972, **44**, 31–38.
8. Opic, B. and Kufner, A. *Hardy-Type Inequalities*. Pitman Research Notes in Mathematics Series, Vol. 211, Longman Scientific and Technical, Harlow, 1990.
9. Persson, L.-E. and Stepanov, V. Weighted integral inequalities with the geometric mean operator. *J. Inequal. Appl.*, 2002, **7**, 727–746.
10. Tomaselli, G. A class of inequalities. *Boll. Un. Mat. Ital.*, 1969, **2**, 622–631.

Uued tingimuste skaalad Hardy võrratuse kehtimiseks

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On uuritud, missugustel tingimustel kehtib üldine Hardy tüüpi võrratus, mis lubab mingi funktsiooni algfunktsiooni normi kaaluga ruumis L^q hinnata selle funktsiooni enda normiga mingi teise kaaluga ruumis L^p. On tuletatud 10 uut tarvilike ja piisavate integraalsete tingimuste skaalat vastavate kaalufunktsioonide jaoks. Need tingimused on saadud järeldusena ekvivalentsusteoreemist, mis väidab 15 integraalse tingimuse ekvivalentsust (neist 5 on varem tõestatud).