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MATHEMATICS

Operator convex functions over C^*-algebras

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Abstract. In this short note we give an exact characterization of C^*-algebras that have the class of convex functions. More precisely, we give a convexity characterization of subhomogeneous C^*-algebras. We use these results to generalize the single function based convexity conditions for commutativity of a C^*-algebra to the single function based convexity conditions for subhomogeneity.

Key words: operator monotone functions, operator convex functions, matrix convex functions.

As for operator monotone matrix functions over C^*-algebras considered in [9], we denote by $K_A(I)$ the set of all A-convex functions (defined on the interval I) for a C^*-algebra A. If $A = B(H)$, the standard C^*-algebra of all bounded linear operators on a Hilbert space H, then $K_A(I) = K_{B(H)}(I)$ is called the set of all operator convex functions. If $A = M_n$, then $K_n(I) = K_A(I) = K_{M_n}(I)$ is called the set of all matrix convex functions of order n on an interval I. The set $K_n(I)$ consists of continuous functions on I satisfying $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for pairs (x, y) of self-adjoint $n \times n$ matrices with their spectra in I and any $0 \leq \lambda \leq 1$. For each positive integer n, the proper inclusion $K_{n+1}(I) \subsetneq K_n(I)$ holds [3]. For an infinite-dimensional Hilbert space, the set of operator convex functions on I can be shown to coincide with the intersection

$$K_\infty(I) = \bigcap_{n=1}^{\infty} K_n(I),$$

or in other words a function is operator convex if and only if it is matrix convex of order n for all positive integers n [5, Chap. 5, Proposition 5.1.5 (ii)].

In this short note we show that for general C^*-algebras the classes of convex functions are the standard classes of matrix and operator convex functions. For every such class we give an exact characterization of C^*-algebras that have this class of convex functions. This can be also used to give a convexity characterization of subhomogeneous C^*-algebras as discussed in the case of monotone functions by [2, Theorem 5; 9, Theorem 2.3]. We use these results to generalize the single function based convexity conditions for commutativity of a C^*-algebra, obtained by Ogasawara [7], Pedersen [10], Wu [15], and Ji and Tomiyama [6], to single function based convexity conditions for subhomogeneity.

It could be also appropriate to mention here that further inspiration for the present work comes from [1,8,11,12] indicating possibilities for a deep interplay of our results with analytic continuation and function spaces, interpolation, moment problems, complete positivity, and order structure in C^*-algebras.

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Lemma 1. *Let A be a C^*-algebra and I be an open interval.*

(1) *If A has an irreducible representation of dimension n, then any A-convex function becomes n-matrix convex, that is $K_A(I) \subseteq K_n(I)$.*

(2) *If $\dim \pi \leq n$ for any irreducible representation π of A, then $K_n(I) \subseteq K_A(I)$.*

(3) *If the set of dimensions of finite dimensional irreducible representations of A is unbounded, then every A-convex function is operator convex, that is $K_A(I) = K_\infty(I)$.*

(4) *If A has an infinite dimensional irreducible representation, then every A-convex function is operator convex, that is $K_A(I) = K_\infty(I)$.*

Proof.

(1) Let $\pi : A \to M_n$ be an n-dimensional irreducible representation of A. Then irreducibility implies that $\pi(A) = M_n$. Thus for any pair $c, d \in M_n$ of self-adjoint elements with spectra in I there exist self-adjoint elements $a, b \in A$ with spectra in I such that $\pi(a) = c$ and $\pi(b) = d$. Then for any $0 \leq \lambda \leq 1$

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b)$$

and hence

$$\pi(f(\lambda a + (1-\lambda)b)) \leq \lambda \pi(f(a)) + (1-\lambda)\pi(f(b))$$

for any $f \in K_A(I)$. By continuity, $\pi(f(x)) = f(\pi(x))$ for any $x \in A$. Thus

$$\begin{aligned}
f(\lambda c + (1-\lambda)d) &= f(\lambda \pi(a) + (1-\lambda)\pi(b)) \\
&= f(\pi(\lambda a + (1-\lambda)b)) \\
&= \pi(f(\lambda a + (1-\lambda)b)) \\
&\leq \pi(\lambda f(a) + (1-\lambda) f(b)) \\
&= \lambda \pi(f(a)) + (1-\lambda)\pi(f(b)) \\
&= \lambda f(\pi(a)) + (1-\lambda) f(\pi(b)) \\
&= \lambda f(c) + (1-\lambda) f(d)
\end{aligned}$$

and therefore $f \in K_n(I)$. Hence, we have proved that $K_A(I) \subseteq K_n(I)$.

(2) Let $f \in K_n(I)$. For any pair $a, b \in A$ of self-adjoint elements with spectra in I and for any irreducible representation $\pi : A \to M_m$, where $m \leq n$, we have $\pi(\lambda a + (1-\lambda)b) = \lambda \pi(a) + (1-\lambda)\pi(b)$ in M_m. Then for any $0 \leq \lambda \leq 1$

$$\begin{aligned}
\pi(f(\lambda a + (1-\lambda)b)) &= f(\pi(\lambda a + (1-\lambda)b)) \\
&= f(\lambda \pi(a) + (1-\lambda)\pi(b)) \\
&\leq \lambda f(\pi(a)) + (1-\lambda) f(\pi(b)) \\
&= \lambda \pi(f(a)) + (1-\lambda)\pi(f(b)) \\
&= \pi(\lambda f(a) + (1-\lambda) f(b)).
\end{aligned}$$

Hence

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b).$$

Thus, $f \in K_A(I)$ and we proved that $K_n(I) \subseteq K_A(I)$.

(3) Let $\{\pi_j \mid j \in \mathbb{N} \setminus \{0\}\}$ be a sequence of irreducible finite dimensional representations of A such that $n_j = \dim \pi_j \to \infty$ when $j \to \infty$. By (1) we have inclusion $K_A(I) \subseteq K_{n_k}(I)$ for any $k \in \mathbb{N} \setminus \{0\}$. Hence

$$K_A(I) \subseteq \bigcap_{k \in \mathbb{N}\setminus\{0\}} K_{n_k}(I) = \bigcap_{k \in \mathbb{N}\setminus\{0\}} K_k(I) = K_\infty(I),$$

and since always $K_\infty(I) \subseteq K_A(I)$ holds, we get the equality $K_A(I) = K_\infty(I)$.

(4) Let $\pi : A \to B(H)$ be an irreducible representation of A on an infinite dimensional Hilbert space H. By Kadison's transitivity theorem (see [13, Ch. 2, Theorem 4.18]), $\pi(A)p = B(H)p$ for every projection $p : H \to H$ of a finite rank $n = \dim pH < \infty$. Let $B = \{a \in A \mid \pi(a)pH \subseteq pH, \pi(a)^* pH \subseteq pH\}$ be the C^*-subalgebra of A consisting of elements mapped by π to operators that, together with their adjoints, leave pH invariant. The restriction of $\pi : B \to pB(H)p$ to B is an n-dimensional representation of B on pH. Thus (1) yields $K_A(I) \subseteq K_B(I) \subseteq K_n(I)$, since B is a C^*-subalgebra of A. As the positive integer n can be chosen arbitrary, we get the inclusion

$$K_A(I) \subseteq \bigcap_{n \in \mathbb{N} \setminus \{0\}} K_n(I) = K_\infty(I).$$

Combining it with $K_\infty(I) \subseteq K_A(I)$ yields the equality $K_A(I) = K_\infty(I)$. □

Theorem 2. *Let A be a C^*-algebra and I be an open interval. Then*

(1) $K_A(I) = K_\infty(I)$ *if and only if either the set of dimensions of finite-dimensional irreducible representations of A is unbounded, or A has an infinite-dimensional irreducible representation.*

(2) $K_A(I) = K_n(I)$ *for some positive integer n if and only if A is n-subhomogeneous.*

Recall that A is said to be subhomogeneous if the dimensions of its irreducible representations are bounded and in particular we call A n-subhomogeneous if the highest dimension is n.

Proof. By Lemma 1 the only part of (1) left to prove is that $K_A(I) = K_\infty(I)$ implies that either the set of dimensions of finite dimensional irreducible representations of A is unbounded, or A has an infinite-dimensional irreducible representation. Suppose on the contrary that

$$n_1 = \sup\{\dim(\pi) \mid \pi \text{ is an irreducible representation of } A\} < \infty.$$

Then $K_A(I) \subseteq K_{n_1}(I)$ by (1) of Lemma 1, and $K_{n_1}(I) \subseteq K_A(I)$ by (2) of Lemma 1. Thus $K_A(I) = K_{n_1}(I)$. But there is a gap between $K_\infty(I)$ and $K_n(I)$ for any n [3]. Hence $K_A(I) \neq K_\infty(I)$, in contradiction to the initial assumption $K_A(I) = K_\infty(I)$.

In part (2), since a C^*-algebra A has sufficiently many irreducible representations, the order $a \leq b$ is equivalent to say that $\pi(a) \leq \pi(b)$ for every irreducible representation of A. Therefore, by (1), (2) of Lemma 1 and the definition of an n-subhomogeneous C^*-algebra we obtain the conclusion. □

Corollary 3. *If a C^*-algebra A is n-homogeneous and I is an open interval, then* $K_A(I) = K_n(I)$.

Let $f_c(x) = x^c$ for $c > 2$ on the positive axis. Then f_c is a continuous convex function, but not 2-convex function by [4, Proposition 3.1].

Theorem 4. *Let A be a C^*-algebra. Then A is commutative if and only if there exists a convex function on the positive axis* $I = [0, \infty)$ *which is not a* 2*-convex function f but an A-convex function.*

Proof. Suppose that A is commutative. Let $f = f_c \quad (c > 2)$. Then f is a convex function which is not 2-convex.

Since A is commutative, there is a locally compact Hausdorff space X such that $A \cong C_0(X)$. For any $x \in X$ $ev_x(f) = f(x)$ for $f \in C_0(X)$.

Then for any $x \in X$, real-valued functions $a, b \in C_0(X)$ with $a(X), b(X) \subset I$, and any $0 \leq \lambda \leq 1$ we have

$$\begin{aligned} ev_x(f(\lambda a + (1-\lambda)b)) &= f(\lambda a(x) + (1-\lambda)b(x)) \\ &\leq \lambda f(a(x)) + (1-\lambda) f(b(x)) \\ &= \lambda f(ev_x(a)) + (1-\lambda) f(ev_x(b)) \\ &= ev_x(\lambda f(a) + (1-\lambda) f(b)), \end{aligned}$$

and $f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b)$. Hence f is an A-convex function.

Conversely, suppose that there exists a convex function on the positive axis $I = [0, \infty)$ which is not 2-convex function f but an A-convex function. We will show that A is commutative. We assume that A is not commutative. There exists an irreducible $\pi\colon A \to B(H)$ such that $\dim H \geq 2$. Take a projection $p \in B(H)$ such that $\dim pH = 2$. Then we have $\pi(A)p = B(H)p$ by Kadison's transitivity theorem.

Let $B = \{a \in A \mid \pi(a)pH \subseteq pH, \pi(a)^*pH \subseteq pH\}$ be the C^*-subalgebra of A consisting of elements mapped by π to operators that, together with their adjoints, leave pH invariant. Then $B \subseteq A$. The restriction of $\pi : B \mapsto pB(H)p$ to B is a 2-dimensional representation of B on pH. Since $B \subseteq A$, $f \in K_B$ f is $B(pH)$-convex, that is, 2-convex. This is a contradiction to the property of f. Therefore, A is commutative. □

Let $g_n(x) = t + \frac{1}{2}t^2 + \frac{1}{3}t^3 + \cdots + \frac{1}{2n}t^{2n}$ for $n \in \mathbb{N}$. Then there exists $\alpha_n > 0$ such that

$$g_n \in K_n((-\alpha_n, \alpha_n)) \backslash K_{n+1}((-\alpha_n, \alpha_n)).$$

Let I be a finite open interval such that $I = (t_0 - c, t_0 + c)$ for some $t_0 \in \mathbb{R}$ and a positive number c. Then $f_n(t) = g_n(\alpha_n c^{-1}(t - t_0))$ is in $K_n(I) \backslash K_{n+1}(I)$. (See [4, Proposition 1.4].)

Corollary 5. *If f_n is an A-convex function on I for a C^*-algebra A, then A is k-subhomogeneous for some $1 \leq k \leq n$.*

Proof. From Lemma 1 and Theorem 2 we have $K_A(I) \subseteq K_n(I)$ or $K_n(I) \subseteq K_A(I)$.

If $K_A(I) \subseteq K_n(I)$, then there exists $n_0 \geq n$ such that $K_A(I) = K_{n_0}(I)$. Since $f_n \in K_n(I) \backslash K_{n+1}(I)$, $n = n_0$. Hence A is n-subhomogeneous by (2) in Theorem 2.

If $K_n(I) \subseteq K_A(I)$, then $K_A(I) = K_{n_0}(I)$ for some $n \geq n_0$. Then A is n_0-subhomogeneous. □

To conclude, we stress that a subhomogeneous C^*-algebra is characterized in both linear versions and non-linear versions of matricial structure of a C^*-algebra by the above results, [9], and [14] as follows.

Theorem 6. *Let I be an open interval. For a C^*-algebra A the following properties are equivalent.*

1. *Every n-matrix convex function on I is A-convex.*
2. *Every n-matrix monotone on I is A-monotone.*
3. *The dimension of every irreducible representation of A is less than or equal to n.*
4. *All n-positive linear maps $\phi\colon A \to B$ and $\psi\colon B \to A$ are completely positive.*

Here a linear positive map $\phi : A \to B$ is said to be n-positive if the multiplicity maps

$$\phi_n = \phi \otimes 1_n \colon A \otimes M_n \to B \otimes M_n$$

are positive. A linear map ϕ is said to be completely positive if for any $n \in \mathbb{N}$ ϕ_n is positive.

Incidentally, one may moreover easily deduce from this result the corresponding characterizations of an n-homogeneous C^*-algebra. A C^*-algebra A is n-subhomogeneous if and only if every n-convex function on I is A-convex and there exists an $(n-1)$-convex function on I which is not A-convex.

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Operaator-kumerad funktsioonid üle C^*-algebrate

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On kirjeldatud C^*-algebrat A, mille korral A-kumerate funktsioonide klass ühtib operaator-kumerate funktsioonide klassiga või mingi maatriks-kumerate funktsioonide klassiga. Osutub, et C^*-algebra A korral langevad A-kumerate funktsioonide klass ja n järku maatriks-kumerate funktsioonide klass kokku parajasti siis, kui A on n-subhomogeenne.

Autorid on näidanud, et teatud tingimustel A-kumera funktsiooni olemasolu kindlustab C^*-algebra A kommutatiivsuse või subhomogeensuse.