MATHEMATICS

Simplified a priori estimate for the time periodic Burgers' equation

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Abstract. We present here a version of the existence and uniqueness result of time periodic solutions to the viscous Burgers' equation with irregular forcing terms (with Sobolev regularity -1 in space). The key result here is an a priori estimate which is simpler than the previously treated case of forcing terms with regularity $-\frac{1}{2}$ in time.

Key words: periodic solutions, Burgers' equation, distributions, Sobolev spaces, time-periodic.

INTRODUCTION

The study of the Burgers' equation has a long history starting with the seminal papers by Burgers [1], Cole [2], and Hopf [7] where the Cole–Hopf transformation was introduced. The Cole–Hopf transformation transforms the homogeneous Burgers' equation into the heat equation.

More recently there have been several articles dealing with the forced Burgers' equation:

$$u_t - \nu u_{xx} + u u_x = f.$$

The vast majority of the articles treat the initial value problem in time with homogeneous Dirichlet or periodic space boundary conditions (see for instance [9]).

Only recently has the question of the time-periodic forced Burgers' equation been tackled ([3,4,8,10]). In most cases [3,8] the authors are chiefly interested in the inviscid limit (the limit when the viscosity ν tends to zero).

The closest related work to ours is that of Jauslin et al. [8] in which the authors show existence and uniqueness of a space and time periodic solution of Burgers' equation for a space and time periodic forcing term which is smooth.

1. DEFINITIONS

In this section we recall some well-known facts and fix some general notations.

We will be concerned with *time-periodic* solutions of Burgers' equation, so we will use the following notation for the one-dimensional torus $\mathbb{T}$:

$$\mathbb{T} = \mathbb{R}/\mathbb{Z}.$$

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1.1. Fractional derivatives

Given a Hilbert space H, we will denote the space of test functions with values in H by $\mathscr{D}(\mathbb{T}, H)$. Its dual space will be denoted by $\mathscr{D}'(\mathbb{T}, H^*)$:

$$\mathscr{D}'(\mathbb{T}, H^*) = (\mathscr{D}(\mathbb{T}, H))^*.$$

For any positive real number s we may define the fractional derivative of order s in the following way on the space of time-periodic distributions $\mathscr{D}'(\mathbb{T}, H^*)$:

$$\mathrm{D}^s u = \sum_{k \in \mathbb{Z}} (2\pi \mathrm{i} k)^s u_k \mathrm{e}^{\mathrm{i}2\pi kt} = \sum_{k \in \mathbb{Z}} |2\pi \mathrm{i} k|^s \, \mathrm{e}^{\mathrm{i}\, \mathrm{sgn}(k)s\frac{\pi}{2}} u_k \mathrm{e}^{\mathrm{i}2\pi kt},$$

where we have used the principal branch of the logarithm. The *sign function* is defined as follows:

$$\mathrm{sgn}(k) := \begin{cases} \frac{k}{|k|} & \text{if } k \neq 0 \\ 0 & \text{if } k = 0. \end{cases}$$

For $s = 0$ we define $\mathrm{D}^0 = \mathrm{Id}$. D^1 coincides with the usual differentiation operator on $\mathscr{D}'(\mathbb{T}, H^*)$. The familiar composition property also holds: $\mathrm{D}^s \circ \mathrm{D}^t = \mathrm{D}^{s+t}$ for any $t, s \geq 0$.

The *adjoint operator* of D^s is defined by using the conjugate of the multiplier of D^s:

$$\mathrm{D}^s_* u = \sum_{k \in \mathbb{Z}} |2\pi \mathrm{i} k|^s \, \mathrm{e}^{-\mathrm{i}\, \mathrm{sgn}(k)s\frac{\pi}{2}} u_k \mathrm{e}^{\mathrm{i}2\pi kt}.$$

D^s and D^s_* are adjoints in the sense that for any $u \in \mathscr{D}'(\mathbb{T}, H^*)$ and $\varphi \in \mathscr{D}(\mathbb{T}, H)$:

$$\langle \mathrm{D}^s u, \varphi \rangle = \langle u, \mathrm{D}^s_* \varphi \rangle$$

and similarly

$$\langle \mathrm{D}^s_* u, \varphi \rangle = \langle u, \mathrm{D}^s \varphi \rangle.$$

1.2. Hilbert transform

The *Hilbert transform* $\mathscr{H}$ is defined using the multiplier $-\mathrm{i}\, \mathrm{sgn}(k)$. For $u \in \mathscr{D}'(\mathbb{T}, H^*)$ let

$$\mathscr{H} u = \sum_{k \in \mathbb{Z}} -\mathrm{i}\, \mathrm{sgn}(k) \, u_k \, \mathrm{e}^{\mathrm{i}2\pi kt}.$$

Simple computations then give:

$$\mathrm{D}^{\frac{1}{2}}_* = \mathrm{D}^{\frac{1}{2}} \circ \mathscr{H} = \mathscr{H} \circ \mathrm{D}^{\frac{1}{2}}.$$

Notice that if H is a function space then $\mathscr{H}$ maps real functions to real functions.

The following properties will be useful in the sequel:

$$\forall u \in \mathrm{H}^{\left(\frac{1}{2}\right)}(\mathbb{T}, H) \quad \left(\mathrm{D}^{\frac{1}{2}} u, \mathrm{D}^{\frac{1}{2}}_* \mathscr{H} u \right)_{\mathrm{L}^2(\mathbb{T}, H)} = -\left\| \mathrm{D}^{\frac{1}{2}} u \right\|^2_{\mathrm{L}^2(\mathbb{T}, H)}, \tag{1}$$

$$\forall u \in \mathrm{L}^2(\mathbb{T}, H) \quad \Re\left((u, \mathscr{H}(u))_{\mathrm{L}^2(\mathbb{T}, H)} \right) = 0,$$

where $\Re$ denotes the real part of the expression.

1.3. Fractional Sobolev spaces

We define fractional Sobolev spaces in the following manner, for any $s \in \mathbb{R}$:

$$\mathrm{H}^{(s)}(\mathbb{T}, H) = \left\{ u \in \mathscr{D}'(\mathbb{T}, H^*); \quad \sum_{k \in \mathbb{Z}} \left|1 + k^2\right|^s \|u_k\|_H^2 < \infty \right\}.$$

Of course $\mathrm{H}^{(0)}(\mathbb{T}, H) = \mathrm{L}^2(\mathbb{T}, H)$. When $s \geq 0$ then for a $u \in \mathrm{L}^2(\mathbb{T}, H)$: $u \in \mathrm{H}^{(s)}(\mathbb{T}, H) \iff \mathrm{D}^s u \in \mathrm{L}^2(\mathbb{T}, H)$. Moreover, $\mathrm{H}^{(s)}(\mathbb{T}, H)$ is then a Hilbert space with the following scalar product:

$$(u, v) := (u, v)_{\mathrm{L}^2(\mathbb{T}, H)} + (\mathrm{D}^s u, \mathrm{D}^s v)_{\mathrm{L}^2(\mathbb{T}, H)}.$$

The following classical result holds: $\left(\mathrm{H}^{(s)}(\mathbb{T}, H)\right)^* = \mathrm{H}^{(-s)}(\mathbb{T}, H^*)$.

1.4. Anisotropic fractional Sobolev spaces

We will now describe some useful Sobolev spaces defined on the interval

$$I = (0, 1).$$

We will also fix a nonnegative real number s:

$$s \geq 0.$$

Let $\mathrm{H}^{(s)}(I)$ denote the usual fractional Sobolev space of real-valued s-times differentiable functions on I. $\mathrm{H}_0^{(s)}(I)$ is the closure of $\mathscr{D}(I)$ in $\mathrm{H}^{(s)}(I)$. In that case we have $\left(\mathrm{H}_0^{(s)}(I)\right)^* = \mathrm{H}^{(-s)}(I)$. We will also use the following notations, for α, β nonnegative real numbers:

$$\mathrm{H}^{(\alpha)(\beta)}(\mathbb{T} \times I) := \mathrm{H}^{(\alpha)}(\mathbb{T}, \mathrm{H}^{(\beta)}(I))$$

and

$$\mathrm{H}^{(\alpha, \beta)}(\mathbb{T} \times I) := \mathrm{H}^{(\alpha)(0)}(\mathbb{T} \times I) \cap \mathrm{H}^{(0)(\beta)}(\mathbb{T} \times I).$$

We also introduce $\mathrm{H}_0^{(\alpha, \beta)}(\mathbb{T} \times I)$ as the closure of $\mathscr{D}(\mathbb{T} \times I)$ in $\mathrm{H}^{(\alpha, \beta)}(\mathbb{T} \times I)$. It is clear that $\mathrm{H}_0^{(\alpha, \beta)}(\mathbb{T} \times I) = \mathrm{H}^{(\alpha)(0)}(\mathbb{T} \times I) \cap \mathrm{L}^2(\mathbb{T}, \mathrm{H}_0^{(\beta)}(I))$.

Below we will use the following notations. The main working space will be $\mathrm{H}_0^{\left(\frac{1}{2}, 1\right)}(\mathbb{T} \times I)$, for which we use the more concise notation:

$$\mathrm{H}_0^{\left(\frac{1}{2}, 1\right)} := \mathrm{H}_0^{\left(\frac{1}{2}, 1\right)}(\mathbb{T} \times I).$$

More generally, we will drop the dependence of the spaces on the domain $\mathbb{T} \times I$, so that L^p stands for $\mathrm{L}^p(\mathbb{T} \times I)$, $\mathrm{H}^{(\alpha, \beta)}$ stands for $\mathrm{H}^{(\alpha, \beta)}(\mathbb{T} \times I)$, and so on.

We will also use the following notations, for $u \in \mathrm{H}_0^{\left(\frac{1}{2}, 1\right)}$:

$$\forall u \in \mathrm{H}_0^{\left(\frac{1}{2}, 1\right)} \qquad \begin{aligned} \widetilde{u} &:= \mathscr{H} u \\ u_{\sqrt{t}*} &:= \mathrm{D}_*^{\frac{1}{2}} u. \end{aligned} \tag{2}$$

2. INTERPOLATION AND REGULARITY

If $s_k(\xi)$ is the Fourier transform $s_k(\xi) = \hat{u}(k,\xi)$ of a distribution u defined on $\mathbb{T} \times \mathbb{R}$, we have the following Hölder inequality for any $\theta \in [0,1]$:

$$\int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |k|^{2\alpha(1-\theta)} |\xi|^{2\beta\theta} |s_k(\xi)|^2 \, d\xi \leq \left(\int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |k|^{2\alpha} |s_k(\xi)|^2 \, d\xi \right)^{1-\theta} \left(\int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |\xi|^{2\beta} |s_k(\xi)|^2 \, d\xi \right)^{\theta}.$$

From this Hölder inequality we deduce

$$\mathrm{H}^{(\alpha,\beta)}(\mathbb{T} \times \mathbb{R}) \hookrightarrow \mathrm{H}^{((1-\theta)\alpha)}(\mathbb{T}, \mathrm{H}^{(\theta\beta)}(\mathbb{R})).$$

So using an extension operator from $\mathrm{H}^{(\theta\beta)}(I)$ to $\mathrm{H}^{(\theta\beta)}(\mathbb{R})$ one can prove the corresponding inclusion:

$$\mathrm{H}^{(\alpha,\beta)}(\mathbb{T} \times I) \hookrightarrow \mathrm{H}^{((1-\theta)\alpha)(\theta\beta)}(\mathbb{T} \times I).$$

For $\alpha = 1/2$ and $\beta = 1$ and $\theta = \frac{1}{3}$ we get:

$$\mathrm{H}_0^{\left(\frac{1}{2},1\right)}(\mathbb{T} \times I) \subset \mathrm{H}^{\left(\frac{1}{2},1\right)}(\mathbb{T} \times I) \subset \mathrm{H}^{\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}(\mathbb{T} \times I).$$

Then the vectorial Sobolev inequalities yield:

$$\mathrm{H}_0^{\left(\frac{1}{2},1\right)}(\mathbb{T} \times I) \subset \mathrm{H}^{\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}(\mathbb{T} \times I) \hookrightarrow \mathrm{L}^4(\mathbb{T}, \mathrm{H}^{\left(\frac{1}{3}\right)}(I)) \hookrightarrow \mathrm{L}^4(\mathbb{T}, \mathrm{L}^4(I)) = \mathrm{L}^4(\mathbb{T} \times I).$$

Here the injection $\mathrm{H}^{\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}(\mathbb{T} \times I) \hookrightarrow \mathrm{L}^4(\mathbb{T}, \mathrm{H}^{\left(\frac{1}{3}\right)})$ is compact and thus the injection $\mathrm{H}_0^{\left(\frac{1}{2},1\right)}(\mathbb{T} \times I) \hookrightarrow \mathrm{L}^4(\mathbb{T} \times I)$ is *compact*.

3. MAIN RESULT: A SIMPLIFIED A PRIORI ESTIMATE

We define Burgers' Operator by:
$$\mathrm{T} = \mathscr{L} + S,$$

where $\mathscr{L}$ and S are defined in the familiar weak form, the bracket being the *duality bracket* between $\mathrm{H}_0^{\left(\frac{1}{2},1\right)}$ and $\mathrm{H}^{\left(-\frac{1}{2},-1\right)}$ (recall the notations (2)):

$$\forall v \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)} \quad \langle \mathscr{L}u, v \rangle := \left(u_{\sqrt{t}}, v_{\sqrt{t}*} \right) + \mu \left(u_x, v_x \right)$$

and

$$\forall v \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)} \quad \langle S(u), v \rangle := -\frac{1}{2} \left(u^2, v_x \right).$$

It turns out that the second definition makes sense because of the embedding $\mathrm{H}_0^{\left(\frac{1}{2},1\right)} \subset \mathrm{L}^4$ (see Fig. 1).

We now prove the main result of this article, namely an a priori estimate on the solutions of the family of equations:
$$(\mathscr{L} + \lambda S)u = f$$

for $f \in \mathrm{H}^{(0,-1)}$ and $0 \leq \lambda \leq 1$.

In the proof we use techniques similar to those in [6]. The proof is simpler (but the result is weaker) than the one obtained in [5].
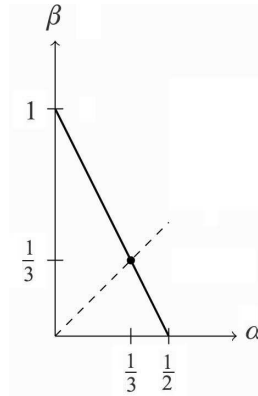
Fig. 1. $H_0^{\left(\frac{1}{2},1\right)}$ is included in $H^{\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}$ which is included in L^6 by the usual Sobolev inclusion theorem. In particular, $H_0^{\left(\frac{1}{2},1\right)}$ is included in L^4, so $u \in H_0^{\left(\frac{1}{2},1\right)} \implies u^2 \in L^2$. As a result the nonlinear term of Burgers' equation may be written as $-(u^2, v_x)$ for a test function $v \in H_0^{\left(\frac{1}{2},1\right)}$ since $v \in H_0^{\left(\frac{1}{2},1\right)} \implies v_x \in L^2$ by definition.

Theorem 1. *Let $f \in H^{(0,-1)}$. The set*

$$\bigcup_{\lambda \in [0,1]} (\mathscr{L} + \lambda S)^{-1}(\{f\})$$

is bounded in $H_0^{\left(\frac{1}{2},1\right)}$.

We will need the following Lemma which may be proved using a scaling argument (a proof is available in [5]).

Lemma 3.1. *There exists a constant $C \in \mathbb{R}$ such that for any $u \in H_0^{\left(\frac{1}{2},1\right)}(Q)$:*

$$\int_Q |u(t,x)|^4 \, dt \, dx \le C^2 \left(\int_Q |u|^2 \, dt \, dx + \int_Q |u_{\sqrt{t}}|^2 \, dt \, dx \right) \cdot \left(\int_Q |u_x|^2 \, dt \, dx \right),$$

which implies that

$$\left| u^2 \right| \le C \|u\| \, |u_x|.$$

Proof of Theorem 1. By definition $\mathscr{L}u + \lambda S(u) = f$ means:

$$\forall v \in H_0^{\left(\frac{1}{2},1\right)} \quad \left(u_{\sqrt{t}}, v_{\sqrt{t*}}\right) + \mu \left(u_x, v_x\right) - \frac{1}{2}\lambda \left(u^2, v_x\right) = \langle f, v \rangle. \tag{3}$$

1. We notice that for smooth u:

$$
\begin{aligned}
\left(u^2, u_x\right) &= \int_Q u^2 u_x \\
&= \frac{1}{3} \int_Q (u^3)_x \\
&= 0
\end{aligned}
$$

and then by density and continuity this holds for all $u \in H_0^{\left(\frac{1}{2},1\right)}$.

2. With $v = u$ in (3) we get:

$$\underbrace{\left(u_{\sqrt{t}}, u_{\sqrt{t}*}\right)}_{=0} + \mu\left(u_x, u_x\right) + \frac{1}{2}\lambda \underbrace{\left(u^2, u_x\right)}_{=0} = \langle f, u \rangle,$$

which gives:

$$|u_x|^2 = \frac{\langle f, u \rangle}{\mu} \le \frac{\|f\| \, |u_x|}{\mu}.$$

From this we deduce that

$$|u_x| \le \frac{\|f\|}{\mu}. \tag{4}$$

3. Pairing in (3) with the Hilbert transform of u, $v = \widetilde{u}$ we get:

$$\left(u_{\sqrt{t}}, \widetilde{u}_{\sqrt{t}*}\right) + \mu\underbrace{\left(u_x, \widetilde{u}_x\right)}_{=0} + \frac{1}{2}\lambda\left(u^2, \widetilde{u}_x\right) = \langle f, \widetilde{u} \rangle.$$

Using identity (1), the fact that $|\widetilde{u}_x| = |u_x|$ and that $\lambda \le 1$ we get:

$$\left|u_{\sqrt{t}}\right|^2 \le \frac{1}{2}\left|\left(u^2, \widetilde{u}_x\right)\right| + \|f\| \, |u_x|.$$

4. We estimate $\left|\left(u^2, \widetilde{u}_x\right)\right|$ using Lemma 3.1:

$$\left|\left(u^2, \widetilde{u}_x\right)\right| \le |u^2| \, |u_x| \le C \|u\| \, |u_x|^2. \tag{5}$$

5. Using the estimate (4) inside (5) we obtain:

$$\left|u_{\sqrt{t}}\right|^2 \le \frac{C}{2}\|f\| \, |u_x|^2 + \|f\| \, |u_x| \le \frac{\|f\|^2}{\mu}\left(\frac{C}{2\mu}\|u\| + 1\right).$$

Since that estimate does not depend on λ the theorem is proved. $\square$

The a priori estimate above may now be used to prove the existence of solutions by a (nonlinear, compact) degree argument using the Leray–Schauder Theorem (cf. [5]).

4. EXISTENCE AND UNIQUENESS OF THE TIME-PERIODIC SOLUTIONS

A weaker result of the main result proved in [5] is

Theorem 2. *For $f \in \mathrm{H}^{(0)(-1)}$ there exists a unique solution $u \in \mathrm{H}_0^{\left(\frac{1}{2}, 1\right)}$ of*

$$\mathrm{T}u = f.$$

In this section we briefly sketch the proof of that result, the full proof being available in [5].
The main ingredient of the proof is the Cole–Hopf transformation, which is essentially defined by:

$$u = \frac{\varphi_x}{\varphi}.$$

In our case there are complications due to the fact that $u \in H_0^{\left(\frac{1}{2},1\right)}$, so u is time-periodic. This change of variable will transform the periodicity problem into an eigenvalue problem (because the Cole–Hopf transformation linearizes Burgers' equation). After working out the details one shows that the uniqueness problem is equivalent to the uniqueness of the *ground state eigenvalue problem*:

Proposition 4.1. *Given* $v \in H_0^{\left(\frac{1}{2},1\right)}$*, the solution set of the following equation in K and* φ

$$
\begin{cases}
\varphi_t - \mu\varphi_{xx} + v\varphi_x + K\varphi = 0 \\
\varphi > 0 \\
\varphi_x|_{\partial Q} = 0 \\
\varphi \in H^{(1,2)} \\
K \in \mathbb{R}
\end{cases}
$$

is $K = 0$ *and* $\varphi = 1$ *if and only if* $Tu = Tv$ *implies* $u = v$ *(that is, the solution to the original Burgers' equation is unique).*

The proof of that proposition essentially hinges on the embedding properties exposed in Section 2 (see Fig. 2).

The remaining part of the proof is concerned with the eigenvalue problem of Proposition 4.1. One first shows that the eigenvalue is zero using a weaker version of the Perron–Frobenius theorem. The second step is to show that the remaining eigenvalue problem is *non degenerate*, namely that the dimension of the eigenspace must be one. This last step makes use of the a priori estimate proved in Theorem 1.

The details of that part of the proof are too lengthy to be exposed here in depth so the interested reader is referred to [5].
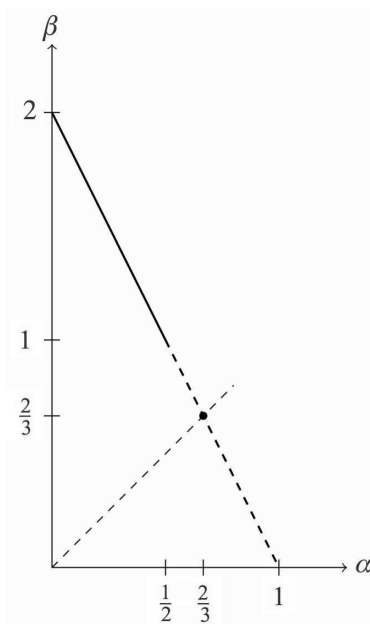

Fig. 2. The first step of the Cole–Hopf Transformation is an integration in x. This function U obtained thus ends up in $H^{(0)(1)} \cap H^{\left(\frac{1}{2}\right)(1)}$, which delimits the plain line on the graph above. But it follows from $Tu \in H^{(0)(-1)}$ that u is actually also in $H^{(1)(-1)}$ so U ends up in $H^{(1)(2)}$ and we have an inclusion in $H^{\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)}$ which is embedded in continuous Hölder functions.

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Lihtsustatud aprioorne hinnang Burgersi võrrandi perioodilise ülesande korral

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On tõestatud Burgersi võrrandi perioodilise ülesande lahendi olemasolu ja ühesus. Tulemuse rangem versioon on varem tõestatud artiklis [5], käesolevas on esitatud nõrgem tulemus, kasutades aprioorset hinnangut lahendile. Sellel hinnangul põhinev lahendi olemasolu ja ühesuse tõestus tugineb Leray-Schauderi teoreemile. Töös [5] saadud tulemus kujutab endast omakorda artiklis [8] esitatud lahendi olemasolu ja ühesuse tingimuste edasiarendust, kusjuures tõestuses kasutatakse täiesti erinevaid meetodeid.