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MATHEMATICS

Neoclassical invariant theory – some lost theorems – a mathematical chat

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Abstract. In this short note I provide an account of some results and ideas, as well as a review of certain aspects and history of the invariant theory, linking it to multilinear forms, multidimensional matrices and geometry, and analysis of symmetric domains.

Key words: invariants, covariants, multilinear forms, hyperdeterminants, symmetric domain.

The first example of an invariant in mathematical literature is probably due to Joseph-Louis Lagrange[1] in his famous book *Analytical Mechanics* [6] first published in 1788. Consider the quadratic form

$$f = a_0x_1^2 + 2a_1x_1x_2 + a_2x_2^2.$$

Then its *discriminant* $D = 2(a_0a_2 - a_1^2)$ is an invariant. If we make a linear transformation

$$x_1 = \alpha x_1' + \beta x_2', \;\; x_2 = \gamma x_1' + \delta x_2',$$

with the *module* $\Delta = \alpha\delta - \beta\gamma$, we get another quadratic form f', with its discriminant being D'. Then there holds

$$D' = \Delta^2 D.$$

Later the same example was given by C. F. Gauß in his *Disquisitiones Arithmeticae*. Subsequently, German authors often quote only him [10].

In the mid-1990s I got interested in invariant theory via my previous preoccupation with multilinear, especially *trilinear* forms, in cooperation with Fernando Cobos (Madrid) and Thomas Kühn (Leipzig). I had several – what I then thought were – bright insights. One of my ideas was a new interpretation of the *symbolic method*, due to Arthur Cayley and developed by Aronhold[2] and Clebsch[3] [10]. In particular, I used

[1] Guiseppe Ludovico Lagrangia in Italian. MacTutor has an interesting biography of Lagrange. He was born in Turin (Torino) but his grandfather was a French officer in the service of the king of Sardinia. For a long time he worked in Berlin, towards the end of his life in Paris. The nationality did not play a very big role in the 18th century.

[2] Siegfried Heinrich Aronhold (1819–1884), German mathematician.

[3] Rudolf Friedrich Alfred Clebsch (1833–1872), outstandig German mathematician, who died young of diphtheria. There is an amusing thing connected with his name. It appears in the book *Le Sceptre d'Ottokar* (English, *King Ottokar's Sceptre*) of the Belgian writer and illustrator Hergé, featuring the young reporter Tintin, pen name of Georges Prosper Remi (1907–1983). It is supposed to be an allegoric tale of Hitler's conquest of Austria in 1938, set in an imaginary Balkan country where they speak a likewise imaginary language modelled on German. Milou (Snowy), the dog of Tintin, is falling down from an airplane. Some farmers watching this from the ground then cry out in their Crypto-German: 'Czwztot on klebcz!' The question is now, is this just incidental, or does it indicate that Remi might have studied algebra?

my ideas for determining invariants of trilinear forms. Binary such forms have essentially only one invariant – the *Cayley hyperdeterminant*, another early instance of invariants, discovered by Cayley in 1845 [1,2], rediscovered many times later, also by me, revived by Gel'fand et al. [5]. For some time all invariants were called hyperdeterminants. So invariant theory was viewed as a generalization of determinant. But I found also many invariants for trilinear forms in higher dimensions, not only D2.

Here is an example; if I ever had a proof, I do not recall it today. For *bibinary-ternary* trilinear form I found [7] the following sextic invariant:

$$\begin{vmatrix} A_{111} & A_{211} & A_{311} \\ A_{112} & A_{212} & A_{312} \\ A_{112} & A_{212} & A_{321} \end{vmatrix} \cdot \begin{vmatrix} A_{122} & A_{222} & A_{322} \\ A_{112} & A_{212} & A_{312} \\ A_{121} & A_{221} & A_{321} \end{vmatrix} + \begin{vmatrix} A_{111} & A_{211} & A_{311} \\ A_{112} & A_{212} & A_{312} \\ A_{122} & A_{222} & A_{322} \end{vmatrix} \cdot \begin{vmatrix} A_{111} & A_{211} & A_{311} \\ A_{121} & A_{221} & A_{321} \\ A_{112} & A_{222} & A_{322} \end{vmatrix};$$

here $A_{jk\ell}$ ($j = 1,2,3,\ k,\ell = 1,2$) is the corresponding 3-dimensional matrix. I suspected then that there exists also a similar *bibinary-quaternary* invariant. None of this has yet been proved. It may also be that I possessed only a heuristic argument based on experiments with Mathematica.

However, all my plans came quickly to an end. On 16 October 1999 I got exposed to a stroke, a rather mild one, according to my doctors, but wiping out large parts of my memory. I remember that the day before I had thought intensively on the problem of determining the volume of the *trilinear ball*. Oh, you know what a ball is! One of my pet ideas had been that this ball is the *analogue of a bounded symmetric domain*! So if you know the size of the ball, you would have the first step towards harmonic analysis of this ball!

Suddenly my bright ideas were all gone with the wind and I was left with a lot of more or less unfinished fragments.

During my last term as a professor, spring term of 2000, I taught a course in Galois theory – my own choice! In front of my students I had to conceal carefully that the professor could not do simple algebra, as moving terms in an equation from one side to the other. Despite this some students afterwards thanked me for my interesting teaching.

Even the other day I hit on a pile of thick folders of manuscripts from this period. At least one of them deals with invariants (which makes me a little bit more optimistic today!), incidently containing also an e-mail from Gel'fand, where he invites me to cooperate with him and his group. However, such a cooperation never took place, but I nevertheless preserve this letter as a kind of 'trophy'. At a certain later stage he wanted me to read a paper by Ernst Fischer [4].[4] I did not understand a word of it.

The rest of my chat will concentrate on one such fragment [8], partly reconstructed with the help of Hjalmar Rosengren (now in Göteborg). It is research done while Genkai Zhang (likewise in Göteborg) visited Lund in the autumn of 1998, because it carries the names of these three authors. I remember Hjalmar exclaiming loud: 'Now I know what the symbolic representation is about!' I do not know why he denies this today.

In the case when the coefficients a_i ($i = 0,1,2$) of f are complex numbers, the group of the above transformations is $\mathrm{GL}(2,\mathbb{C})$. If they are only real, one gets instead its *real form* $\mathrm{GL}(2,\mathbb{R})$. Then we deal with functions on the *Riemann sphere* S^2, which, as is well known, can be identified to the *complex projective plane* $P^2(\mathbb{C})$; at least this was known to me at the time [9].

It is natural to ask what happens when one instead considers the real form $G = \mathrm{SU}(1,1)$ consisting of all unimodular 2×2 matrices, i.e.

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$$

if and only if $\alpha\delta - \beta\gamma = 1$, $\gamma = \bar{\beta}$, $\delta = \bar{\alpha}$.

[4] This is the Fischer of 'Riesz-Fischer'. Once during a meeting Frederick Riesz said that he had always wanted to encounter Fischer. He was very amused when people told him: 'But you have been sitting besides him the whole hour!'

This group has a natural action (via Moebius transformations) on the unit disc $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$. A function, analytic in $\mathbb{D}$,

$$f = f(z) = \sum_{j=1}^{\infty} a_j z^j,$$

belongs to the *weighted Bergman space* $A = A^\nu$ $(\nu > -1)$ [3] if and only if the following norm is finite:

$$\|f\|_A^2 = \int_{\mathbb{D}} |f(z)|^2 d\mu(z) = \sum_{j=1}^{\infty} |a_j|^2 \frac{j!}{(\nu)_j},$$

where

$$d\mu = d\mu_\nu(z) = \frac{\nu - 1}{\pi}(1 - |z|^2)^{\nu-2} dx\, dy.$$

Here we shall only deal with invariants $J = J(f)$ which are polynomials of some bidegree (p, q), i.e.

$$J(cf) = c^p \bar{c}^q J(f) \text{ for } c \in \mathbb{C},$$

with the expansion

$$J(f) = \sum_{j_1,\dots,j_p,k_1,\dots,k_q=0}^{\infty} C_{j_1,\dots,j_p,k_1,\dots,k_q} a_{j_1} \dots a_{j_p} \bar{a}_{k_1} \dots \bar{a}_{k_q}.$$

Here $C_{j_1,\dots,j_p,k_1,\dots,k_q} \in \mathbb{C}$ and the a_j are the Taylor coefficients of f.

If we introduce the functions

$$\begin{aligned} u(\bar{z}_1, \dots \bar{z}_p, \bar{w}_1, \dots \bar{w}_q) &= \sum_{j_1,\dots,j_p,k_1,\dots,k_q=0}^{\infty} C_{j_1,\dots,j_p,k_1,\dots,k_q} \bar{z}_1^{j_1} \dots \bar{z}_p^{j_p} \\ &\times \bar{w}_1^{k_1} \dots \bar{w}_q^{k_q} \|z_1\|_{A^\nu}^{j_1} \dots \|z_p\|_{A^\nu}^{j_p} \Big/ (\|w_1^{k_1}\|_{A^\nu} \dots \|w_q^{k_q}\|_{A^\nu}), \end{aligned}$$

we can write this as an integral:

$$\begin{aligned} J(f) &= \int_{\mathbb{D}^{p+q}} u(\bar{z}_1, \dots \bar{z}_p, \bar{w}_1, \dots, \bar{w}_q) f(z_1) \dots f(z_p) f(w_1) \dots f(w_q) \\ &\times d\mu(z_1) \dots d\mu(z_p)\, d\mu(w_1) \dots d\mu(w_q). \end{aligned}$$

It turns out that u is a *covariant*. Indeed, this shows that *there is a correspondence between invariants and covariants*.

Finally we mention the following result:

Theorem 1. *If $p = q = 1$, then, up to a scalar the only invariant for G is the square of the norm $\|f\|_{A^\nu}$.*

References

1. Cayley, A. On the theory of linear transformations. *Cambridge Math. J.*, 1845, **4**, 193–209.
2. Crilly, T. and Crilly, A. J. *Arthur Cayley: Mathematician Laureate of the Victorian Age*. JHU Press, 2006.
3. Engliš, M., Hille, S. C., Rosengren, H., and Zhang, G. A new kind of Hankel-Toeplitz type operator connected with the complementary series. *Arab J. Math. Sci.*, 2000, **6**, 49–80.
4. Fischer, E. S. Über die Cayleysche Eliminationsmethode. *Math. Z.*, 1927, **26**, 497–550.

5. Gel'fand, I. M., Kapranov, M. M., and Zelevinsky, A. V. *Discriminants, Resultants and Multidimensional Determinants*. Birkhäuser, Boston, 1994.
6. Lagrange, J.-L. *Mécanique Analytique* (Analytical Mechanics) (4. ed., 2 vols. Paris: Gauthier-Villars et fils, 1888–89. First Edition: 1788.)
7. Peetre, J. *On Quaternary-Bibinary Forms*. Manuscript, Kåseberga, August 31, 1994.
8. Peetre, J., Rosengren, H., and Zhang, G. *Neoclassical Invariant Theory*. New edition of 1998 version, 2007.
9. Peetre, J. and Zhang, G. Harmonic analysis on the quantized Riemann sphere. *Internat. J. Math. Math. Sci.*, 1993, **16**, 225–243.
10. Timmerding, H. E. Invariantentheorie. In *Repertorium der höheren Mathematik. I. Analysis. Erste Hälfte. Algebra, Differential- und Integralrechnung* (Pascal, E., ed.). B. G. Teubner, Leipzig, 1910.

Neoklassikaline invariantide teooria. Kaotatud teoreemid. Matemaatilised mälestused

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On vaadeldud invariantide teooria mõningaid tulemusi, ideid, aspekte ja selle teooria ajalugu. On näidatud, kuidas invariantide teooriat saab rakendada multilineaarsete vormide, ruumiliste maatriksite ja sümmeetriliste piirkondade uurimiseks. Autor on kirjutanud isiklikest mälestustest seoses invariantide teooria uurimisega.