Proceedings of the Estonian Academy of Sciences,
2010, **59**, 1, 29–33
doi: 10.3176/proc.2010.1.05
Available online at www.eap.ee/proceedings

MATHEMATICS

The Besicovitch covering theorem and near-minimizers for the couple (L^2, BV)

Irina Asekritova[a] and Natan Kruglyak[b*]

[a] Department of Mathematics, Växjö University, SE-351 95 Växjö, Sweden; irina.asekritova@vxu.se
[b] Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden

Received 10 November 2009, accepted 2 February 2010

Abstract. Let Ω be a rectangle in $\mathbb{R}^2$. A new algorithm for the construction of a near-minimizer for the couple $(L^2(\Omega), BV(\Omega))$ is presented. The algorithm is based on the Besicovitch covering theorem and analysis of local approximations of the given function $f \in L^2(\Omega)$.

Key words: covering theorems, near-minimizers.

The theory of real interpolation is based on the notion of Peetre's K-functional, which is defined for the couple of Banach spaces (X_0, X_1) by the formula

$$K(t,x;X_0,X_1) = \inf_{u \in X_1} (\|x-u\|_{X_0} + t\,\|u\|_{X_1}),$$

where $x \in X_0 + X_1$ and $t > 0$.

It is well known that the K-functional is connected (see [1,2]) to the more general $L_{p,q}$-functional, defined for $0 < p,\, q < \infty$ by the expression

$$L_{p,q}(t,x;X_0,X_1) = \inf_{u \in X_1} (\|x-u\|_{X_0}^p + t\,\|u\|_{X_1}^q),\ t > 0.$$

Definition 1. *An element $x_t \in X_1$ will be called a near-minimizer for the $L_{p,q}$-functional of the element $x \in X_0 + X_1$ at the point $t > 0$ if*

$$\|x-x_t\|_{X_0}^p + t\,\|x_t\|_{X_1}^q \le cL_{p,q}(t,x;X_0,X_1).$$

In the case when $c = 1$ the element x_t will be called the exact minimizer for the $L_{p,q}$-functional.

Let us give one example connected to inverse problems (see, for example, [3]). Let $A : X \to Y$ be a given linear bounded operator which maps the Hilbert space X to the Hilbert space Y. Suppose that we observe the quantity

$$y = Ax + \eta, \tag{1}$$

where $\eta \in Y$ is a 'noise' and we are interested in the reconstruction of the exact solution $x \in X$ of (1).

* Corresponding author, natan.kruglyak@liu.se

An attempt to define an approximate solution of (1) as $A^{-1}y$ does not lead to success even in the case when A is injective and y belongs to the image of the operator A, since such a procedure usually increases the level of noise. For example, this happens even in the finite-dimensional case when A has rather small singular numbers. Hence more sophisticated ideas are used to solve this problem. One of the most popular methods to construct an approximate solution to (1) is the classical Tikhonov regularization strategy, which suggests that as an approximate solution we take an element $x_t \in X$ which minimizes the Tikhonov functional

$$T_A(t,y) = \|y - Ax\|_Y^2 + t\,\|x\|_X^2$$

for some parameter $t > 0$ chosen on the basis of a priori information on the noise η and the exact solution x.

In fact, the Tikhonov method is deeply connected to the problem of constructing a minimizer for the $L_{2,2}$-functional for the Hilbert couple $(Y, A(X))$, where by $A(X)$ we denote the image of the space X with the norm

$$\|u\|_{A(X)} = \inf_{u=Ax} \|x\|_X .$$

Indeed, from the equality

$$\begin{aligned} L_{2,2}(t,y;Y,A(X)) &= \inf_{u\in A(X)} (\|y-u\|_Y^2 + t\,\|u\|_{A(X)}^2) \\ &= \inf_{x\in X} \left(\|y-Ax\|_Y^2 + t\,\|x\|_X^2\right) = \inf_{x\in X} T(t,y) \end{aligned}$$

it follows that if the element $x_t \in X$ is the minimizer for the Tikhonov functional, then

$$L_{2,2}(t,y;Y,A(X)) = \|y - Ax_t\|_Y^2 + t\,\|x_t\|_X^2 = \|y - Ax_t\|_Y^2 + t\,\|Ax_t\|_{A(X)}^2 ,$$

and therefore Ax_t is a minimizer for the $L_{2,2}$-functional. Moreover, in the case when A is an injective operator the element x_t can be reconstructed as $A^{-1}u_t$, where u_t is a minimizer for the $L_{2,2}$-functional of the element y for the Hilbert couple $(Y, A(X))$.

The Tikhonov method provides regularization only for the case of Hilbert spaces, however, during the last years regularization in the case when X, Y are not Hilbert spaces have attained a great importance. For example, a ROF (Rudin–Osher–Fatemi) model, which is very popular in image processing, suggests (see the books [4,5]) reduction of the level of noise in the 'noisy' image $f \in L^2$ by taking instead of f an element f_t which minimizes the $L_{2,1}$-functional of the couple (L^2, BV)

$$L_{2,1}(t,f;L^2,BV) = \inf_{g\in BV} (\|f-g\|_{L^2}^2 + t\,\|g\|_{BV}). \tag{2}$$

However, to construct the exact minimizer and even a near-minimizer for the functional (2) is not a simple task and several different approaches have been suggested (see, for example, [6–9], and the book [4]). The most popular methods are based on nonlinear elliptic partial differential equations and on wavelet theory.

Below we will present a new algorithm for constructing a near-minimizer for (2). This algorithm is a limiting case of the general algorithm (see [10,11]) and is based on the theory of local approximations (see [12,13]) and the Besicovitch covering theorem [14].

The algorithm consists of three steps (see Fig. 1). In the first step we analyse the local approximations of the given function $f \in L^2$ and construct the initial family of cubes[1] $\{Q_x\}$. In the second step we apply the Besicovitch covering theorem to the family $\{Q_x\}$ and obtain a family of cubes $\{K_i\}_{i\in I}$ consisting of a finite number of cubes. In the third step we use the family of cubes $\{K_i\}_{i\in I}$ to construct a partition of unity and to define a near-minimizer f_t.

Let us consider the algorithm in detail. Let $f \in L^2 = L^2(\Omega)$, where $\Omega \subset R^2$ is a rectangle, and let $t > 0$ be a given number. We will construct a near-minimizer for (2), i.e. an element f_t such that

$$\|f - f_t\|_{L^2}^2 + t\,\|f_t\|_{BV} \le cL_{2,1}(t,f;L^2,BV)$$

with the constant $c > 1$ independent of $f \in L^2$ and $t > 0$.

[1] Here and below we consider only cubes with sides parallel to the coordinate axes.

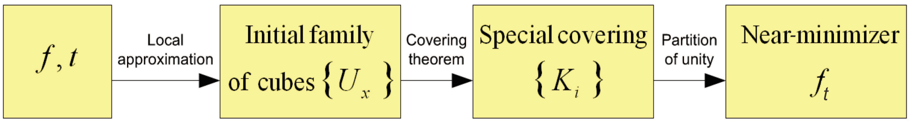

Fig. 1. The three steps of the algorithm.

If $t \geq t_*$, where

$$t_* = \left(\int_\Omega |f(s) - f_{\Omega)}|^2 ds \right)^{\frac{1}{2}}, \; f_\Omega = \frac{1}{|\Omega|} \int_\Omega f(s) ds, \tag{3}$$

then it is possible to prove that the near-minimizer can be defined as a constant function on Ω equal to f_Ω. Hence it is enough to consider the case when $t < t_*$.

In the first step of the algorithm we construct the initial family of cubes $\{Q_x = Q(x, r_x)\}_{x \in \Omega}$. Here and below by $Q(x, r)$ we denote a cube Q with the centre at the point x and radius r equal to the half side length of Q.

To construct the cube $Q(x, r_x)$ for $x \in \Omega$, we consider the function

$$\varphi_x(r) = \left(\int_{Q(x,r) \cap \Omega} |f(s) - f_{Q(x,r)}|^2 ds \right)^{\frac{1}{2}},$$

where

$$f_{Q(x,r)} = \frac{1}{|Q(x,r) \cap \Omega|} \int_{Q(x,r) \cap \Omega} f(s) ds.$$

It is clear that the function $\varphi_x(r)$ is a nondecreasing continuous function with values from zero to t_* defined in (3). As $t < t_*$, therefore for each $x \in \Omega$ there exists a number $r = r_x$ such that (see Fig. 2)

$$\varphi_x(r_x) = \left(\int_{Q(x,r_x) \cap \Omega} |f(s) - f_{Q(x,r_x)}|^2 ds \right)^{\frac{1}{2}} = t.$$

As the initial family of cubes $\{Q_x\}_{x \in \Omega}$we will take the constructed family $\{Q(x, r_x)\}_{x \in \Omega}$. Note that from the condition $\varphi_x(r_x) = t$ it follows that if the subfamily $\{Q(x_i, r_{x_i})\}_{i \in I}$ of $\{Q(x, r_x)\}_{x \in \Omega}$ consists of pairwise

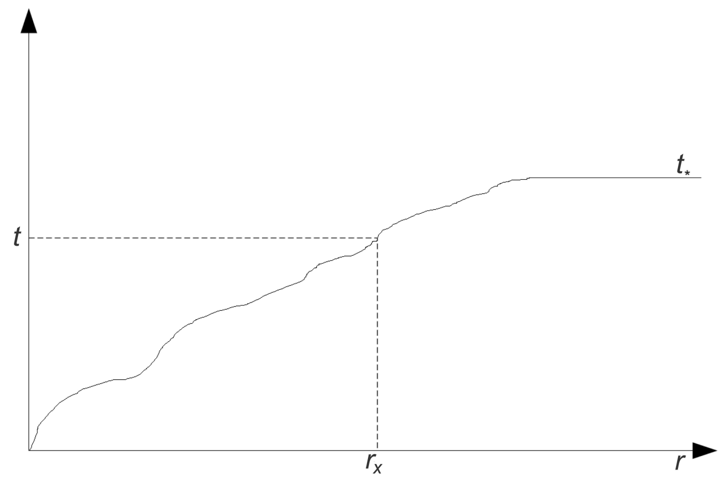

Fig. 2. Graph of the function $\varphi_x(r)$.

disjoint cubes, then the number of cubes $|I|$ in the subfamily satisfies the inequality

$$t^2 \cdot |I| = \sum_{i \in I} \int_{Q(x_i, r_{x_i}) \cap \Omega} \left| f(s) - f_{Q(x_i, r_{x_i})} \right|^2 ds \leq \|f\|_{L^2}^2 \tag{4}$$

and therefore is finite.

Now we will apply the Besicovitch covering theorem to the initial family of cubes $\{Q_x\}_{x \in \Omega}$. Let us remind the formulation of the Besicovitch theorem (see [15]).

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Suppose that for any $x \in \Omega$ the cube Q_x with the centre in x is given. Then there exists a subfamily $\{Q_{x_i}\}$ which covers Ω, i.e. $\Omega \subset \cup Q_{x_i}$, and which can be split into a finite number of subfamilies of pairwise disjoint cubes $\pi_k = \{Q_{x_i}\}_{i \in I_k}$. Moreover, the number of subfamilies π_k can be estimated only by the dimension n.*

From (4) we see that each subfamily π_k consists of a finite number of cubes and therefore the whole family of constructed cubes $\{K_i = Q_{x_i}\}_{i \in I}$ is finite and covers Ω.

Below we will assume that cubes K_i are indexed in agreement with the decrease in their volume. Now we are ready to give the formula for the near-minimizer f_t for the L-functional (2). On $K_1 \cap \Omega$ we will take f_t equal to f_{K_1}, then we define f_t on $(K_2 \cap \Omega) \backslash K_1$ by the constant f_{K_2} on $(K_3 \cap \Omega) \backslash (K_1 \cup K_2)$, etc. As the number of the cubes in the family $\{K_i\}$ is finite, the process of constructing the function f_t consists of a finite number of steps. The constructed near-minimizer can be written as

$$f_t = f_{K_1} \psi_1 + f_{K_2} \psi_2 + \ldots + f_{K_N} \psi_N,$$

where the functions

$$\psi_1 = \chi_{K_1 \cap \Omega}, \; \psi_2 = \chi_{K_2 \cap \Omega \backslash K_1}, \; \ldots \;, \; \psi_N = \chi_{K_N \cap \Omega \backslash \cup_{i<N} K_i}$$

form a partition of unity of Ω. Note that the constructed near-minimizer is not a continuous function.

Remark 3. Construction of the family of cubes $\{K_i\}$ in the Besicovitch covering theorem is constructive: as the first cube K_1 we take the largest or almost the largest cube in the family $\{Q_x\}_{x \in \Omega}$. Then from the family $\{Q_x\}_{x \in \Omega}$ we exclude cubes with centres in K_1. As the cube K_2 we take the largest or almost the largest cube from the family of remaining cubes, etc.

REFERENCES

1. Bergh, J. and Löfström, J. *Interpolation Spaces. An Introduction*. Springer-Verlag, Berlin-New York, 1976.
2. Brudnyi, Yu. A. and Krugljak, N. *Interpolation Functors and Interpolation Spaces 1*. North Holland, Amsterdam, 1991.
3. Kirsch, A. *An Introduction to the Mathematical Theory of Inverse Problems*. Springer, New York, 1996.
4. Chan, T. and Shen, J. *Image Processing and Analysis: Variational, PDE, Wavelet, and Stochastic Methods*. SIAM, Philadelphia, 2005.
5. Meyer, Y. *Oscillating Pattern in Image Processing and Nonlinear Evolution Equations.* University Lecture Series, AMS Providence, 2002, **22**.
6. Bechler, P., DeVore, R., Kamot, A., Petrova, G., and Wojtaszczyk, P. Greedy wavelet projections are bounded on BV. *Trans. AMS*, 2007, **359**, 619–635.
7. Cohen, A., DeVore, R., Petrushev, P., and Xu, H. Nonlinear approximation and the space $BV(R^2)$. *Amer. J. Math.*, 1999, **121**, 587–628.
8. Cohen, A., Dahmen, W., Daubechies, I., and DeVore, R. Harmonic analysis of the space BV. *Rev. Mat. Iberoamericana*, 2003, **19**, 235–263.
9. Daubechies, I., Teschke, G., and Vese, L. Iteratively solving linear inverse problems with general convex constraints. *Inverse Probl. Imaging*, 2007, **1**, 29–46.
10. Kruglyak, N. Smooth analogs of Calderon–Zygmund decompositions, quantitative covering theorems and K-functional for the couple $(L^p, \dot{W}_q^k)$. *Algebra i Analiz*, 1996, **8**, 110–160 (in Russian); English translation in *St.-Petersburg Math. Journal*, 1997, **8**, 617–649.

11. Kruglyak, N. The K-functional and Calderón–Zygmund type decompositions. *Contemporary Math.*, 2007, **446**, 183–194.
12. Brudnyi, Yu. A. Spaces determined by local approximations. *Trudy Mosk. Matem. Ob-va*, 1971, **24**, 69–132.
13. Triebel, H. *Theory of Function Spaces II*. Monographs in Mathematics, 1992, **84**.
14. Besicovitch, A. S. A general form of the covering principle and relative differentiation of additive functions II. *Proc. Cambridge Philos. Soc.*, 1946, **42**, 1–10.
15. de Guzman, M. *Differentiation of Integrals in R^n*. Lecture Notes in Mathematics, Springer-Verlag, New York, 1975, 481.

Besicovitchi katmisteoreem ja lähi-minimiseerijad paarile (L^2, BV)

Irina Asekritova ja Natan Kruglyak

On esitatud uudne algoritm (Peetre K-funktsionaali üldistava) $L_{p,q}$-funktsionaali lähi-minimiseerija konstrueerimiseks paarile $(L^2(\Omega), BV(\Omega))$, kus Ω on ristkülik ruumis $\mathbb{R}^2$. Algoritm toetub Besicovitchi katmisteoreemile.