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MATHEMATICS

Some properties of biconcircular gradient vector fields

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Abstract. We consider a Riemannian manifold carrying a biconcircular gradient vector field X, having as generative a closed torse forming U. The existence of such an X is determined by an exterior differential system in involution depending on two arbitrary functions of one argument. The Riemannian manifold is foliated by Einstein surfaces tangent to X and U. Properties of the biconcircular vector field X are investigated.

Key words: differential geometry, biconcircular gradient vector field, skew-symmetric Killing vector field, closed torse forming.

1. INTRODUCTION

Let (M,g) be a Riemannian (or pseudo-Riemannian) C^∞-manifold, and ∇, dp, and $\flat : TM \to T^*M$ be the Levi-Civita connection, the soldering form of M (i.e. the canonical vector-valued 1-form of M), and the musical isomorphism defined by g, respectively.

A vector field X on M such that

$$\nabla X = U^\flat \otimes X + X^\flat \otimes U, \tag{1.1}$$

where U is a certain vector field, called the *generative* of X, is defined as a *biconcircular gradient* (abbr. *BC gradient*) vector field. In consequence of (1.1), X is a self-adjoint vector field (i.e., $dX^\flat = 0$).

If U is a closed torse forming [8,9]

$$\nabla_Z U = aZ + g(Z,U)U, \ a = \text{const.}, \tag{1.2}$$

then the existence of such an X is determined by an exterior differential system in involution (in the sense of Cartan [1]) and depends on two arbitrary functions of one argument. In these conditions, we prove that a manifold (M,g) which carries such an X is foliated by Einstein surfaces M_X tangent to X and U.

If $\mathscr{L}_U$ is the Lie derivative, we also find

$$\mathscr{L}_U \nabla U = 0, \ [U,X] = aX, \tag{1.3}$$

i.e., U is an *affine* vector field and defines an *infinitesimal homothety* of X.

We also consider the skew-symmetric Killing vector field V defined by

$$\nabla V = X \wedge U,$$

($\wedge$: wedge product) and prove that V is a 2-*exterior concurrent* vector field. Finally two examples are given.

2. PRELIMINARIES

Let (M,g) be a Riemannian C^∞-manifold and ∇ be the covariant differential operator with respect to the metric tensor g. We assume that M is oriented and ∇ is the Levi-Civita connection. Let ΓTM be the set of sections of the tangent bundle and $\flat : TM \to T^*M$ and $\natural = \flat^{-1}$ the classical musical isomorphisms defined by g.

As usual, we denote by $C^\infty M$ and $\Gamma\Lambda^1 TM$ the algebra of smooth functions on M and the set of 1-forms on M, respectively.

Following [6], we denote by $A^q(M,TM) = \Gamma\mathrm{Hom}(\Lambda^q TM, TM)$ the set of vector-valued q-forms, $q < \dim M$, and by

$$d^\nabla : A^q(M,TM) \to A^{q+1}(M,TM)$$

the covariant derivative operator with respect to ∇ (in general $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$). The vector-valued 1-form $dp \in A^1(M,TM)$ is the identity vector-valued 1-form, called the *soldering form* of M (see [2]). Since ∇ is symmetric, we have $d^\nabla(dp) = 0$.

A vector field Y such that

$$d^\nabla(\nabla Y) = \nabla^2 Y = \pi \wedge dp \in A^2(M,TM) \tag{2.1}$$

for some 1-form π (called the *concurrence form*) is defined as *exterior concurrent* vector field [4,8].

If R is the Ricci tensor of ∇, we have

$$R(Y,Z) = -(n-1)\lambda g(Y,Z),\ Z \in \Gamma TM, \tag{2.2}$$

where $n = \dim M$ and $\pi = \lambda Y^\flat$ ($\lambda \in C^\infty M$ is a conformal scalar).

A vector field U such that

$$\nabla U = a dp + u \otimes U,\ u \in \Gamma\Lambda^1 TM,\ a \in C^\infty M, \tag{2.3}$$

is called a *torse forming* [9].

Let $O = \{e_A; A = 1,\dots,n\}$ be a local field of adapted vectorial frames over M and let $O^* = \{\omega^A\}$ be its associated coframe. Then the soldering form dp of M is expressed by $dp = \omega^A \otimes e_A$ and Cartan structure equations written in an indexless manner are

$$\nabla e = \theta \otimes e, \tag{2.4}$$

$$d\omega = -\theta \wedge \omega, \tag{2.5}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{2.6}$$

In the above equations, θ (resp. Θ) are the *local connection forms* in the tangent bundle TM (resp. the *curvature forms* on M).

3. PROPERTIES OF BICONCIRCULAR GRADIENT VECTOR FIELDS

A vector field X on a Riemannian (or pseudo-Riemannian) manifold (M,g) is said to be *biconcircular* (abbr. BC) if its covariant differential ∇X has no zero components only in two directions.

An example of a BC vector field is given by the skew-symmetric Killing vector field (in the sense of Rosca [8]).

In the present paper we consider a BC vector field X such that

$$\nabla X = U^\flat \otimes X + X^\flat \otimes U, \tag{3.1}$$

where U is a certain vector field called the *generative* of X. It is easy to prove that

$$g(\nabla_Z X, Z') = g(\nabla_{Z'} X, Z),\ Z, Z' \in \Gamma TM, \tag{3.2}$$

which shows that X is a gradient vector field in the sense of Okumura (see [7]). Using Cartan's structure equations, it follows that

$$dX^\flat = 0. \tag{3.3}$$

In the current paper we assume that U is a closed torse forming [4], i.e.

$$\nabla U = adp + U^\flat \otimes U \Leftrightarrow \nabla_Z U = aZ + g(Z,U)U,\ a = \text{const}. \tag{3.4}$$

From (3.1) and (3.4) we derive

$$\mathscr{L}_U X = (\|U\|^2 - a)X, \tag{3.5}$$

which, as is known, proves that X admits an *infinitesimal transformation* U.

Since

$$dU^\flat = 0, \tag{3.6}$$

it follows from (3.3) and (3.6) that M receives a foliation.

Operating on (3.1) and (3.4) by d^∇, we derive by a standard calculation

$$\begin{cases} d^\nabla(\nabla X) = \nabla^2 X = -aX^\flat \wedge dp, \\ d^\nabla(\nabla U) = \nabla^2 U = -aU^\flat \wedge dp, \end{cases} \tag{3.7}$$

which proves that X and U are *exterior concurrent* vector fields. Then, by reference to [8], the Ricci tensors of X and U are expressed by

$$\begin{cases} R(X,X) = (n-1)ag(X,X) \Rightarrow \text{Ric}X = (n-1)a, \\ R(U,U) = (n-1)ag(U,U) \Rightarrow \text{Ric}U = (n-1)a. \end{cases} \tag{3.8}$$

We recall that a (pseudo)-Riemannian manifold N is said to be *Einstein* if its Ricci tensor is given by $R = cg$, for some constant c (see [5]).

It follows from (3.8) that if M is compact, then the constant a is positive. In order to simplify, we set

$$l_X = \|X\|^2,\ l_U = \|U\|^2,\ s = g(X,U). \tag{3.9}$$

We obtain

$$\begin{cases} dl_X = 2l_X U^\flat + sX^\flat, \\ dl_U = (a + 2l_U)U^\flat, \\ ds = (a + 2l_U)X^\flat + 2sU^\flat. \end{cases} \tag{3.10}$$

Denote now by $\sum$ the exterior differential system which defines the BC gradient vector field X under consideration.

By (3.3), (3.6), and (3.10) the characteristic numbers of $\sum$ (i.e. Cartan's numbers) are $r = 5$, $s_0 = 3$, $s_1 = 2$. Since $r = s_0 + s_1$, it follows that $\sum$ is in involution and by Cartan's test we conclude that the existence of X depends on two arbitrary functions of one argument.

Further, we denote by $D_X = \{X, U\}$ the 2-dimensional distribution spanned by X and U.

Since the property of exterior concurrency is invariant by linearity, it follows that if X', $X'' \in D_X$, then

$$\nabla_{X''} X' \in D_X. \tag{3.11}$$

Summing up, we conclude from (3.11) and (3.8) that the manifold (M,g) carrying X is foliated by Einstein surfaces M_X tangent to D_X.

Theorem 1. *Let (M,g) be a Riemannian manifold carrying a BC gradient vector field X with closed torse forming generative U. The existence of such an X is determined by an exterior differential system in involution depending on two arbitrary functions of one argument.*

Any manifold (M,g) which carries such an X is foliated by Einstein surfaces M_X tangent to X and U.

In another order of ideas, if we take the Lie derivative of ∇U with respect to U and since $a = \text{const.}$, we get

$$\mathscr{L}_U \nabla U = 0, \tag{3.12}$$

which means that U is an *affine* vector field.

Further, we define a vector field V such that

$$\nabla V = X \wedge U = U^\flat \otimes X - X^\flat \otimes U. \tag{3.13}$$

We find

$$d^\nabla(\nabla V) = \nabla^2 V = aX^\flat \wedge dp + 2(X^\flat \wedge U^\flat) \otimes U, \tag{3.14}$$

$$d^\nabla(\nabla^2 V) = \nabla^3 V = 2a(X^\flat \wedge U^\flat) \wedge dp, \tag{3.15}$$

i.e., V is a 2-exterior concurrent vector field.

We also remark that V is a Killing vector field, i.e.

$$g(\nabla_Z V, Z') + g(\nabla_{Z'} V, Z) = 0. \tag{3.16}$$

From the general formula

$$dV^\flat(U,X) = g(\nabla_U V, X) - g(U, \nabla_X V),$$

we also derive

$$dV^\flat(U,X) = \|X\|^2\,\|U\|^2 - 2g(U,X)^2.$$

Next we consider the skew-symmetric Killing vector field W having U as generative [3], i.e.

$$\nabla W = W \wedge U. \tag{3.17}$$

Then, by Rosca's Lemma [8] it follows that

$$dW^\flat = aU^\flat \wedge W^\flat. \tag{3.18}$$

It should be noticed that, since $a = \text{const.}$, $[W,U]$ is also a Killing vector field.

Theorem 2. *Let (M,g) be a Riemannian manifold carrying a BC gradient vector field X, having as generative a closed torse forming U. Then*

i) *the generative U of the BC vector field X is an affine vector field*;

ii) *the wedge product $X \wedge U$ of X and U defines a 2-exterior concurrent vector field V, which is a Killing vector field*;

iii) *if W is a skew-symmetric vector field having U as generative*, *then $[W,U]$ is also a Killing vector field.*

4. EXAMPLES

We shall determine the BC gradient vector fields on two Riemannian manifolds.

1. We take the upper half space $x^n > 0$ in the sense of Poincaré's representation as the model of the hyperbolic n-space form $\mathbf{H}^n$. The metric of $\mathbf{H}^n$ is given by

$$g_{ij}(x) = \frac{1}{(x^n)^2}\delta_{ij},\ \forall x \in \mathbf{H}^n,\ \forall i,j \in \{1,...,n\}.$$

The Christoffel's symbols with respect to g are

$$\Gamma^i_{ni} = -\Gamma^n_{\lambda\lambda} = -\frac{1}{x^n},\ i \in \{1,...,n\},\ \lambda \in \{1,...,n-1\},$$

the other being zero.

The vector field $\xi = x^n \frac{\partial}{\partial x^n}$ is a closed torse forming (see [4]).

We determine the BC gradient vector fields on $\mathbf{H}^n$ having ξ as generative. Equation (3.1) can be written as

$$\nabla X = u \otimes X + v \otimes \xi,$$

where $u = \xi^\flat$ and $v = X^\flat$.

Let

$$X = \sum_{\lambda=1}^{n-1} f^\lambda \frac{\partial}{\partial x^\lambda} + f\frac{\partial}{\partial x^n}.$$

Then $v = \frac{1}{(x^n)^2}(\sum_{\lambda=1}^{n-1} f^\lambda dx^\lambda + f dx^n)$ and $u = \frac{1}{x^n}dx^n$.

In particular, we have

$$\nabla_{\frac{\partial}{\partial x^n}} X = u\left(\frac{\partial}{\partial x^n}\right)X + v\left(\frac{\partial}{\partial x^n}\right)x^n\frac{\partial}{\partial x^n},$$

i.e.,

$$\nabla_{\frac{\partial}{\partial x^n}}\left(f^\lambda \frac{\partial}{\partial x^\lambda} + f\frac{\partial}{\partial x^n}\right) = \frac{1}{x^n}\left(f^\lambda \frac{\partial}{\partial x^\lambda} + f\frac{\partial}{\partial x^n}\right) + \frac{f}{x^n}\frac{\partial}{\partial x^n},$$

or, equivalently,

$$\frac{\partial f^\lambda}{\partial x^n}\frac{\partial}{\partial x^\lambda} + f^\lambda \Gamma^k_{n\lambda}\frac{\partial}{\partial x^k} + \frac{\partial f}{\partial x^n}\frac{\partial}{\partial x^n} + f\Gamma^k_{nn}\frac{\partial}{\partial x^k} = \frac{1}{x_n}\left(f^\lambda \frac{\partial}{\partial x^\lambda} + 2f\frac{\partial}{\partial x^n}\right).$$

Thus we have

$$\frac{\partial f^\lambda}{\partial x^n}\frac{\partial}{\partial x^\lambda} - \frac{f^\lambda}{x^n}\frac{\partial}{\partial x^\lambda} + \frac{\partial f}{\partial x^n}\frac{\partial}{\partial x^n} - \frac{f}{x^n}\frac{\partial}{\partial x^n} = \frac{1}{x_n}\left(f^\lambda \frac{\partial}{\partial x^\lambda} + 2f\frac{\partial}{\partial x^n}\right).$$

It follows that

$$\begin{cases} \dfrac{\partial f^\lambda}{\partial x^n} = 2\dfrac{f^\lambda}{x^n}, \\ \dfrac{\partial f}{\partial x^n} = 3\dfrac{f}{x^n}. \end{cases}$$

By integrating we get

$$\begin{cases} f^\lambda = c^\lambda(x^1,...,x^{n-1})(x^n)^2, \\ f = a(x^1,...,x^{n-1})(x^n)^3. \end{cases}$$

On the other hand, for $\mu \in \{1,...,n-1\}$, we have

$$\nabla_{\frac{\partial}{\partial x^\mu}} \left(f^\lambda \frac{\partial}{\partial x^\lambda} + f \frac{\partial}{\partial x^n} \right) = u\left(\frac{\partial}{\partial x^\mu}\right) X + v\left(\frac{\partial}{\partial x^\mu}\right) x^n \frac{\partial}{\partial x^n},$$

i.e.,

$$\frac{\partial f^\lambda}{\partial x^\mu}\frac{\partial}{\partial x^\lambda} + f^\lambda \Gamma^k_{\mu\lambda}\frac{\partial}{\partial x^k} + \frac{\partial f}{\partial x^\mu}\frac{\partial}{\partial x^n} + f\Gamma^k_{n\mu}\frac{\partial}{\partial x^k} = \frac{f^\mu}{x^n}\frac{\partial}{\partial x^n},$$

or, equivalently,

$$\frac{\partial f^\lambda}{\partial x^\mu}\frac{\partial}{\partial x^\lambda} + \frac{f^\mu}{x^n}\frac{\partial}{\partial x^n} + \frac{\partial f}{\partial x^\mu}\frac{\partial}{\partial x^n} - \frac{f}{x^n}\frac{\partial}{\partial x^\mu} = \frac{f^\mu}{x^n}\frac{\partial}{\partial x^n}.$$

It follows that

$$\begin{cases} \dfrac{\partial f^\lambda}{\partial x^\mu} = \dfrac{f}{x^n}\delta_{\lambda\mu} \Longrightarrow c^\lambda = c^\lambda(x^\lambda), \\ \dfrac{\partial f}{\partial x^\mu} = 0 \Longrightarrow a = \text{const.} \end{cases}$$

For $\lambda = \mu$, we get $\frac{\partial c^\mu}{dx^\mu} = a \Longleftrightarrow c^\mu = ax^\mu + b^\mu$.
Consequently,

$$X = \sum_{\lambda=1}^{n-1} (ax^\lambda + b^\lambda)(x^n)^2 \frac{\partial}{\partial x^\lambda} + a(x^n)^3 \frac{\partial}{\partial x^n}.$$

2. Let T^{n-1} be an $(n-1)$-dimensional flat torus with the coordinate system $(x^1,...,x^{n-1})$ and $\mathbf{R}$ a real line with coordinate x^n. Consider the warped product $M = \mathbf{R} \times_\sigma T^{n-1}$, with $\sigma(x^n) = e^{-x^n}$. Then the components of the Riemannian metric on M are

$$g_{\lambda\mu} = e^{-2x^n}\delta_{\lambda\mu},\ g_{n\lambda} = 0,\ \lambda,\mu \in \{1,...,n-1\};\ g_{nn} = 1,$$

and Christoffel's symbols are

$$\Gamma^\lambda_{n\lambda} = -1,\ \Gamma^n_{\lambda\lambda} = e^{-2x^n},$$

the other being zero.

We can prove that $\xi = \frac{\partial}{\partial x^n}$ is a closed torse forming (see [4]). The BC gradient vector fields on M having ξ as generative are defined by

$$\nabla X = u \otimes X + v \otimes \xi,$$

with $u = \xi^\flat$ and $v = X^\flat$.

If we put

$$X = \sum_{\lambda=1}^{n-1} f^\lambda \frac{\partial}{\partial x^\lambda} + f\frac{\partial}{\partial x^n},$$

then $v = e^{-2x^n} f^\lambda dx^\lambda + f dx^n$.

We have

$$\nabla_{\frac{\partial}{\partial x^n}} X = u\left(\frac{\partial}{\partial x^n}\right) X + v\left(\frac{\partial}{\partial x^n}\right)\frac{\partial}{\partial x^n},$$

i.e.,

$$\frac{\partial f^\lambda}{\partial x^n}\frac{\partial}{\partial x^\lambda} - f^\lambda\frac{\partial}{\partial x^\lambda} + \frac{\partial f}{\partial x^n}\frac{\partial}{\partial x^n} = f^\lambda\frac{\partial}{\partial x^\lambda} + 2f\frac{\partial}{\partial x^n},$$

or, equivalently,

$$\begin{cases} \dfrac{\partial f^\lambda}{\partial x^n} = 2f^\lambda, \\ \dfrac{\partial f}{\partial x^n} = 2f, \end{cases} \iff \begin{cases} f^\lambda = c^\lambda e^{2x^n},\ c^\lambda = c^\lambda(x^1,...,x^{n-1}), \\ f = ae^{2x^n},\ a = a(x^1,...,x^{n-1}). \end{cases}$$

For $\mu \in \{1,...,n-1\}$, we have

$$\nabla_{\frac{\partial}{\partial x^\mu}} X = u\left(\frac{\partial}{\partial x^\mu}\right) X + v\left(\frac{\partial}{\partial x^\mu}\right)\frac{\partial}{\partial x^n},$$

i.e.,

$$\frac{\partial f^\lambda}{\partial x^\mu}\frac{\partial}{\partial x^\lambda} + e^{-2x^n} f^\mu \frac{\partial}{\partial x^n} + \frac{\partial f}{\partial x^\mu}\frac{\partial}{\partial x^n} - f\frac{\partial}{\partial x^\mu} = e^{-2x^n} f^\mu \frac{\partial}{\partial x^n},$$

or, equivalently,

$$\begin{cases} \dfrac{\partial f^\lambda}{\partial x^\mu} = f\delta_{\lambda\mu}, \\ \dfrac{\partial f}{\partial x^\mu} = 0. \end{cases}$$

The last equation implies $a = \text{const.}$; then $f^\mu = (ax^\mu + b^\mu)e^{2x^n}$.

Consequently,

$$X = \sum_{\mu=1}^{n-1} (ax^\mu + b^\mu)e^{2x^n} \frac{\partial}{\partial x^\mu} + ae^{2x^n}\frac{\partial}{\partial x^n}.$$

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Bikaasringse gradientvektorvälja mõned omadused

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On vaadeldud Riemanni muutkonda (M,g), millel on määratud bikaasringne gradientvektorväli X, st X rahuldab tingimust $\nabla X = U^\flat \otimes X + X^\flat \otimes U$, kus ∇ on kovariantne diferentsiaal, U on vektorväli muutkonnal M ja $\flat : TM \to T^*M$ on puutujavektorkonna ning kaaspuutujavektorkonna vaheline isomorfism. Tingimusel, et vektorväli U rahuldab tingimust $\nabla_Z U = aZ + g(Z,U)U$, kus a on konstant, on uuritud bikaasringse gradientvektorvälja X olemasolu ja näidatud, et X on involutsiooniga välisdiferentsiaalvõrrandisüsteemi lahendiks, kusjuures selle võrrandisüsteemi iga lahend sõltub kahest parameetrist ning parameetriteks on siledad ühemuutuja funktsioonid muutkonnal M. On tõestatud, et kui Riemanni muutkonnal M eksisteerib selline bikaasringne gradientvektorväli X, et U rahuldab eelmainitud tingimust, siis muutkond M on foliatsioon, mille iga kiht on Einsteini pind, ja vektorväljad X,U on selle pinna puutujavektorväljad.