Proceedings of the Estonian Academy of Sciences,
2009, **58**, 2, 98–107
doi: 10.3176/proc.2009.2.02
Available online at www.eap.ee/proceedings

MATHEMATICS

Removing the input derivatives in the generalized bilinear state equations

Tanel Mullari[a*], Ülle Kotta[a], Palle Kotta[a], Maris Tõnso[a], and Alan S. I. Zinober[b]

[a] Institute of Cybernetics at Tallinn University of Technology, Akadeemia tee 21, 12618 Tallinn, Estonia; kotta@cc.ioc.ee, palle@cc.ioc.ee, mtonso@staff.ttu.ee
[b] Department of Applied Mathematics, The University of Sheffield, Sheffield S10 2TN, UK; a.zinober@sheffield.ac.uk

Received 16 October 2008, revised 22 December 2008, accepted 7 January 2009

Abstract. The paper suggests constraints on the coefficients a_i, b_i, c_{ij} of the bilinear continuous-time input-output model that yield generalized state equations with input derivative order lower than that in the input-output equations. In the limiting case when one removes the input derivatives altogether, these conditions provide a solution of the realizability problem. The new state coordinates are found step by step. We first find a coordinate transformation allowing the reduction of the maximal order of the input time derivatives by one and write the corresponding state equations. At the second step we find the next coordinate transformation to lower the maximal order of input time derivative in the new state equations, etc. At each step we check, what condition the coefficients should satisfy to make the next step possible.

Key words: control systems, bilinear systems, differential input-output equations, state-space realization.

1. INTRODUCTION

In many situations the continuous-time input-output (i/o) model is obtained from experimental data using identification procedures. It is clear from the theoretical results [1–6] that an arbitrarily structured i/o model does not necessarily have a state-space realization. Using such a model is highly undesirable in further analysis and/or control design, since practically all existing control theory for nonlinear systems is based on a state-space description. Motivated by the above and relying on the necessary and sufficient realizability conditions stated in [1,4,5], our long-range goal is to find the subclass(es) of i/o models, each of which is guaranteed to have a classical state-space description.

One approach to identify a nonlinear i/o system is to use a model structure that can be considered as a general approximator to the nonlinear mapping f, such as neural networks. Due to the complex structure of these models, they can be difficult to identify. Another approach is to choose a certain simple structure for the model. In many cases a simple structure provides a reasonable approximation and is easy to identify. Examples of simple structures are bilinear and quadratic models, which are simple nonlinear extension of the linear model. In this paper we will concentrate on bilinear i/o equations with a single input u and a single output y:

* Corresponding author, tanel@parsek.yf.ttu.ee

Preliminary results of this paper were presented at the 7th IFAC Symposium on Nonlinear Control Systems (NOLCOS) 2007, Pretoria, South Africa.

$$y^{(n)} = \sum_{i=1}^{n} a_i y^{(n-i)} + \sum_{\alpha=1}^{s+1} b_\alpha u^{(s+1-\alpha)} + \sum_{i=1}^{n} \sum_{\alpha=1}^{s+1} c_{i\alpha} y^{(n-i)} u^{(s+1-\alpha)}. \tag{1}$$

The bilinear system gets its name from the fact that, if you fix the input, the system is linear in the output and if you fix the output, it is linear in the input. Bilinear models are capable of modelling certain nonlinear dynamics, whilst having a relatively simple structure that makes them attractive for identification and analysis [7].

It is known from a previous study [8] that the bilinear i/o differential equation (1) is, in general, not realizable in the classical state-space form. In [8], necessary and sufficient realizability conditions for low-order bilinear i/o equations with $s = n-1$ were given directly in terms of bilinear equation parameters. It turned out that the conditions have a complex structure with many branches. For the 4th-order i/o bilinear equation

$$y^{(4)} = \sum_{i=1}^{4} a_i y^{(4-i)} + \sum_{\alpha=1}^{4} b_\alpha u^{(4-\alpha)} + \sum_{i=1}^{4} \sum_{\alpha=1}^{4} c_{i\alpha} y^{(4-i)} u^{(4-\alpha)}, \tag{2}$$

three independent realizability conditions are shown diagrammatically in Fig. 1.

In Fig. 1 '$\circ$' indicates a parameter that must be zero and '$\bullet$' indicates an unrestricted parameter; in the second set (right upper), A indicates that these, in general, nonzero coefficients have equal value.

The purpose of this paper is to demonstrate that with a simple problem reformulation, one can abandon the branching in the solution. Observe that in Fig. 1 only the third case corresponds to the situation where the highest input derivative s is equal to $n-1$.

$$\begin{array}{cccc}
a_1\bullet & a_2\bullet & a_3\bullet & a_4\bullet \\
\\
b_1\circ & b_2\circ & b_3\bullet & b_4\bullet \\
\\
c_{11}\circ & c_{12}\circ & c_{13}\bullet & c_{14}\bullet \\
c_{21}\circ & c_{22}\circ & c_{23}\bullet & c_{24}\bullet \\
c_{31}\circ & c_{32}\circ & c_{33}\bullet & c_{34}\bullet \\
c_{41}\circ & c_{42}\circ & c_{43}\bullet & c_{44}\bullet
\end{array}
\qquad\qquad
\begin{array}{cccc}
a_1\bullet & a_2\bullet & a_3\bullet & a_4\bullet \\
\\
b_1\circ & b_2\bullet & b_3\bullet & b_4\bullet \\
\\
c_{11}\circ & c_{12}\circ & c_{13}A & c_{14}\bullet \\
c_{21}\circ & c_{22}A & c_{23}\bullet & c_{24}\bullet \\
c_{31}\circ & c_{32}\bullet & c_{33}\bullet & c_{34}\bullet \\
c_{41}\circ & c_{42}\bullet & c_{43}\bullet & c_{44}\bullet
\end{array}$$

$$\begin{array}{cccc}
a_1\bullet & a_2\bullet & a_3\bullet & a_4\bullet \\
\\
b_1\bullet & b_2\bullet & b_3\bullet & b_4\bullet \\
\\
c_{11}\circ & c_{12}\circ & c_{13}\circ & c_{14}A \\
c_{21}\circ & c_{22}\circ & c_{23}C & c_{24}\bullet \\
c_{31}\circ & c_{32}B & c_{33}\bullet & c_{34}\bullet \\
c_{41}D & c_{42}\bullet & c_{43}\bullet & c_{44}\bullet
\end{array}$$

$$A - D + C - B = 0$$

Fig. 1. Realizability conditions for the 4th-order bilinear i/o equation.

Defining the generalized state variables as the time derivatives of the output, $x_i = y^{(i-1)}$, $i = 1,\ldots,n$, the i/o equation yields the generalized state equations

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
&\vdots \\
\dot{x}_{n-1} &= x_n, \\
\dot{x}_n &= \sum_{i=1}^{n} a_i x_{n+1-i} + \sum_{\alpha=1}^{s+1} b_\alpha u^{(s+1-\alpha)} + \sum_{i=1}^{n}\sum_{\alpha=1}^{s+1} c_{i\alpha} x_{n+1-i} u^{(s+1-\alpha)},
\end{aligned} \tag{3}$$

containing in addition to inputs also a certain number of their time derivatives. Note that the generalized state equations (3) were first introduced by Fliess in [9]. We are looking for conditions allowing us to reduce the highest order of input time derivative in (3) with the help of a suitable generalized state coordinate transformation that depends also on the inputs and their time derivatives

$$\tilde{x} = \phi(x, u, \ldots, u^{(s-1)}). \tag{4}$$

In [4] the problem of lowering the order of input derivative in the generalized state equation was studied for the general case, and the result was formulated in terms of the commutativity of certain vector fields. Though transparent and inherently simple, the result yields little insight regarding which structures of model (1) allow lowering the order of input derivative in (3). The same holds for [10] where algebraic conditions in terms of one-forms were given. The objective of this paper is to study the problem further for the subclass of i/o *bilinear* models and to suggest constraints on the parameters a_i, b_i, c_{ij} of the bilinear model (1) that lead to generalized state equations with input derivative lower than in equation (1). More precisely, we will prove that if certain combinations of coefficients in (1) are zero, then the order can be lowered by two, or respectively, by three. We also suggest a conjecture for the general case. Our analysis is based on algorithmic necessary and sufficient conditions [3]; see also [11] for the general i/o equation (1).

Note that the results, similar to those of [12], were obtained earlier for discrete-time bilinear i/o equations [8]. Though both papers provide the necessary and sufficient realizability conditions for low-order bilinear systems and suggest a few realizable subclasses together with the corresponding state equations for the arbitrary-order bilinear systems, there is no formal similarity in the realizability conditions for the discrete- and continuous-time cases. For example, the continuous-time second-order bilinear system is always realizable, unlike the 2nd-order discrete-time bilinear system. Although, in general, the number of nonzero coefficients in both cases is approximately the same, their placement is rather different. This is the result of the very different properties of differential and shift operators.

2. MAIN RESULT

Note that the order of input derivative in (3) can be always lowered by one.

Theorem 1. *Using a generalized state transformation* (4), *the maximal order of input derivative in equations* (3) *can be reduced by two iff in the bilinear i/o equations*

$$c_{11} = c_{12} - c_{21} = 0 \tag{5}$$

and by three iff

$$c_{11} = c_{12} = c_{13} = c_{21} = c_{22} = c_{31} = -c_{23} + c_{32} - c_{41} + c_{14} = 0. \tag{6}$$

Proof. As the first step we will define new generalized state variables $x_i^{[1]}$ that allow us to reduce the maximal order of input derivative by one. According to [11], $x_i^{[1]}$ must be invariants of a vector field

$$-L_f \frac{\partial}{\partial u^{(s)}} = \left[\frac{\partial}{\partial u^{(s)}}, f\right] = \frac{\partial}{\partial u^{(s-1)}} + \left(b_1 + \sum_{i=1}^{n} c_{i1} y^{(n-i)}\right) \frac{\partial}{\partial y^{(n-1)}}. \tag{7}$$

Vector field (7) has the following invariants:

$$
\begin{aligned}
x_1^{[1]} &= y, \ldots, x_{n-1}^{[1]} = y^{(n-2)}, \\
x_n^{[1]} &= \frac{1}{c_{11}} \exp\left(-c_{11} u^{(s-1)}\right) \left[b_1 + \sum_{i=1}^{n} c_{i1} y^{(n-i)} \right].
\end{aligned}
\tag{8}
$$

From (8), the output and its derivatives are expressed in terms of invariants (7)

$$
\begin{aligned}
y &= x_1^{[1]}, \ldots, y^{(n-2)} = x_{n-1}^{[1]}, \\
y^{(n-1)} &= x_n^{[1]} \exp\left(c_{11} u^{(s-1)}\right) - \frac{1}{c_{11}} \left[b_1 + \sum_{i=2}^{n} c_{i1} x_{n+1-i}^{[1]} \right].
\end{aligned}
\tag{9}
$$

Using the generalized state variables (8) and taking into account expressions (9), we get the first $(n-1)$ generalized state equations as follows:

$$
\begin{aligned}
\dot{x}_1^{[1]} &= x_2^{[1]}, \ldots, \dot{x}_{n-2}^{[1]} = x_{n-1}^{[1]}, \\
\dot{x}_{n-1}^{[1]} &= x_n^{[1]} \exp\left(c_{11} u^{(s-1)}\right) - \frac{1}{c_{11}} \left[b_1 + \sum_{i=2}^{n} c_{i1} x_{n+1-i}^{[1]} \right].
\end{aligned}
\tag{10}
$$

Due to the nonlinearity of the last equation in (10) with respect to $u^{(s-1)}$, we cannot lower the order of input derivative in the general case further up to $s-2$ [3,11]. Consequently, the first necessary condition to reduce the highest order of input time derivative by two is

$$
c_{11} = 0. \tag{11}
$$

Under the restriction (11) the independent invariants of the vector field (7) become

$$
\begin{aligned}
x_1^{[1]} &= y, \ldots, x_{n-1}^{[1]} = \ldots y^{(n-2)}, \\
x_n^{[1]} &= y^{(n-1)} - \left(b_1 + \sum_{i=2}^{n} c_{i1} y^{(n-i)} \right) u^{(s-1)}
\end{aligned}
$$

and the output derivatives, written in terms of $x_i^{[1]}$, are:

$$
\begin{aligned}
y &= x_1^{[1]}, \ldots, y^{(n-2)} = x_{n-1}^{[1]}, \\
y^{(n-1)} &= x_n^{[1]} + \left(b_1 + \sum_{i=2}^{n} c_{i1} x_{(n+1-i)}^{[1]} \right) u^{(s-1)}.
\end{aligned}
$$

The generalized state equations now are as follows:

$$
\begin{aligned}
\dot{x}_1^{[1]} &= x_2^{[1]}, \ldots, \dot{x}_{n-2}^{[1]} = x_{n-1}^{[1]}, \\
\dot{x}_{n-1}^{[1]} &= x_n^{[1]} + \left(b_1 + \sum_{i=2}^{n} c_{i1} x_{(n+1-i)}^{[1]} \right) u^{(s-1)}, \\
\dot{x}_n^{[1]} &= \sum_{i=2}^{n} a_i x_{n+1-i}^{[1]} + \sum_{\alpha=3}^{s+1} b_\alpha u^{(s+1-\alpha)} + \sum_{i=2}^{n} \sum_{\alpha=3}^{s+1} c_{i\alpha} x_{n+1-i}^{[1]} u^{(s+1-\alpha)} \\
&\quad + \left(b_2 + \sum_{i=2}^{n} c_{i2} x_{n+1-i}^{[1]} - \sum_{i=3}^{n} c_{i1} x_{n-i}^{[1]} \right) u^{(s-1)} \\
&\quad \times \left(a_1 + \sum_{\alpha=3}^{s+1} c_{1\alpha} u^{(s+1-\alpha)} \right) \left[x_n^{[1]} + \left(b_1 + \sum_{i=2}^{n} c_{i1} x_{(n+1-i)}^{[1]} \right) u^{(s-1)} \right] \\
&\quad + (c_{12} - c_{21}) \left[x_n^{[1]} + \left(b_1 + \sum_{i=2}^{n} c_{i1} x_{(n+1-i)}^{[1]} \right) u^{(s-1)} \right] u^{(s-1)}.
\end{aligned}
\tag{12}
$$

Again, the last equation in (12) is nonlinear with respect to $u^{(s-1)}$. This means that we cannot eliminate $u^{(s-1)}$ in the subsequent coordinate transformation except in case when

$$c_{12} = c_{21} \tag{13}$$

or

$$b_1 = c_{i1} = 0, \quad \forall i = 2, \ldots, n. \tag{14}$$

Conditions (14), taken together with (11), mean that equation (1) does not contain the variable $u^{(s)}$; the highest order of input derivative is $s-1$. Therefore, in this case, without loss of generality, we can take in equation (1) simply $s-1$ instead of s and *this leads us again to equation (1), etc.* So, it is necessary to continue along the first branch (13) only, and conditions (5) are necessary and sufficient for lowering the highest order of input time derivative by two.

To eliminate the variables $u^{(s-1)}$ from equations (12) under the restriction (13), we define the new state variables $x_i^{[2]}$ as the independent invariants of the vector field

$$
\begin{aligned}
-L_{f^{[1]}} \frac{\partial}{\partial u^{(s)}} &= \left[\frac{\partial}{\partial u^{(s)}}, f^{[1]} \right] = -\frac{\partial}{\partial u^{(s-2)}} + \left(b_1 + \sum_{i=2}^{n} c_{i1} x_{n+1-i}^{[1]} \right) \frac{\partial}{\partial x_{n-1}^{[1]}} \\
&\quad + \left[\left(b_2 + \sum_{i=2}^{n} c_{i2} x_{n+1-i}^{[1]} - \sum_{i=3}^{n} c_{i1} x_{n-1}^{[1]} \right) \right. \\
&\quad \left. + \left(a_1 + \sum_{\alpha=3}^{s+1} c_{1\alpha} u^{(s+1-\alpha)} \right) \left(b_1 + \sum_{i=2}^{n} c_{i1} x_{n+1-i}^{[1]} \right) \right] \frac{\partial}{\partial x_n^{[1]}},
\end{aligned}
\tag{15}
$$

where $f^{[1]}$ is the total derivative operator corresponding to system (12):

$$f^{[1]} = \sum_{i=1}^{n} \dot{x}_i^{[1]} \frac{\partial}{\partial x_i^{[1]}} + \sum_{\alpha=1}^{s} u^{(\alpha+1)} \frac{\partial}{\partial u^{(\alpha)}}. \tag{16}$$

The first $(n-1)$ independent invariants of this vector field are

$$\begin{aligned} x_1^{[2]} &= x_1^{[1]}, \ldots, x_{n-2}^{[2]} = x_{n-2}^{[1]}, \\ x_{n-1}^{[2]} &= \frac{1}{c_{21}} \exp\left(-c_{21}u^{(s-2)}\right)\left(b_1 + \sum_{i=2}^{n} c_{i1}x_{n+1-i}^{[1]}\right). \end{aligned}$$

Consequently, we obtain the first $(n-1)$ generalized state equations:

$$\begin{aligned} \dot{x}_1^{[2]} &= x_2^{[2]}, \ldots, \dot{x}_{n-3}^{[2]} = x_{n-2}^{[2]}, \\ \dot{x}_{n-2}^{[2]} &= x_{n-1}^{[2]} \exp\left(c_{21}u^{(s-2)}\right) - \frac{1}{c_{21}}\left(b_1 + \sum_{i=3}^{n} c_{i1}x_{n+1-i}^{[2]}\right). \end{aligned} \tag{17}$$

Due to the nonlinearity of the second equation in (17), with respect to $u^{(s-2)}$ one cannot eliminate the variables $u^{(s-2)}$ via the subsequent coordinate transformation except in the case when $c_{21} = 0$, which, together with conditions (5), gives necessary conditions for the reduction of the maximal input derivative order by three:

$$c_{11} = c_{12} = c_{21} = 0. \tag{18}$$

Under (18) the generalized state equations become

$$\begin{aligned} \dot{x}_1^{[1]} &= x_2^{[1]}, \ldots, \dot{x}_{n-2}^{[1]} = x_{n-1}^{[1]}, \\ \dot{x}_{n-1}^{[1]} &= x_n^{[1]} + \left(b_1 + \sum_{i=3}^{n} c_{i1}x_{(n+1-i)}^{[1]}\right)u^{(s-1)}, \\ \dot{x}_n^{[1]} &= \sum_{i=2}^{n} a_i x_{n+1-i}^{[1]} + \sum_{\alpha=3}^{s+1} b_\alpha u^{(s+1-\alpha)} + \sum_{i=2}^{n}\sum_{i=3}^{s+1} c_{i\alpha}x_{n+1-i}^{[1]}u^{(s+1-\alpha)} \\ &\quad + \left(b_2 + \sum_{i=2}^{n} c_{i2}x_{n+1-i}^{[1]} - \sum_{i=3}^{n} c_{i1}x_{n-1}^{[1]}\right)u^{(s-1)} \\ &\quad + \left(a_1 + \sum_{\alpha=3}^{s+1} c_{1\alpha}u^{(s+1-\alpha)}\right)\left[x_n^{[1]} + \left(b_1 + \sum_{i=3}^{n} c_{i1}x_{(n-1-i)}^{[1]}\right)u^{(s-1)}\right], \end{aligned}$$

and the vector field (15) takes the form

$$\begin{aligned} -L_{f^{[1]}}\frac{\partial}{\partial u^{(s)}} &= \frac{\partial}{\partial u^{(s-2)}} + \left(b_1 + \sum_{i=3}^{n} c_{i1}x_{(n+1-i)}^{[1]}\right)\frac{\partial}{\partial x_{n-1}^{[1]}} \\ &\quad + \left[\left(b_2 + \sum_{i=2}^{n} c_{i2}x_{n+1-i}^{[1]} - \sum_{i=3}^{n} c_{i1}x_{n-1}^{[1]}\right)\right. \\ &\quad + \left(a_1 + \sum_{\alpha=4}^{s+1} c_{1\alpha}u^{(s+1-\alpha)}\right)\left(b_1 + \sum_{i=3}^{n} c_{i1}x_{(n+1-i)}^{[1]}\right) \\ &\quad \left. + \left(b_1 + \sum_{i=3}^{n} c_{i1}x_{(n+1-i)}^{[1]}\right)c_{13}u^{(s-2)}\right]\frac{\partial}{\partial x_n^{[1]}} \end{aligned}$$

with the following independent invariants

$$
\begin{aligned}
x_1^{[2]} &= x_1^{[1]}, \ldots, x_{n-2}^{[2]} = x_{n-2}^{[1]}, \\
x_{n-1}^{[2]} &= x_{n-1}^{[1]} - \left(b_1 + \sum_{i=3} c_{i1} x_{(n+1-i)}^{[1]} \right) u^{(s-2)}, \\
x_n^{[2]} &= x_n^{[1]} - \left\{ \left(b_1 + \sum_{i=3}^{n} c_{i1} x_{(n+1-i)}^{[1]} \right) \left(a_1 + \sum_{\alpha=4}^{(s+1)} c_{1\alpha} u^{(s+1-\alpha)} \right) \right. \\
&\quad \left. + \left[b_2 + c_{n2} x_{n-1}^{[1]} + \sum_{i=3}^{(n-1)} (c_{i2} - c_{i+1,1}) x_{n+1-i}^{[1]} \right] \right\} u^{(s-2)} \\
&\quad + \frac{1}{2} \left(b_1 + \sum_{i=3}^{n} c_{i1} x_{(n+1-i)}^{[1]} \right) (c_{22} - c_{31} - c_{13}) \left(u^{(s-2)} \right)^2 .
\end{aligned} \tag{19}
$$

The invariants (19) define the new generalized state variables $x_i^{[2]}$. The corresponding first $(n-1)$ generalized state equations are as follows:

$$
\begin{aligned}
\dot{x}_1^{[2]} &= x_2^{[2]}, \ldots, \dot{x}_{n-3}^{[2]} = x_{n-2}^{[2]}, \\
\dot{x}_{n-2}^{[2]} &= x_{n-1}^{[2]} + \left(b_1 + \sum_{i=3}^{n} c_{i1} x_{n-1-i}^{[2]} \right) u^{(s-2)}, \\
\dot{x}_{n-1}^{[2]} &= x_n^{[2]} + \left[\left(b_1 + \sum_{i=3}^{n} c_{i1} x_{n-1-i}^{[2]} \right) \left(a_1 + \sum_{\alpha=4}^{s+1} c_{1\alpha} u^{(s+1-\alpha)} \right) \right. \\
&\quad \left. + b_2 + c_{n2} x_{n-i}^{[2]} + \sum_{i=3}^{n-1} (c_{i2} - c_{i+1,1}) x_{n+1-i}^{[2]} \right] u^{(s-2)} \\
&\quad + \frac{1}{2} \left(b_1 + \sum_{i=3}^{n} c_{i1} x_{n-1-i}^{[2]} \right) (c_{22} - 3c_{31} + c_{13}) \left(u^{(s-2)} \right)^2 .
\end{aligned}
$$

The nonlinearity of the last equation with respect to $u^{(s-2)}$ does not allow us to eliminate this variable via the subsequent coordinate transformations unless linearity is guaranteed by requiring $b_1 = c_{i1} = 0$, $\forall i$, or $c_{22} - 3c_{31} + c_{13} = 0$. The first case means that equation (1) does not contain the terms with $u^{(s)}$, so without loss of generality we may assume that we start with equation (1) with $(s-1)$ as the highest time derivative order of input. Therefore, it is necessary to continue with the second condition: $c_{22} - 3c_{31} + c_{13} = 0$.

The expression of the nth generalized state equation is extremely complicated and contains the quadratic and cubic terms in $u^{(s-2)}$. The coefficients of these terms will be zero either if

$$b_1 = c_{31} = c_{41} = 0, \quad c_{22} = c_{13}, \tag{20}$$

or

$$c_{13} = c_{31} = c_{22} = 0, \quad -c_{23} + c_{32} - c_{41} + c_{14} = 0. \tag{21}$$

The condition (20), together with conditions (18), will mean again that in equation (1) the highest order of input time derivative is $s-1$ and so, we may omit this branch. The conditions (21) together with (18) will yield conditions (6). □

Remark. When deriving the conditions in [12] shown for the case $n=4$ in Fig. 1, we assumed $s=n-1$ and for the case $n=4$ three possible realizable structures resulted. Note that the results for the 4th-order bilinear i/o equation, shown in Fig. 1, follow also from Theorem 1, being more general and simple.

- The first diagram in Fig. 1 corresponds to the situation when $s=1$, that is when the 2nd- and the 3rd-order input derivatives are missing in the bilinear i/o equation. In this case $b_1=b_2=c_{1j}=c_{2j}=0$ for $j=1,\dots,4$ as shown in the first diagram. There are no more additional restrictions since the input derivative can be lowered by one.
- The second diagram in Fig. 1 corresponds to the situation when $s=2$, that is when the 3rd-order input derivative is missing in the bilinear i/o equation. In this case $b_1=c_{11}=c_{21}=c_{31}=c_{41}=0$. Moreover, according to Theorem 1, additional conditions (5) have to be satisfied, in this case yielding $c_{11}=c_{12}-c_{21}=0$.
- Finally, the third diagram in Fig. 1 corresponds to the situation when $s=3$. According to Theorem 1, conditions (6) have to be satisfied now, yielding the results shown in the third diagram.

3. ASSISTANCE OF THE COMPUTER ALGEBRA SYSTEM MATHEMATICA IN FURTHER PROBLEM SOLUTION

The application of a computer algebra system (CAS) like Mathematica is well documented in control-related literature. Mostly, calculations are carried out to provide automatic reliable solutions of problems whose theory is well understood. Our goal is different. The purpose of this section is to report how the CAS Mathematica assisted us to formulate the conjecture, given below, and to prove it for small n and s values.

Conjecture 1. *Using a generalized state transformation* (4), *the input derivatives in equations* (3) *can be removed iff in the bilinear i/o equations* (1),

$$c_{ij}=0,\ \ \forall\, i+j\le 2s-2$$

and

$$\sum_{i=s-2}^{\min(n,2s-2)}(-1)^i c_{i,2s-1-i}=0.$$

The CAS Mathematica assisted us to prove this conjecture for small n and s values in the following way. It is known [10] that the input derivatives can be removed from the generalized state equations (3) iff the subspaces of one-forms

$$\begin{aligned}\mathscr{H}_1 &= \operatorname{span}\{\mathrm{d}x_1,\dots,\mathrm{d}x_n,\mathrm{d}u,\mathrm{d}u^{(1)},\dots,\mathrm{d}u^{(s)}\},\\ \mathscr{H}_k &= \operatorname{span}\{\omega\in\mathscr{H}_{k-1}\mid\dot\omega\in\mathscr{H}_{k-1}\},\ \ k\ge 2\end{aligned}$$

for $k=3,\dots,s+2$ are completely integrable.

Note that integrability can be checked by the Frobenius Theorem.

Theorem 2. (Frobenius). *Let* $V=\operatorname{span}_{\mathscr{K}}\{\,\omega_1,\,\dots,\,\omega_r\}$ *be a subspace of* $\mathscr{E}$. *V is closed if and only if*

$$\mathrm{d}\omega_i\wedge\omega_1\wedge\ldots\wedge\omega_r=0,\ \ \text{for any } i=1,\dots,r. \tag{22}$$

In (22) '$\wedge$' denotes the wedge product. Under conditions (22) there exists locally a system of coordinates $\{\zeta_1, \ldots, \zeta_r\}$ such that V is generated by $\{\mathrm{d}\zeta_1, \ldots, \mathrm{d}\zeta_r\}$. In this case V is said to be completely integrable [13].

So, we fix certain n and s values and calculate $\mathscr{H}_3$. If it turns out to be nonintegrable, we find the restrictions on bilinear equation parameters a_i, b_i, c_{ij} that would be necessary and sufficient for integrability. Next, we compute $\mathscr{H}_4$ under the restrictions on the system parameters found in the previous step and check its integrability. If $\mathscr{H}_4$ turns out to be non-integrable, it would yield an additional set of restrictions on the system parameters. In this way, we find step by step all the restrictions on system parameters for the given fixed pair of n and s values. We run through n values from 3 to 8 and combine each n to s values from $1, \ldots, n-1$. Using these results, we end up with a computer-generated proof of our conjecture.

4. CONCLUSION

The necessary and sufficient conditions in terms of bilinear i/o equation parameters, under which the input derivatives can be removed from the generalized state equations that correspond to the bilinear i/o equation, have been found. For input derivative values from 1 to 3, the result has been proved theoretically. For s values from 4 to 7, the result has been proved with the assistance of the CAS Mathematica. For higher-order s values, the result is still only a conjecture.

Our long-range goal is directed towards the development of a general subclass of realizable i/o models like the one given in [14] for the discrete-time case. By simple inspection of the i/o equation structure it allows us to decide if the equation has a state-space description or not.

ACKNOWLEDGEMENT

This work was partially supported by the Estonian Science Foundation (grant No. 6922) and The Royal Society.

REFERENCES

1. Van der Schaft, A. J. On realization of nonlinear systems described by higher-order differential equations. *Math. Syst. Theory*, 1987, **19**, 239–275.
2. Crouch, P. and Lamnabhi-Lagarrique, F. State space realizations of nonlinear systems defined by input-output differential equations. In *Analysis and Optimization Systems* (Bensousan, A. and Lion, J., eds), *Lecture Notes in Control and Inform. Sci.*, 1988, **111**, 138–149.
3. Glad, S. T. Nonlinear state space and input-output descriptions using differential polynomials. In *New Trends in Nonlinear Control Theory* (Descusse, J., Fliess, M., Isidori, A., and Leborne, P., eds), *Lecture Notes in Control and Inform. Sci.*, 1989, **122**, 182–189.
4. Delaleau, E. and Respondek, W. Lowering the orders of derivatives of controls in generalized state space systems. *J. Math. Systems Estim. Control*, 1995, **5**, 1–27.
5. Moog, C. H., Zheng, Y.-F., and Liu, P. Input-output equivalence of nonlinear systems and their equations. In *Proceedings of the 15th FIFAC World Congress, Barcelona* (Camacho, E. F., Basañez, L., and de la Puente, J. A., eds). Elsevier, 2002, 265–270.
6. Kotta, Ü. and Mullari, T. Equivalence of different realization methods for higher order nonlinear input-output differential equations. *European J. Control*, 2005, **11**, 185–193.
7. Verdult, V. and Verhaegen, V. Bilinear state space systems for nonlinear dynamical modelling. *Theory Bio. Sci.*, 2000, **119**, 1–9.
8. Kotta, Ü., Nõmm, S., and Zinober, A. S. I. Classical state space realizability of input-output bilinear models. *Internat. J. Control*, 2003, **76**, 1224–1232.
9. Fliess, M. Generalized controller canonical form for linear and nonlinear dynamics. *IEEE TAC*, 1990, **35**, 994–1001.
10. Kotta, Ü. Removing input derivatives in generalized state space systems: a linear algebraic approach. In *Proceedings of the 4th International Conference on Actual Problems of Electronic Instrument Engineering*. Novosibirsk, 1998, 391–396.
11. Mullari, T. and Kotta, Ü. Simplification of the generalized state equations. *Kybernetika*, 2006, **42**, 617–628.

12. Kotta, Ü., Mullari, T., Zinober, A. S. I., and Kotta, P. State space realizations of bilinear continuous-time input-output equations. *Internat. J. Control*, 2007, **80**, 1607–1615.
13. Choquet-Bruhat, Y., DeWitt-Morette, C., and Dillard-Bleick, M. *Analysis, Manifolds and Physics, Part I: Basics*. North-Holland, Amsterdam, 1989.
14. Kotta, Ü. and Sadegh, N. Two approaches for state space realization of Narma models: bridging the gap. *Math. Computer Modelling Dynam. Systems*, 2002, **8**, 21–32.

Sisendite tuletiste eemaldamine üldistatud bilineaarsetes olekuvõrrandites

Tanel Mullari, Ülle Kotta, Palle Kotta, Maris Tõnso ja Alan S. I. Zinober

On esitatud piirangud pideva ajaga bilineaarsetes sisend-väljundmudelites sisalduvatele kordajatele a_i, b_i, ja c_{ij}, mis võimaldavad leida sobivad olekukoordinaadid, et üldistatud olekuvõrrandid sisaldaksid madalamat järku sisendite ajalisi tuletisi kui sisend-väljundvõrrandid. Piirjuhul, kui vabanetakse kõigist sisendi ajalistest tuletistest, lubavad need tingimused lahendada realisatsiooniülesande. Uued olekukoordinaadid määratakse samm-sammult: esmalt leitakse koordinaatteisendus, mis lubab alandada sisendi ajalise tuletise maksimaalset järku ühe võrra, ja kirjutatakse välja uued olekuvõrrandid saadud koordinaatides. Teisel sammul leitakse uus koordinaatteisendus, mis lubab alandada sisendi ajalise tuletise maksimaalset järku ühe võrra juba uutes olekuvõrrandites jne. Igal sammul selgitatakse, milliseid tingimusi peavad kordajad rahuldama, et järgmine samm oleks võimalik.