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MATHEMATICS

On the Pringsheim convergence of double series

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Abstract. Several aspects of the convergence of a double series in the sense of Pringsheim are considered in analogy with some well-known results for single series. They include various tests for absolute convergence and also criteria for convergence of the Cauchy product. Some errors in the works of earlier authors are corrected.

Key words: summability theory, double series, convergence test, Pringsheim convergence, boundedly convergent, regularly convergent, Cauchy product, matrix transformation.

1. INTRODUCTION

Since Pringsheim introduced the notion of convergence of a numerical double series in terms of the convergence of the double sequence of its rectangular partial sums in [9], several authors have contributed to this topic during the last century. However, an exhaustive treatment giving analogues of all well-known convergence aspects of single series seems to be unavailable. The purpose of this article is to fill in some of the gaps in such a treatment, and also to point out some errors in previous attempts to obtain results exactly analogous to those of a single series. In Section 2, we give some tests for absolute convergence of a double series including analogues of Cauchy's Condensation Test, Abel's kth Term Test, Limit Comparison Test, Ratio Test, Ratio Comparison Test, and Raabe's Test. In Section 3, we give necessary and sufficient conditions on a double sequence $(a_{k,\ell})$ in order that the Cauchy product double series $\sum_{k,\ell} a_{k,\ell} * b_{k,\ell}$ would be convergent/boundedly convergent/regularly convergent whenever a double series $\sum_{k,\ell} b_{k,\ell}$ is convergent/boundedly convergent/regularly convergent. We also show that if two double series are boundedly convergent, then the Cauchy product double series is Cesàro summable and its Cesàro sum is equal to the product of the sums of the given double series. We compare our results with those obtained previously and give several examples to which our results apply. Although we shall consider, for simplicity, only double series whose terms are real numbers, the treatment carries over to multiple series whose terms may be complex numbers.

Throughout this article, $\mathbb{N}$, $\mathbb{N}^2$, $\mathbb{N}_0$, $\mathbb{N}_0^2$, and $\mathbb{R}$ will denote the set of all positive integers, of all pairs of positive integers, of all nonnegative integers, of all pairs of nonnegative integers, and of all real numbers, respectively. We shall use the partial order on $\mathbb{N}_0^2$ given by '$(k_1,\ell_1) \le (k_2,\ell_2)$ if and only if $k_1 \le k_2$ and $\ell_1 \le \ell_2$'. Monotonicity of a double sequence is defined in terms of this partial order. We shall adopt

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Pringsheim's definition of convergence of a double series $\sum_{k,\ell} a_{k,\ell}$ of real numbers: If $A_{m,n} := \sum_{k=1}^{m} \sum_{\ell=1}^{n} a_{k,\ell}$ for $(m,n) \in \mathbb{N}^2$, then $\sum_{k,\ell} a_{k,\ell}$ is said to be *convergent* if the double sequence $(A_{m,n})$ of its *partial sums* is convergent in the sense of Pringsheim, that is, there is $A \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is $(m_0,n_0) \in \mathbb{N}^2$ satisfying $(m,n) \geq (m_0,n_0) \Longrightarrow |A_{m,n} - A| < \varepsilon$. When every $a_{k,\ell}$ is nonnegative, $\sum_{k,\ell} a_{k,\ell}$ is convergent if and only if $(A_{m,n})$ is bounded above. For each fixed $k \in \mathbb{N}$, the series $\sum_{\ell} a_{k,\ell}$ is called a *row-series*, and for each fixed $\ell \in \mathbb{N}$, the series $\sum_{k} a_{k,\ell}$ is called a *column-series* corresponding to the double series $\sum_{k,\ell} a_{k,\ell}$.

2. ABSOLUTE CONVERGENCE

If a double series is absolutely convergent, then evidently the corresponding row-series and the column-series are all absolutely convergent. However, the converse is not true, as can be seen by considering $\sum_{k,\ell} a_{k,\ell}$, where $a_{k,k} := 1$ and $a_{k,\ell} := 0$ for $k \neq \ell$. The following result gives necessary and sufficient conditions for the absolute convergence of a double series.

Lemma 2.1. *A double series $\sum_{k,\ell} a_{k,\ell}$ is absolutely convergent if and only if the following conditions hold*:

(i) *There are $(k_0,\ell_0) \in \mathbb{N}^2$ and $\alpha_0 > 0$ such that*

$$\sum_{k=k_0}^{m} \sum_{\ell=\ell_0}^{n} |a_{k,\ell}| \leq \alpha_0 \quad \text{for all } (m,n) \geq (k_0,\ell_0).$$

(ii) *Each row-series as well as each column-series is absolutely convergent.*

We shall provide a variety of conditions each of which imply condition (i) of the above lemma. These yield convergence tests for double series which are analogous to well-known convergence tests for single series. (See, for example, Chapter 9 of [6] and the exercises therein.)

The following test shows that we can study the convergence of certain double series by considering only some of its terms.

Theorem 2.2 (Cauchy's Condensation Test)**.** *Let $(a_{k,\ell})$ be a monotonically decreasing double sequence of nonnegative numbers. Then $\sum_{k,\ell=1}^{\infty} a_{k,\ell}$ converges if and only if $\sum_{k,\ell=0}^{\infty} 2^{k+\ell} a_{2^k,2^\ell}$ converges.*

Proof. Given $(m,n) \in \mathbb{N}^2$, let $i,j \in \mathbb{N}_0$ be such that $2^i \leq m < 2^{i+1}$ and $2^j \leq n < 2^{j+1}$. Since $a_{k,\ell} \geq 0$ for all $(k,\ell) \in \mathbb{N}^2$, we have

$$\sum_{k=0}^{i-1} \sum_{\ell=0}^{j-1} \left(\sum_{u=2^k}^{2^{k+1}-1} \sum_{v=2^\ell}^{2^{\ell+1}-1} a_{u,v} \right) \leq \sum_{k=1}^{m} \sum_{\ell=1}^{n} a_{k,\ell} \leq \sum_{k=0}^{i} \sum_{\ell=0}^{j} \left(\sum_{u=2^k}^{2^{k+1}-1} \sum_{v=2^\ell}^{2^{\ell+1}-1} a_{u,v} \right),$$

and since $(a_{k,\ell})$ is monotonically decreasing, we obtain

$$\frac{1}{4} \sum_{k=1}^{i} \sum_{\ell=1}^{j} 2^{k+\ell} a_{2^k,2^\ell} = \sum_{k=0}^{i-1} \sum_{\ell=0}^{j-1} 2^{k+\ell} a_{2^{k+1},2^{\ell+1}} \leq \sum_{k=1}^{m} \sum_{\ell=1}^{n} a_{k,\ell} \leq \sum_{k=0}^{i} \sum_{\ell=0}^{j} 2^{k+\ell} a_{2^k,2^\ell}.$$

This shows that if the partial sums of $\sum_{k,\ell=0}^{\infty} 2^{k+\ell} a_{2^k,2^\ell}$ are bounded, then so are the partial sums of $\sum_{k,\ell=1}^{\infty} a_{k,\ell}$, and if the partial sums of $\sum_{k,\ell=1}^{\infty} a_{k,\ell}$ are bounded, then so are the partial sums $\sum_{k,\ell=1}^{\infty} 2^{k+\ell} a_{2^k,2^\ell}$. Further, if $\sum_{k,\ell=1}^{\infty} a_{k,\ell}$ is convergent, then the row-series $\sum_{\ell=1}^{\infty} a_{1,\ell}$ and the column-series $\sum_{k=1}^{\infty} a_{k,1}$ are convergent, where $(a_{k,1})$ and $(a_{1,\ell})$ are monotonically decreasing sequences of nonnegative numbers. In this case, the series $\sum_{k=0}^{\infty} 2^k a_{2^k,1}$ and $\sum_{\ell=0}^{\infty} 2^\ell a_{1,2^\ell}$ are convergent by Cauchy's Condensation Test for single series. Hence the desired result follows. $\square$

Example 2.3. Let $p \in \mathbb{R}$ and $a_{k,\ell} := 1/(k+\ell)^p$ for $(k,\ell) \in \mathbb{N}^2$. By Theorem 2.2, $\sum_{k,\ell=1}^{\infty} a_{k,\ell}$ converges if and only if the double series $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ converges, where $b_{k,\ell} := 2^{k+\ell}/(2^k+2^\ell)^p$. If $p \le 2$, then

$$b_{k,k} = \frac{2^{2k}}{(2 \cdot 2^k)^p} = \frac{1}{2^p} 2^{k(2-p)} \ge \frac{1}{2^p} \quad \text{for } k \in \mathbb{N},$$

and so the double series $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ diverges. If $p > 2$, then

$$b_{k,\ell} = \frac{2^{k+\ell}}{(2^k+2^\ell)^p} \le \frac{2^{k+\ell}}{2^p(2^{k+\ell})^{p/2}} = \frac{1}{2^p}\left(2^{(2-p)/2}\right)^{k+\ell} \quad \text{for } \ (k,\ell) \in \mathbb{N}^2,$$

and so the double series $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ converges. It follows that the double series $\sum_{k,\ell=1}^{\infty} 1/(k+\ell)^p$ converges if and only if $p > 2$.

If $\sum_{k,\ell} a_{k,\ell}$ is convergent, then $a_{k,\ell} \to 0$ as $k,\ell \to \infty$. This (k,ℓ)th Term Test is useful for establishing the divergence of a double series. The following variant of this test is analogous to Abel's kth Term Test for a single series.

Theorem 2.4 (Abel's (k,ℓ)th Term Test). *Suppose $(a_{k,\ell})$ is a monotonically decreasing double sequence of nonnegative numbers. If the double series $\sum_{k,\ell} a_{k,\ell}$ is convergent, then $k\ell\, a_{k,\ell} \to 0$ as $k,\ell \to \infty$.*

Proof. Given $(k,\ell) \in \mathbb{N}^2$, let $i_k, j_\ell \in \mathbb{N}_0$ be such that $2^{i_k} \le k < 2^{i_k+1}$ and $2^{j_\ell} \le \ell < 2^{j_\ell+1}$, and note that

$$0 \le k\ell\, a_{k,\ell} \le 2^{i_k+1} 2^{j_\ell+1} a_{2^{i_k},2^{j_\ell}} = 4 \cdot 2^{i_k} 2^{j_\ell} a_{2^{i_k},2^{j_\ell}}.$$

By Theorem 2.2, $\sum_{i,j=0}^{\infty} 2^{i+j} a_{2^i,2^j}$ is convergent, and so $2^{i+j} a_{2^i,2^j} \to 0$ as $i,j \to \infty$. Hence $k\ell\, a_{k,\ell} \to 0$ as $k,\ell \to \infty$. $\square$

Examples 2.5.

(i) Let $p,q \in \mathbb{R}$ satisfy $p > 0$, $q > 0$ and $(1/p)+(1/q) \ge 1$, and define $a_{k,\ell} := 1/(k^p+\ell^q)$ for $(k,\ell) \in \mathbb{N}^2$. For $k \in \mathbb{N}$ and $\ell = [k^{p/q}]$, the integer part of $k^{p/q}$, we have

$$k\ell\, a_{k,\ell} = \frac{k\ell}{k^p+\ell^q} > \frac{k(k^{p/q}-1)}{k^p+(k^{p/q})^q} = \frac{1}{2} k^{1-p+(p/q)}(1-k^{-p/q}),$$

which does not tend to 0 as $k \to \infty$ since $1-p+(p/q) \ge 0$ and $p/q > 0$. Hence by Theorem 2.4, the double series $\sum_{k,\ell} 1/(k^p+\ell^q)$ diverges.

(ii) The converse of Theorem 2.4 does not hold. Define $a_{k,\ell} := 1/k\ell(\ln k)(\ln \ell)$ for $(k,\ell) \in \mathbb{N}^2$. Then $(a_{k,\ell})$ is a monotonically decreasing double sequence of nonnegative numbers and $k\ell\, a_{k,\ell} \to 0$ as $k,\ell \to \infty$. However,

$$\sum_{k=1}^{m} \sum_{\ell=1}^{n} a_{k,\ell} = \left(\sum_{k=1}^{m} \frac{1}{k(\ln k)}\right)\left(\sum_{\ell=1}^{n} \frac{1}{\ell(\ln \ell)}\right) \to \infty \quad \text{as} \quad m,n \to \infty.$$

Theorem 2.6 (Limit Comparison Test). *Let $(a_{k,\ell})$ and $(b_{k,\ell})$ be double sequences such that $a_{k,\ell} > 0$, $b_{k,\ell} > 0$ for all $(k,\ell) \in \mathbb{N}^2$, each row-series as well as each column-series corresponding to both $\sum_{k,\ell} a_{k,\ell}$ and $\sum_{k,\ell} b_{k,\ell}$ is convergent, and $\lim_{k,\ell} a_{k,\ell}/b_{k,\ell} = r$, where $r \in \mathbb{R}$ and $r \ne 0$. Then $\sum_{k,\ell} a_{k,\ell}$ is convergent if and only if $\sum_{k,\ell} b_{k,\ell}$ is convergent.*

Proof. Let the double series $\sum_{k,\ell} b_{k,\ell}$ be convergent. Then there is $\beta > 0$ such that $\sum_{k=1}^{m}\sum_{\ell=1}^{n} b_{k,\ell} \le \beta$ for all $(m,n) \in \mathbb{N}^2$. Since $a_{k,\ell}/b_{k,\ell} \to r$ as $k,\ell \to \infty$, there is $(k_0,\ell_0) \in \mathbb{N}^2$ such that $a_{k,\ell} \le (r+1)b_{k,\ell}$ for all $(k,\ell) \ge (k_0,\ell_0)$. Hence for all $(m,n) \ge (k_0,\ell_0)$, we have

$$\sum_{k=k_0}^{m}\sum_{\ell=\ell_0}^{n} a_{k,\ell} \le (r+1)\sum_{k=k_0}^{m}\sum_{\ell=\ell_0}^{n} b_{k,\ell} \le (r+1)\beta.$$

By Lemma 2.1, it follows that the double series $\sum_{k,\ell} a_{k,\ell}$ is convergent.

Conversely, let the double series $\sum_{k,\ell} a_{k,\ell}$ be convergent. Since $\lim_{k,\ell} b_{k,\ell}/a_{k,\ell} = 1/r$, the convergence of the double series $\sum_{k,\ell} b_{k,\ell}$ follows from the first part of the proof by interchanging $a_{k,\ell}$ and $b_{k,\ell}$. $\square$

We shall now develop several convergence tests involving ratios of 'consecutive' terms of a double series.

Theorem 2.7 (Ratio Test)**.** *Let $(a_{k,\ell})$ be a double sequence of nonzero numbers such that either $|a_{k,\ell+1}|/|a_{k,\ell}| \to a$ or $|a_{k+1,\ell}|/|a_{k,\ell}| \to \tilde{a}$ as $k,\ell \to \infty$, where $a,\tilde{a}, \in \mathbb{R}\cup\{\infty\}$.*

(i) *Suppose each row-series as well as each column-series corresponding to $\sum_{k,\ell} a_{k,\ell}$ is absolutely convergent. If $a < 1$ or $\tilde{a} < 1$, then $\sum_{k,\ell} a_{k,\ell}$ is absolutely convergent.*

(ii) *If $a > 1$ or $\tilde{a} > 1$, then $\sum_{k,\ell} a_{k,\ell}$ is divergent.*

Proof.

(i) Assume that $a < 1$. Then there are $\alpha \in (0,1)$ and $(k_0,\ell_0) \in \mathbb{N}^2$ such that $|a_{k,\ell+1}| \le \alpha|a_{k,\ell}|$ for all $(k,\ell) \ge (k_0,\ell_0)$. Hence

$$|a_{k,\ell}| \le \alpha|a_{k,\ell-1}| \le \cdots \le \alpha^{\ell-\ell_0}|a_{k,\ell_0}| \quad \text{for all } (k,\ell) \ge (k_0,\ell_0+1).$$

Since $\alpha < 1$, we have $\sum_{\ell=1}^{n}\alpha^\ell \le 1/(1-\alpha)$ for all $n \in \mathbb{N}$. Also, since the series $\sum_k a_{k,\ell_0}$ is assumed to be absolutely convergent, there is $\beta > 0$ such that $\sum_{k=1}^{m}|a_{k,\ell_0}| \le \beta$ for all $m \in \mathbb{N}$. Hence we obtain

$$\sum_{k=k_0}^{m}\sum_{\ell=\ell_0+1}^{n}|a_{k,\ell}| \le \frac{\alpha^{-\ell_0}\beta}{1-\alpha} \quad \text{for all } (m,n) \ge (k_0,\ell_0+1).$$

By Lemma 2.1, it follows that $\sum_{k,\ell} a_{k,\ell}$ is absolutely convergent. A similar argument holds if $\tilde{a} < 1$ instead of $a < 1$.

(ii) Assume that $a \in \mathbb{R}$ with $a > 1$ or $a = \infty$. Then there are $\alpha \in (1,\infty)$ and $(k_0,\ell_0) \in \mathbb{N}^2$ such that $|a_{k,\ell+1}|/|a_{k,\ell}| \ge \alpha$ for all $(k,\ell) \ge (k_0,\ell_0)$. Hence

$$|a_{k,\ell}| \ge \alpha|a_{k,\ell-1}| \ge \cdots \ge \alpha^{\ell-\ell_0}|a_{k,\ell_0}| > 0 \quad \text{for all } (k,\ell) \ge (k_0,\ell_0+1).$$

For a fixed $k \in \mathbb{N}$ with $k \ge k_0$, there is $\ell_1 \in \mathbb{N}$ such that $|a_{k,\ell}| \ge \alpha^{\ell-\ell_0}|a_{k,\ell_0}| \ge 1$ for all $\ell \ge \ell_1$. Hence $a_{k,\ell} \not\to 0$ as $k,\ell \to \infty$, so that $\sum_{k,\ell} a_{k,\ell}$ is divergent. The same conclusion holds if $\tilde{a} \in \mathbb{R}$ with $\tilde{a} > 1$ or $\tilde{a} = \infty$. $\square$

Remarks 2.8.

(i) The proof of part (ii) of Theorem 2.7 shows that if $a > 1$, then the row-series $\sum_\ell a_{k,\ell}$ diverges for each (fixed) large k. Hence if each row-series converges and the limit a exists, then $a \le 1$. Similarly, if $\tilde{a} > 1$, then the column-series $\sum_k a_{k,\ell}$ diverges for each (fixed) large ℓ. Hence if each column-series converges and the limit $\tilde{a}$ exists, then $\tilde{a} \le 1$.

(ii) Suppose each row-series as well as each column-series corresponding to $\sum_{k,\ell} a_{k,\ell}$ is absolutely convergent. If $a = \tilde{a} = 1$, then the double series may converge or may diverge, as Example 2.3 shows.

(iii) For a double sequence $(a_{k,\ell})$ of positive numbers, consider the limits $b_k := \lim_\ell a_{k,\ell+1}/a_{k,\ell}$ for a fixed $k \in \mathbb{N}$, and $c_\ell := \lim_k a_{k+1,\ell}/a_{k,\ell}$ for a fixed $\ell \in \mathbb{N}$, whenever they exist. Biermann (page 123 of [2]) and Vorob'ev (§4 of Chapter 13 in [11]) claimed that if b_k exists and is less than 1 for each $k \in \mathbb{N}$, and if c_ℓ exists and is less than 1 for each $\ell \in \mathbb{N}$, then the double series $\sum_{k,\ell} a_{k,\ell}$ is absolutely convergent. Although the Ratio Test for single series shows that each row-series as well as each column-series corresponding to $\sum_{k,\ell} a_{k,\ell}$ is absolutely convergent, the double series $\sum_{k,\ell} a_{k,\ell}$ may not be absolutely convergent. For example, define $a_{k,\ell} := 1/(2^k 2^{\ell/2^k})$ for (k,ℓ) in $\mathbb{N}^2$. Then $b_k < 1$ for each $k \in \mathbb{N}$ and $c_\ell < 1$ for each $\ell \in \mathbb{N}$. However, since for each fixed $k \in \mathbb{N}$,

$$\sum_{\ell=1}^{\infty} a_{k,\ell} = \frac{1}{2^k} \sum_{\ell=1}^{\infty} \left(\frac{1}{2^{1/2^k}}\right)^{\ell} = \frac{1}{2^k(2^{1/2^k}-1)},$$

and since $1/2^k(2^{1/2^k}-1) \to 1/\ln 2$ as $k \to \infty$, we see that the iterated series $\sum_k \sum_\ell a_{k,\ell}$ diverges, and so the double series $\sum_{k,\ell} a_{k,\ell}$ also diverges. Thus the claims of Biermann and Vorob'ev are incorrect.

(iv) Suppose $a_{k,\ell} > 0$ for all $(k,\ell) \in \mathbb{N}^2$, and each row-series as well as each column-series corresponding to $\sum_{k,\ell} a_{k,\ell}$ is convergent. A rather involved version of the Ratio Test is given by Baron in §2 of [1] as follows. If $\lim_{k,\ell} a_{k,\ell}$ exists and if the limit

$$d := \lim_{k,\ell} \frac{a_{k+1,\ell} + a_{k,\ell+1} - a_{k+1,\ell+1}}{a_{k,\ell}}$$

exists with $d < 1$, then the double series $\sum_{k,\ell} a_{k,\ell}$ is convergent. Let us compare Baron's version of the Ratio Test with our Theorem 2.7. Suppose both the limits a and $\tilde{a}$ stated in Theorem 2.7 exist and are in $\mathbb{R}$. Then the limit d exists and

$$d = \lim_{k,\ell} \left(\frac{a_{k+1,\ell}}{a_{k,\ell}} + \frac{a_{k,\ell+1}}{a_{k,\ell}} - \frac{a_{k+1,\ell+1}}{a_{k+1,\ell}} \frac{a_{k+1,\ell}}{a_{k,\ell}} \right) = \tilde{a} + a - a\tilde{a}.$$

Since, in view of (i) above, we have $a \leq 1$ and $\tilde{a} \leq 1$, and since $1 - d = (1-a)(1-\tilde{a})$, we see that $d < 1$ if and only if $a < 1$ and $\tilde{a} < 1$. Thus if one of a and $\tilde{a}$ is equal to 1 and the other is not, then $d = 1$, and hence Theorem 2.7 is applicable, but Baron's version of the Ratio Test is not. For example, if $a_{k,\ell} := 1/k^2 2^\ell$ for $(k,\ell) \in \mathbb{N}^2$, then $a = 1/2$, $\tilde{a} = 1$, and $d = 1$. Now suppose that the limit a exists and it is a real number other than 1. If the limit d exists, then it can be seen that the limit $\tilde{a}$ exists and is equal to $(d-a)/(1-a)$. Thus if $a < 1$ and $\tilde{a}$ does not exist, then the limit d cannot exist, and hence Theorem 2.7 is applicable, but Baron's version of the Ratio Test is not. For example, if $a_{k,\ell} := 1/2^{(k^2+k\ell+\ell)/k}$, then $a = 1/2$, while the limits $\tilde{a}$ and d do not exist.

Now we consider an analogue of the Ratio Comparison Test for single series. (See, for example, Theorem 6 in Chapter 5 of [4].)

Theorem 2.9 (Ratio Comparison Test)**.** *Let $(a_{k,\ell})$ and $(b_{k,\ell})$ be double sequences with $b_{k,\ell} > 0$ for all $(k,\ell) \in \mathbb{N}^2$.*

(i) *Suppose each row-series as well as each column-series corresponding to $\sum_{k,\ell} |a_{k,\ell}|$ is convergent. If $|a_{k,\ell+1}| b_{k,\ell} \leq |a_{k,\ell}| b_{k,\ell+1}$ and $|a_{k+1,\ell}| b_{k,\ell} \leq |a_{k,\ell}| b_{k+1,\ell}$ whenever k and ℓ are large, and if $\sum_{k,\ell} b_{k,\ell}$ is convergent, then so is $\sum_{k,\ell} |a_{k,\ell}|$.*

(ii) *If $|a_{k,\ell+1}| b_{k,\ell} \geq |a_{k,\ell}| b_{k,\ell+1} > 0$ whenever ℓ is large and $k \in \mathbb{N}$, and $|a_{k+1,\ell}| b_{k,\ell} \geq |a_{k,\ell}| b_{k+1,\ell} > 0$ whenever k is large and $\ell \in \mathbb{N}$, and if $\sum_{k,\ell} b_{k,\ell}$ is divergent, then so is $\sum_{k,\ell} |a_{k,\ell}|$.*

Proof.

(i) Let $k_0, \ell_0 \in \mathbb{N}$ be such that $|a_{k,\ell+1}|b_{k,\ell} \le |a_{k,\ell}|b_{k,\ell+1}$ and $|a_{k+1,\ell}|b_{k,\ell} \le |a_{k,\ell}|b_{k+1,\ell}$ for all $(k,\ell) \ge (k_0,\ell_0)$. Then

$$\frac{|a_{k,\ell}|}{b_{k,\ell}} \le \frac{|a_{k,\ell-1}|}{b_{k,\ell-1}} \le \cdots \le \frac{|a_{k,\ell_0}|}{b_{k,\ell_0}} \le \frac{|a_{k-1,\ell_0}|}{b_{k-1,\ell_0}} \le \cdots \le \frac{|a_{k_0,\ell_0}|}{b_{k_0,\ell_0}} \quad \text{for } (k,\ell) \ge (k_0,\ell_0).$$

Let $\sum_{k,\ell} b_{k,\ell}$ be convergent. Since $\beta := \sup\left\{\sum_{k=1}^m \sum_{\ell=1}^n b_{k,\ell} : (m,n) \in \mathbb{N}^2\right\} < \infty$, we obtain

$$\sum_{k=k_0}^m \sum_{\ell=\ell_0}^n |a_{k,\ell}| \le \frac{|a_{k_0,\ell_0}|}{b_{k_0,\ell_0}} \sum_{k=k_0}^m \sum_{\ell=\ell_0}^n b_{kl} \le \beta \frac{|a_{k_0,\ell_0}|}{b_{k_0,\ell_0}} \quad \text{for all } m \ge k_0 \text{ and } n \ge \ell_0.$$

Hence by Lemma 2.1, the double series $\sum_{k,\ell} |a_{k,\ell}|$ converges.

(ii) Suppose $k_0 \in \mathbb{N}$ is such that $|a_{k+1,\ell}|b_{k,\ell} \ge |a_{k,\ell}|b_{k+1,\ell} > 0$ for $k \ge k_0$ and $\ell \in \mathbb{N}$, and $\ell_0 \in \mathbb{N}$ is such that $|a_{k,\ell+1}|b_{k,\ell} \ge |a_{k,\ell}|b_{k,\ell+1} > 0$ for $\ell \ge \ell_0$ and $k \in \mathbb{N}$. Let $\sum_{k,\ell} b_{k,\ell}$ be divergent. If $\sum_k b_{k,\ell}$ diverges for some $\ell \in \mathbb{N}$, then by the Ratio Comparison Test for single series, $\sum_k |a_{k,\ell}|$ also diverges for that ℓ. Similarly, if $\sum_\ell b_{k,\ell}$ diverges for some $k \in \mathbb{N}$, then $\sum_\ell |a_{k,\ell}|$ also diverges for that k. In these cases, condition (ii) of Lemma 2.1 is not satisfied, and so the double series $\sum_{k,\ell} |a_{k,\ell}|$ diverges. In the remaining case, the set $\left\{\sum_{k=k_0}^m \sum_{\ell=\ell_0}^n b_{k,\ell} : (m,n) \in \mathbb{N}^2\right\}$ is unbounded. Reversing the inequality signs in (i) above, we obtain

$$\sum_{k=k_0}^m \sum_{\ell=\ell_0}^n |a_{k,\ell}| \ge \frac{|a_{k_0,\ell_0}|}{b_{k_0,\ell_0}} \sum_{k=k_0}^m \sum_{\ell=\ell_0}^n b_{kl},$$

which tends to ∞ as $m,n \to \infty$. Hence by Lemma 2.1, $\sum_{k,\ell} |a_{k,\ell}|$ diverges. □

Remarks 2.10.

(i) The following example shows that both the inequalities $|a_{k,\ell+1}|b_{k,\ell} \le |a_{k,\ell}|b_{k,\ell+1}$ and $|a_{k+1,\ell}|b_{k,\ell} \le |a_{k,\ell}|b_{k+1,\ell}$ are needed in part (i) of Theorem 2.9. Define

$$a_{k,\ell} := \frac{1}{(k+\ell)^2} \quad \text{and} \quad b_{k,\ell} = \frac{1}{2^k(k+\ell)^2} \quad \text{for } (k,\ell) \in \mathbb{N}^2.$$

Although each row-series as well as each column-series converges, the double series $\sum_{k,\ell} a_{k,\ell}$ diverges, as we have seen in Example 2.3. However, the double series $\sum_{k,\ell} b_{k,\ell}$ converges, since $1/(2^k(k+\ell)^2) \le 1/(2^k\ell^2)$ for all $(k,\ell) \in \mathbb{N}^2$. Here the first inequality mentioned above holds but the second does not. To see that both the inequalities $|a_{k,\ell+1}|b_{k,\ell} \ge |a_{k,\ell}|b_{k,\ell+1}$ and $|a_{k+1,\ell}|b_{k,\ell} \ge |a_{k,\ell}|b_{k+1,\ell}$ are needed in part (ii) of Theorem 2.9, we just interchange the roles of $a_{k,\ell}$ and $b_{k,\ell}$ in (i) above.

(ii) The requirement '$|a_{k,\ell+1}|b_{k,\ell} \le |a_{k,\ell}|b_{k,\ell+1}$ and $|a_{k+1,\ell}|b_{k,\ell} \le |a_{k,\ell}|b_{k+1,\ell}$ whenever k and ℓ are large', in part (i) of Theorem 2.9, is less stringent than the requirement '$|a_{k+p,\ell+1+q}|b_{k+r,\ell+s} \le |a_{k+p,\ell+q}|b_{k+r,\ell+1+s}$ and $|a_{k+1+p,\ell+q}|b_{k+r,\ell+s} \le |a_{k+1+p,\ell+q}|b_{k+1+r,\ell+s}$ for all $k,\ell,p,q,r,s \in \mathbb{N}$' imposed by Biermann for a similar result in [2], page 124.

As a consequence of the Ratio Comparison Test, we obtain an analogue of Raabe's Test for single series. It is useful in some cases when $a = \tilde{a} = 1$ in Theorem 2.7.

Theorem 2.11. *Let $(a_{k,\ell})$ be a double sequence.*

(i) *Suppose each row-series as well as each column-series corresponding to $\sum_{k,\ell} |a_{k,\ell}|$ is convergent. If there is $p > 1$ such that*

$$|a_{k,\ell+1}| \le \left(1 - \frac{p}{\ell}\right)|a_{k,\ell}| \quad \textit{and} \quad |a_{k+1,\ell}| \le \left(1 - \frac{p}{k}\right)|a_{k,\ell}|$$

whenever k and ℓ are large, then $\sum_{k,\ell} |a_{k,\ell}|$ is convergent.

(ii) *If $|a_{k,\ell+1}| \geq (1-1/\ell)|a_{k,\ell}| > 0$ for some $k \in \mathbb{N}$ and all large $\ell \in \mathbb{N}$, or if $|a_{k+1,\ell}| \geq (1-1/k)|a_{k,\ell}| > 0$ for some $\ell \in \mathbb{N}$ and all large $k \in \mathbb{N}$, then $\sum_{k,\ell}|a_{k,\ell}|$ is divergent.*

Proof.

(i) Suppose there is $p > 1$ with the stated properties. Using the inequality $1 - px \leq (1-x)^p$ for $x \in [0,1]$, we obtain

$$|a_{k,\ell+1}| \leq \left(1 - \frac{p}{\ell}\right)|a_{k,\ell}| \leq \left(1 - \frac{1}{\ell}\right)^p |a_{k,\ell}| \leq \left(\frac{\ell}{\ell+1}\right)^p |a_{k,\ell}|$$

whenever k and ℓ are large, and hence

$$|a_{k,\ell+1}|\frac{1}{k^p \ell^p} \leq |a_{k,\ell}|\frac{1}{k^p(\ell+1)^p}.$$

Similarly, we obtain

$$|a_{k+1,\ell}|\frac{1}{k^p \ell^p} \leq |a_{k,\ell}|\frac{1}{(k+1)^p \ell^p}$$

whenever k and ℓ are large. By part (i) of Theorem 2.9 with $b_{k,\ell} := 1/(k^p \ell^p)$ for $(k,\ell) \in \mathbb{N}^2$, we obtain the desired result.

(ii) Suppose the assumption in (ii) holds. Then by Raabe's Test for single series, $\sum_k |a_{k,\ell}|$ diverges for some $\ell \in \mathbb{N}$ or $\sum_\ell |a_{k,\ell}|$ diverges for some $k \in \mathbb{N}$. In any case, condition (ii) of Lemma 2.1 is not satisfied, and so the double series $\sum_{k,\ell}|a_{k,\ell}|$ is divergent. □

Examples 2.12.

(i) Let $a_{1,1} := 1$, $a_{k+1,1} := (2k-1)a_{k,1}/(2k+2)$ for $k \in \mathbb{N}$ and $a_{k,\ell+1} := (2\ell-1)a_{k,\ell}/(2\ell+2)$ for $(k,\ell) \in \mathbb{N}^2$. By part (i) of Theorem 2.11 with $p = 5/4$, we see that the double series $\sum_{k,\ell} a_{k,\ell}$ is convergent.

(ii) Let $a_{1,1} := 1$, $a_{k+1,1} := k a_{k,1}/(k+1)$ for $k \in \mathbb{N}$ and $a_{k,\ell+1} := \ell a_{k,\ell}/(\ell+1)$ for $(k,\ell) \in \mathbb{N}^2$. By part (ii) of Theorem 2.11, we see that the double series $\sum_{k,\ell} a_{k,\ell}$ is divergent.

In both examples, Theorem 2.7 is not applicable, since $a = \tilde{a} = 1$.

We deduce the following 'limit' version of Raabe's Test from Theorem 2.11.

Theorem 2.13. *Let $(a_{k,\ell})$ be a double sequence of nonzero numbers.*

(i) *Suppose $\ell(1 - |a_{k,\ell+1}/a_{k,\ell}|) \to \alpha$ and $k(1 - |a_{k+1,\ell}/a_{k,\ell}|) \to \tilde{\alpha}$ as $k,\ell \to \infty$, where $\alpha, \tilde{\alpha} \in \mathbb{R} \cup \{\infty\}$. Suppose each row-series as well as each column-series corresponding to $\sum_{k,\ell}|a_{k,\ell}|$ is convergent. If $\alpha > 1$ and $\tilde{\alpha} > 1$, then $\sum_{k,\ell}|a_{k,\ell}|$ is convergent.*

(ii) *If for some $k \in \mathbb{N}$, the limit $\lim_{\ell\to\infty} \ell(1 - |a_{k,\ell+1}/a_{k,\ell}|)$ exists and is less than 1, or if for some $\ell \in \mathbb{N}$, the limit $\lim_{k\to\infty} k(1 - |a_{k+1,\ell}/a_{k,\ell}|)$ exists and is less than 1, then $\sum_{k,\ell}|a_{k,\ell}|$ is divergent.*

Remark 2.14. Suppose $a_{k,\ell} > 0$ for all $(k,\ell) \in \mathbb{N}^2$, and each row-series as well as each column-series corresponding to $\sum_{k,\ell} a_{k,\ell}$ is convergent. A rather involved version of Raabe's Test for double series is given by Baron in §3 of [1] as follows. If $\lim_{k,\ell} a_{k,\ell}$ exists and if the limit

$$r := \lim_{k,\ell}\left[(k+\ell)\left(1 - \frac{a_{k+1,\ell} + a_{k,\ell+1} - a_{k+1,\ell+1}}{a_{k,\ell}}\right) + \frac{a_{k+1,\ell+1}}{a_{k,\ell}}\right]$$

exists with $r > 1$, then the double series $\sum_{k,\ell} a_{k,\ell}$ is convergent. Let us compare Baron's version of Raabe's Test with our Theorem 2.13. Suppose both the limits α and $\tilde{\alpha}$ stated in Theorem 2.13 exist and are in $\mathbb{R}$. Then since $a_{k,\ell+1}/a_{k,\ell} \to 1$ and $a_{k+1,\ell}/a_{k,\ell} \to 1$ as $k,\ell \to \infty$, we see that the limit r exists and

$$\begin{aligned} r &= \lim_{k,\ell}\left[k\left(1 - \frac{a_{k+1,\ell}}{a_{k,\ell}}\right) + \ell\left(1 - \frac{a_{k,\ell+1}}{a_{k,\ell}}\right) - \frac{a_{k+1,\ell}}{a_{k,\ell}}\ell\left(1 - \frac{a_{k+1,\ell+1}}{a_{k+1,\ell}}\right)\right. \\ &\quad \left. - \frac{a_{k,\ell+1}}{a_{k,\ell}}k\left(1 - \frac{a_{k+1,\ell+1}}{a_{k,\ell+1}}\right) + \frac{a_{k+1,\ell+1}}{a_{k,\ell+1}}\frac{a_{k,\ell+1}}{a_{k,\ell}}\right] \\ &= \tilde{\alpha} + \alpha - \alpha - \tilde{\alpha} + 1 = 1. \end{aligned}$$

In all such cases, Theorem 2.13 is applicable, but Baron's version of Raabe's Test is not. For instance, in Example 2.12 (i), we have $\alpha = 2 = \tilde{\alpha}$, but $r = 1$.

3. CAUCHY PRODUCT

The Cauchy product of sequences (a_k) and (b_k) with $k \in \mathbb{N}_0$ is defined to be the sequence $(a_k * b_k)$, where $a_k * b_k := \sum_{i=0}^{k} a_i b_{k-i}$ for $k \in \mathbb{N}_0$, and the Cauchy product of single series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ is defined to be the double series $\sum_{k=0}^{\infty} a_k * b_k$. Analogously, the *Cauchy product* of double sequences $(a_{k,\ell})$ and $(b_{k,\ell})$ with $(k,\ell) \in \mathbb{N}_0^2$ is defined to be the sequence $(a_{k,\ell} * b_{k,\ell})$, where

$$a_{k,\ell} * b_{k,\ell} := \sum_{i=0}^{k} \sum_{j=0}^{\ell} a_{i,j} b_{k-i,\ell-j} \quad \text{for } (k,\ell) \in \mathbb{N}_0^2,$$

and the Cauchy product of double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ and $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ is defined to be the double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell}$. A classical result of Mertens states that if one of the given single series is absolutely convergent and the other is convergent, then their Cauchy product series is convergent. Another result due to Abel states that if both the given single series and their Cauchy product series are convergent, then the sum of the Cauchy product series is equal to the product of the sums of the given series. It has been known for long that the exact analogue of Mertens' result does not hold for double series. (See the examples given on page 1036 of [10] and on page 190 of [5].) The example below shows that the exact analogue of Abel's result does not hold for double series.

Example 3.1. Consider double sequences $(a_{k,\ell})$ and $(b_{k,\ell})$ defined by $a_{0,\ell} := 1$ and $a_{1,\ell} := -1$ for $\ell \in \mathbb{N}_0$, whereas $a_{k,\ell} := 0$ for $k \in \mathbb{N}_0 \setminus \{0,1\}$ and $\ell \in \mathbb{N}_0$, while $b_{k,0} := 1$ and $b_{k,1} := -1$ for $k \in \mathbb{N}_0$, whereas $b_{k,\ell} := 0$ for $\ell \in \mathbb{N}_0 \setminus \{0,1\}$ and $k \in \mathbb{N}_0$. Then $\sum_{k,\ell} a_{k,\ell}$ and $\sum_{k,\ell} b_{k,\ell}$ are convergent and the sum of each is equal to 0. Also, it is easy to see that $a_{0,0} * b_{0,0} = 1$ and $a_{k,\ell} * b_{k,\ell} = 0$ for all $(k,\ell) \in \mathbb{N}_0^2 \setminus \{(0,0)\}$, so that $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell}$ is convergent and its sum is 1.

We shall now prove analogues of the theorems of Mertens and Abel for double series which are *boundedly convergent*, that is, which are convergent and their partial sums are bounded. In fact, we shall show that Mertens' result admits a converse for such double series in respect of absolute convergence. Our proofs will be based on the following result for a transformation of a double sequence by a 4-fold infinite matrix. It is an analogue of the well-known Kojima–Schur Theorem (given, for instance, in Theorem 2.3.7 of [3]) for boundedly convergent double sequences.

Let $\alpha_{m,n,k,\ell} \in \mathbb{R}$ and consider the matrix $\boldsymbol{A} := (\alpha_{m,n,k,\ell})$. We say that $\boldsymbol{A}$ maps a double sequence $x := (x_{k,\ell})$ to the double sequence $\boldsymbol{A}x$ defined by

$$[\boldsymbol{A}x]_{m,n} := \sum_{k,\ell} \alpha_{m,n,k,\ell} x_{k,\ell},$$

provided the double series on the right side converges for each fixed (m,n).

Lemma 3.2. *A matrix $\boldsymbol{A} := (\alpha_{m,n,k,\ell})$ maps each bounded convergent double sequence to a bounded convergent double sequence if and only if*

(i) $\sup_{m,n} \sum_{k,\ell} |\alpha_{m,n,k,\ell}| < \infty$,

(ii) *the limit* $\alpha := \lim_{m,n} \sum_{k,\ell} \alpha_{m,n,k,\ell}$ *exists,*

(iii) *the limit* $\alpha_{k,\ell} := \lim_{m,n} \alpha_{m,n,k,\ell}$ *exists for each fixed* (k,ℓ), *and*

(iv) $\lim_{m,n} \sum_k |\alpha_{m,n,k,\ell} - \alpha_{k,\ell}| = 0$ *for each fixed* ℓ,
$\lim_{m,n} \sum_\ell |\alpha_{m,n,k,\ell} - \alpha_{k,\ell}| = 0$ *for each fixed* k.

In this event, the double series $\sum_{k,\ell} \alpha_{k,\ell}$ *is absolutely convergent, and for any bounded convergent double sequence* $(x_{k,\ell})$, *we have the limit formula*

$$\lim_{m,n} [\boldsymbol{A}x]_{m,n} = \left(\alpha - \sum_{k,\ell} \alpha_{k,\ell}\right) \lim_{m,n} x_{m,n} + \sum_{k,\ell} \alpha_{k,\ell} x_{k,\ell}.$$

See [8], especially conditions $(c_1),(d_3)$, and (d_4) in §3, condition 20. $S.BC \to BC$ and limit formula (11.1) in §6, and a remark in §7 about the necessity of S. conditions. (See also Theorem 4.1.2 of [12].)

Theorem 3.3. *Let $(a_{k,\ell})$ be a double sequence. Then the Cauchy product double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell}$ is boundedly convergent for every boundedly convergent double series $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ if and only if the double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ is absolutely convergent. In this event, we have $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell} = \left(\sum_{k,\ell=0}^{\infty} a_{k,\ell}\right)\left(\sum_{k,\ell=0}^{\infty} b_{k,\ell}\right)$.*

Proof. For $m,n \in \mathbb{N}_0$, let

$$A_{m,n} := \sum_{k=0}^{m}\sum_{\ell=0}^{n} a_{k,\ell}, \quad B_{m,n} := \sum_{k=0}^{m}\sum_{\ell=0}^{n} b_{k,\ell}, \quad \text{and} \quad C_{m,n} := \sum_{k=0}^{m}\sum_{\ell=0}^{n} a_{k,\ell} * b_{k,\ell}.$$

Then

$$\begin{aligned} C_{m,n} &= \sum_{k=0}^{m}\sum_{\ell=0}^{n}\Big(\sum_{i=0}^{k}\sum_{j=0}^{\ell} a_{k-i,\ell-j} b_{i,j}\Big) = \sum_{k=0}^{m}\sum_{\ell=0}^{n} a_{m-k,n-\ell}\Big(\sum_{i=0}^{k}\sum_{j=0}^{\ell} b_{i,j}\Big) \\ &= \sum_{k=0}^{m}\sum_{\ell=0}^{n} a_{m-k,n-\ell} B_{k,\ell} \quad \text{for } (m,n) \in \mathbb{N}_0^2. \end{aligned}$$

Now the double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell}$ (respectively, $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$) is boundedly convergent if and only if the double sequence $(C_{m,n})$ (respectively, $(B_{m,n})$) is boundedly convergent. Consider the matrix $\boldsymbol{A} := (\alpha_{m,n,k,\ell})$, where

$$\alpha_{m,n,k,\ell,} := \begin{cases} a_{m-k,n-\ell} & \text{if } 0 \le k \le m \text{ and } 0 \le \ell \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $[\boldsymbol{A}(B_{k,\ell})]_{m,n} = C_{m,n}$ for all $(m,n) \in \mathbb{N}_0^2$. It is clear that the matrix $\boldsymbol{A}$ satisfies condition (i) of Lemma 3.2 if and only if

$$\sup_{m,n} \textstyle\sum_{k=0}^{m}\sum_{\ell=0}^{n} |a_{m-k,n-\ell}| = \sup_{m,n} \sum_{k=0}^{m}\sum_{\ell=0}^{n} |a_{k,\ell}| < \infty,$$

that is, if and only if the double series $\sum_{k,\ell} a_{k,\ell}$ is absolutely convergent, and in that case, we have

$$\begin{aligned} &\alpha := \lim_{m,n} \textstyle\sum_{k,\ell} \alpha_{m,n,k,\ell} = \lim_{m,n} \sum_{k=0}^{m}\sum_{\ell=0}^{n} a_{k,\ell} = \sum_{k,\ell} a_{k,\ell}, \\ &\alpha_{k,\ell} := \lim_{m,n} \alpha_{m,n,k,\ell} = \lim_{m,n} a_{m,n} = 0 \quad \text{for each fixed } (k,\ell) \in \mathbb{N}_0^2, \\ &\lim_{m,n} \textstyle\sum_{k} |\alpha_{m,n,k,\ell} - \alpha_{k,\ell}| = \lim_{n} \sum_{k=0}^{\infty} |a_{k,n-\ell}| = 0 \quad \text{for each fixed } \ell \in \mathbb{N}_0, \\ &\lim_{m,n} \textstyle\sum_{\ell} |\alpha_{m,n,k,\ell} - \alpha_{k,\ell}| = \lim_{m} \sum_{\ell=0}^{\infty} |a_{m-k,\ell}| = 0 \quad \text{for each fixed } k \in \mathbb{N}_0, \end{aligned}$$

that is, conditions (ii), (iii), and (iv) of Lemma 3.2 are automatically satisfied. Hence the desired result follows from Lemma 3.2 with $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell} = \left(\sum_{k,\ell=0}^{\infty} a_{k,\ell} - 0\right) \lim_{m,n} B_{m,n} + 0 = \left(\sum_{k,\ell=0}^{\infty} a_{k,\ell}\right)\left(\sum_{k,\ell=0}^{\infty} b_{k,\ell}\right)$. □

Remark 3.4. It is interesting to compare the above analogue of Mertens' theorem with the following result stated by Sheffer in Theorem 3 of [10]. Let $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ be a convergent double series. Then the Cauchy product double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell}$ is convergent for every absolutely convergent double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ if and only if the partial sums of $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ are bounded, and in that event, $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell} = \left(\sum_{k,\ell=0}^{\infty} a_{k,\ell}\right)\left(\sum_{k,\ell=0}^{\infty} b_{k,\ell}\right)$.

We proceed to prove an analogue of Abel's theorem for boundedly convergent double series. Its proof is based on the following result which uses a matrix transformation considered in Lemma 3.2.

Lemma 3.5. *Let* $(a_{m,n})$ *with* $(m,n) \in \mathbb{N}_0^2$ *be a bounded convergent double sequence and* $a := \lim_{m,n} a_{m,n}$*. If* $\tilde{a}_{m,n} := \left(\sum_{k=0}^{m}\sum_{\ell=0}^{n} a_{k,\ell}\right)/(m+1)(n+1)$*, then* $(\tilde{a}_{m,n})$ *is a bounded convergent double sequence and* $\lim_{m,n}\tilde{a}_{m,n} = a$*. Further, if* $(b_{m,n})$ *is a bounded convergent double sequence and* $b := \lim_{m,n} b_{m,n}$*, then*

$$\lim_{m,n} \frac{a_{m,n} * b_{m,n}}{(m+1)(n+1)} = ab.$$

Proof. Consider the matrix $\boldsymbol{L} := (\lambda_{m,n,k,\ell})$, where

$$\lambda_{m,n,k,\ell,} := \begin{cases} 1/(m+1)(n+1) & \text{if } 0 \le k \le m \text{ and } 0 \le \ell \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $[\boldsymbol{L}(a_{k,\ell})]_{m,n} = \tilde{a}_{m,n}$ for all $(m,n) \in \mathbb{N}_0^2$. Also,

$$\sup_{m,n} \textstyle\sum_{k,\ell} |\lambda_{m,n,k,\ell}| = \lim_{m,n} \sum_{k,\ell} \lambda_{m,n,k,\ell} = \lim_{m,n} \sum_{k=0}^{m}\sum_{\ell=0}^{n} \lambda_{m,n,k,\ell} = 1,$$

$$\lambda_{k,\ell} := \lim_{m,n} \lambda_{m,n,k,\ell} = \lim_{m,n} \frac{1}{(m+1)(n+1)} = 0 \quad \text{for each fixed } (k,\ell) \in \mathbb{N}_0^2,$$

$$\lim_{m,n} \textstyle\sum_{k} |\lambda_{m,n,k,\ell} - \lambda_{k,\ell}| = \lim_{n} \dfrac{1}{(n+1)} = 0 \quad \text{for each fixed } \ell \in \mathbb{N}_0 \text{ and}$$

$$\lim_{m,n} \textstyle\sum_{\ell} |\lambda_{m,n,k,\ell} - \lambda_{k,\ell}| = \lim_{m} \dfrac{1}{(m+1)} = 0 \quad \text{for each fixed } k \in \mathbb{N}_0,$$

that is, conditions (i)–(iv) of Lemma 3.2 are satisfied, and so by the limit formula, we obtain $\lim_{m,n}\tilde{a}_{m,n} = (1-0)a + 0 = a$.

Next, let $(b_{m,n})$ be a bounded convergent double sequence and $b := \lim_{m,n} b_{m,n}$. Consider the matrix $\boldsymbol{A} := (\alpha_{m,n,k,\ell})$, where

$$\alpha_{m,n,k,\ell,} := \begin{cases} a_{m-k,n-\ell}/(m+1)(n+1) & \text{if } 0 \le k \le m \text{ and } 0 \le \ell \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $[\boldsymbol{A}(b_{k,\ell})]_{m,n} = (a_{m,n} * b_{m,n})/(m+1)(n+1)$ for all $(m,n) \in \mathbb{N}_0^2$. Also, since $\beta := \sup\left\{|a_{m,n}| : (m,n) \in \mathbb{N}_0^2\right\} < \infty$, we obtain

$$\sup_{m,n} \textstyle\sum_{k,\ell} |\alpha_{m,n,k,\ell}| = \sup_{m,n} \sum_{k=0}^{m}\sum_{\ell=0}^{n} \alpha_{m,n,k,\ell} \le \beta,$$

$$\alpha := \lim_{m,n} \textstyle\sum_{k,\ell} \alpha_{m,n,k,\ell} = \lim_{m,n} \sum_{k=0}^{m}\sum_{\ell=0}^{n} \alpha_{m,n,k,\ell} = \lim_{m,n} \tilde{a}_{m,n} = a,$$

$$\alpha_{k,\ell} := \lim_{m,n} \alpha_{m,n,k,\ell} = \lim_{m,n} \frac{a_{m,n}}{(m+1)(n+1)} = 0 \quad \text{for each fixed } (k,\ell) \in \mathbb{N}_0^2,$$

$$\lim_{m,n} \textstyle\sum_{k} |\alpha_{m,n,k,\ell} - \alpha_{k,\ell}| \le \lim_{n} \dfrac{\beta}{(n+1)} = 0 \quad \text{for each fixed } \ell \in \mathbb{N}_0 \text{ and}$$

$$\lim_{m,n} \textstyle\sum_{\ell} |\alpha_{m,n,k,\ell} - \alpha_{k,\ell}| \le \lim_{m} \dfrac{\beta}{(m+1)} = 0 \quad \text{for each fixed } k \in \mathbb{N}_0,$$

that is, conditions (i)–(iv) of Lemma 3.2 are satisfied, and so by the limit formula, we obtain $\lim_{m,n}(a_{m,n} * b_{m,n})/(m+1)(n+1) = (a-0)b + 0 = ab$. □

A double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ is said to be *Cesàro summable* if

$$\lim_{m,n} \frac{1}{(m+1)(n+1)} \sum_{k=0}^{m}\sum_{\ell=0}^{n} A_{k,\ell}$$

exists, where $A_{k,\ell}$ is the (k,ℓ)th partial sum of the double series. In that event, the above limit is called the *Cesàro sum* of the double series. It follows from Lemma 3.5 that a boundedly convergent double series is Cesàro summable, and its Cesàro sum is equal to its sum.

Theorem 3.6. *Let $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ and $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ be boundedly convergent double series. Then the Cauchy product double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell}$ is Cesàro summable and its Cesàro sum is equal to AB, where A and B are the sums of the given double series. In particular, if the double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell}$ is boundedly convergent, then its sum is equal to AB.*

Proof. We use the notation introduced in the proof of Theorem 3.3. We have

$$C_{m,n} = \sum_{k=0}^{m} \sum_{\ell=0}^{n} a_{k,\ell} * b_{k,\ell} = a_{m,n} * B_{m,n} = A_{m,n} * b_{m,n} \quad \text{for } (m,n) \in \mathbb{N}_0^2.$$

Replacing $a_{k,\ell}$ and $b_{k,\ell}$ by $A_{m,n}$ and $b_{m,n}$, we obtain

$$\sum_{m=0}^{p} \sum_{n=0}^{q} C_{m,n} = \sum_{m=0}^{p} \sum_{n=0}^{q} A_{m,n} * b_{m,n} = A_{p,q} * B_{p,q} \quad \text{for } (p,q) \in \mathbb{N}_0^2.$$

Hence by Lemma 3.5, we have

$$\lim_{p,q} \frac{1}{(p+1)(q+1)} \sum_{m=0}^{p} \sum_{n=0}^{q} C_{m,n} = \lim_{p,q} \frac{A_{p,q} * B_{p,q}}{(p+1)(q+1)} = AB.$$

Thus $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell}$ is Cesàro summable and its Cesàro sum is equal to AB. The last sentence in the statement of the theorem follows easily. □

Remark 3.7. In Theorem 1 of [5], Cesari gives the following analogue of Abel's theorem. Let $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ and $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ be convergent double series such that

$$\lim_{k+\ell \to \infty} a_{k,\ell} = 0 = \lim_{k+\ell \to \infty} b_{k,\ell}.$$

Then the double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell}$ is restrictedly Cesàro summable to AB in the following sense: For any positive real numbers r, s with $r < s$,

$$\lim_{\substack{m,n\to\infty \\ nr<m<ns}} \sigma_{m,n} = AB, \quad \text{where } \sigma_{m,n} := \frac{1}{(m+1)(n+1)} \sum_{k=0}^{m} \sum_{\ell=0}^{n} C_{k,\ell}.$$

We shall now consider a notion of convergence which is stronger than bounded convergence. Let us recall that a double sequence $(a_{k,\ell})$ is said to be *regularly convergent* if it is convergent, and further, for each fixed $k \in \mathbb{N}$, the sequence given by $\ell \mapsto a_{k,\ell}$ is convergent and for each fixed $\ell \in \mathbb{N}$, the sequence given by $k \mapsto a_{k,\ell}$ is convergent. A double series is *regularly convergent* if the double sequence of its (rectangular) partial sums is regularly convergent, that is, the double series is convergent, and further, each corresponding row-series as well as each corresponding column-series is convergent. It is easy to see that a regularly convergent double sequence is bounded, and a regularly convergent double series is boundedly convergent. We have the following analogue of Theorem 3.3 for regularly convergent double series.

Theorem 3.8. *Let $(a_{k,\ell})$ be a double sequence. Then the Cauchy product double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell}$ is regularly convergent for every regularly convergent double series $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ if and only if the double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ is absolutely convergent. In this event, we have $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell} = \left(\sum_{k,\ell=0}^{\infty} a_{k,\ell}\right)\left(\sum_{k,\ell=0}^{\infty} b_{k,\ell}\right)$.*

Proof. An argument along the lines given in the proof of Theorem 3.3 yields the desired result if we note the following. A matrix $\boldsymbol{A} := (\alpha_{m,n,k,\ell})$ maps each regularly convergent double sequence to a regularly convergent double sequence if and only if conditions (i)–(iii) of Lemma 3.2 hold, and also the following condition holds:

(iv)' the limit $\beta_k := \lim_{m,n} \sum_\ell \alpha_{m,n,k,\ell}$ exists for each fixed k,
the limit $\gamma_\ell := \lim_{m,n} \sum_k \alpha_{m,n,k,\ell}$ exists for each fixed ℓ,

where the convergence indicated in each of the conditions (ii), (iii), and (iv)' is regular. In this event, the double series $\sum_{k,\ell} \alpha_{k,\ell}$ is absolutely convergent, the series $\sum_k \beta_k$ and $\sum_\ell \gamma_\ell$ are absolutely convergent, and for any regularly convergent double sequence $(x_{k,\ell})$, we have the limit formula

$$\begin{aligned}\lim_{m,n} [\boldsymbol{A}x]_{m,n} = {} & \left(\alpha + \textstyle\sum_{k,\ell}\alpha_{k,\ell} - \sum_k\beta_k - \sum_\ell\gamma_\ell\right) \lim_{m,n} x_{m,n} + \textstyle\sum_{k,\ell}\alpha_{k,\ell}x_{k,\ell} \\ & + \textstyle\sum_k \left[\left(\beta_k - \sum_\ell \alpha_{k,\ell}\right)\left(\lim_\ell x_{k,\ell}\right)\right] + \sum_\ell \left[\left(\gamma_\ell - \sum_k \alpha_{k,\ell}\right)\left(\lim_k x_{k,\ell}\right)\right].\end{aligned}$$

See [8], especially conditions $(c_1), (d_1), (d_2), (d_3), (f_1), (f_2)$, and (f_3) in §3, condition 132. *S.RC* $\to$ *RC* and limit formula (9.1) in §6. (See also Theorem 4.1.1 of [12].) In our case, with the matrix $\boldsymbol{A}$ as defined in the proof of Theorem 3.3, we have $\alpha_{k,\ell} = \beta_k = \gamma_\ell = 0$ for all $(k,\ell) \in \mathbb{N}_0^2$ and $\alpha = \sum_{k,\ell} a_{k,\ell}$. ☐

Remarks 3.9.

(i) The 'if' part of the above theorem was proved in Theorem 3 of [5]. The 'only if' part of the above theorem as well as of Theorem 3.3 can be strengthened as follows. If the Cauchy product double series $\sum_{k,\ell=0}^\infty a_{k,\ell} * b_{k,\ell}$ is boundedly convergent for every regularly convergent double series $\sum_{k,\ell=0}^\infty b_{k,\ell}$, then the double series $\sum_{k,\ell=0}^\infty a_{k,\ell}$ is absolutely convergent. To see this, one only has to note that condition (i) of Lemma 3.2 is a necessary condition for a matrix $\boldsymbol{A}$ to map each regularly convergent double sequence to a bounded convergent double sequence. (See condition (c_1) in §3, condition 134. *S.RC* $\to$ *BC* of §6, and a remark in §7 about the necessity of S. conditions.)

(ii) In view of (i) above, it is worthwhile to observe that if $\sum_{k,\ell=0}^\infty a_{k,\ell}$ is an absolutely convergent double series not all of whose terms are equal to zero, then there is a boundedly convergent double series $\sum_{k,\ell=0}^\infty b_{k,\ell}$ such that the double series $\sum_{k,\ell=0}^\infty a_{k,\ell} * b_{k,\ell}$ is not regularly convergent. To see this, let $\sum_{k,\ell=0}^\infty a_{k,\ell}$ be absolutely convergent and $a_{k_0,\ell_0} \neq 0$ for some $(k_0,\ell_0) \in \mathbb{N}_0^2$. Define $a_k := a_{k,\ell_0}$ for $k \in \mathbb{N}_0$. We note that $\sum_{k=0}^\infty a_k$ is an absolutely convergent series and $a_{k_0} \neq 0$, so that there is a series $\sum_{k=0}^\infty b_k$ having bounded partial sums such that the series $\sum_{k=0}^\infty a_k * b_k$ is divergent. Now define

$$b_{k,\ell} := \begin{cases} b_k & \text{if } k \in \mathbb{N}_0 \text{ and } \ell = 0, \\ -b_k & \text{if } k \in \mathbb{N}_0 \text{ and } \ell = \ell_0 + 1, \\ 0 & \text{if } k \in \mathbb{N}_0 \text{ and } \ell \in \mathbb{N}_0 \setminus \{0, \ell_0+1\}. \end{cases}$$

Then

$$a_{k,\ell_0} * b_{k,\ell_0} = \sum_{i=0}^{k} \sum_{j=0}^{\ell_0} a_{i,j} b_{k-i,\ell_0-j} = \sum_{i=0}^{k} a_{i,\ell_0} b_{k-i,0} = a_k * b_k.$$

Hence $\sum_{k=0}^\infty a_{k,\ell_0} * b_{k,\ell_0}$ diverges, and so $\sum_{k,\ell=0}^\infty a_{k,\ell} * b_{k,\ell}$ is not regularly convergent.

Examples 3.10.

(i) Let $x, y \in (-1,0)$ with $x + y = -1$. Define

$$a_{k,\ell} := x^k y^\ell \quad \text{and} \quad b_{k,\ell} := \binom{k+\ell}{k} \quad \text{for } (k,\ell) \in \mathbb{N}_0^2.$$

Then $\sum_{k,\ell=0}^\infty a_{k,\ell}$ converges absolutely and $\sum_{k,\ell=0}^\infty b_{k,\ell}$ converges regularly (but not absolutely). The Cauchy product series is $\sum_{k,\ell=0}^\infty c_{k,\ell}$, where

$$c_{k,\ell} = \sum_{i=0}^{k} \sum_{j=0}^{\ell} \binom{i+j}{i} x^k y^\ell = \left[\binom{k+\ell+2}{k+1} - 1\right] x^k y^\ell \quad \text{for } (k,\ell) \in \mathbb{N}_0^2,$$

the last equality being a consequence of Pascal's 3rd identity.

(ii) Let $x, y \in (-1,1]$, $a, \tilde{a} > 0$, and $b, \tilde{b} \in (-1,0]$. Define

$$a_{k,\ell} := \binom{a}{k}\binom{\tilde{a}}{\ell} x^k y^\ell \quad \text{and} \quad b_{k,\ell} := \binom{b}{k}\binom{\tilde{b}}{\ell} x^k y^\ell \quad \text{for } (k,\ell) \in \mathbb{N}_0^2.$$

Then $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ converges absolutely and $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ converges regularly (but not absolutely when $x = 1$ or $y = 1$). The Cauchy product series is $\sum_{k,\ell=0}^{\infty} c_{k,\ell}$, where

$$c_{k,\ell} = \sum_{i=0}^{k}\sum_{j=0}^{\ell} \binom{a}{i}\binom{b}{k-i}\binom{\tilde{a}}{j}\binom{\tilde{b}}{\ell-j} x^k y^\ell = \binom{a+b}{k}\binom{\tilde{a}+\tilde{b}}{\ell} x^k y^\ell,$$

for $(k,\ell) \in \mathbb{N}_0^2$, the last equality being a consequence of Vandermonde's Convolution Formula.

(iii) Let $p \in (0,2]$. Define $a_{k,\ell} := 1/2^{k+\ell}$ and $b_{k,\ell} := (-1)^{k+\ell}/(k+\ell+1)^p$ for $(k,\ell) \in \mathbb{N}_0^2$. Then $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ converges absolutely and $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ converges regularly (but not absolutely). The Cauchy product series is $\sum_{k,\ell=0}^{\infty} c_{k,\ell}$, where

$$c_{k,\ell} = \sum_{i=0}^{k}\sum_{j=0}^{\ell} \frac{(-1)^{i+j}}{2^{k-i+\ell-j}(i+j+1)^p} = \frac{1}{2^{k+\ell}}\sum_{i=0}^{k}\sum_{j=0}^{\ell} \frac{(-2)^{i+j}}{(i+j+1)^p}$$

for $(k,\ell) \in \mathbb{N}_0^2$.

In each of the above cases, the Cauchy product double series $\sum_{k,\ell=0}^{\infty} c_{k,\ell}$ is regularly convergent by Theorem 3.8.

As we have seen at the beginning of this section, absolute convergence of $\sum_{k,\ell=0}^{\infty} a_{k,\ell}$ and convergence of $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ do not guarantee convergence of the Cauchy product double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell}$. In fact, the next result gives precise conditions on the double sequence $(a_{k,\ell})$ for such convergence.

Theorem 3.11. *Let $(a_{k,\ell})$ be a double sequence. Then the Cauchy product double series $\sum_{k,\ell=0}^{\infty} a_{k,\ell} * b_{k,\ell}$ is convergent for every convergent double series $\sum_{k,\ell=0}^{\infty} b_{k,\ell}$ if and only if the set $\{(k,\ell) \in \mathbb{N}_0^2 : a_{k,\ell} \neq 0\}$ is finite.*

Proof. As in the case of Theorem 3.8, an argument along the lines given in the proof of Theorem 3.3 yields the desired result if we note the following. One of the six necessary and sufficient conditions for a matrix $\boldsymbol{A} := (\alpha_{m,n,k,\ell})$ to map each convergent double sequence to a convergent double sequence is the following: For each fixed $k \in \mathbb{N}_0$, there is $p_0 \in \mathbb{N}_0$ such that $\alpha_{m,n,k,\ell} = 0$ whenever $m,n,\ell \geq p_0$, and for each fixed $\ell \in \mathbb{N}_0$, there is $q_0 \in \mathbb{N}_0$ such that $\alpha_{m,n,k,\ell} = 0$ whenever $m,n,k \geq q_0$. In our case, with the matrix $\boldsymbol{A}$ as defined in the proof of Theorem 3.3, this condition entails that the set $\{(k,\ell) \in \mathbb{N}_0^2 : a_{k,\ell} \neq 0\}$ is finite, and then the remaining five conditions are automatically satisfied. See [8], especially conditions $(\mathrm{a}_1), (\mathrm{a}_2), (\mathrm{b}_1), (\mathrm{b}_2), (\mathrm{d}_1)$, and (d_3) in §3, condition 14. *S.C* $\rightarrow$ *C* in §6. (See also Theorem 4.1.3 of [12].) □

Remark 3.12. We conclude this section by making some comments about the companion problem of determining conditions on a double sequence $(a_{k,\ell})$ (with $(k,\ell) \in \mathbb{N}^2$) in order that, for every convergent/boundedly convergent/regularly convergent double series $\sum_{k,\ell=1}^{\infty} b_{k,\ell}$, the double series $\sum_{k,\ell=1}^{\infty} a_{k,\ell} b_{k,\ell}$ would be convergent/boundedly convergent/regularly convergent, as per the assumption on $\sum_{k,\ell=1}^{\infty} b_{k,\ell}$. For this purpose, let us say that a sequence (a_k) is of *bounded variation* if the series $\sum_{k=1}^{\infty} |a_k - a_{k+1}|$ is convergent, and that a double sequence $(a_{k,\ell})$ is of *bounded bivariation* if the double series $\sum_{k,\ell=1}^{\infty} |a_{k,\ell} - a_{k+1,\ell} - a_{k,\ell+1} + a_{k+1,\ell+1}|$ is convergent. In the case of regular convergence, necessary and sufficient conditions on $(a_{k,\ell})$ are as follows: $(a_{k,\ell})$ is of bounded bivariation and $(a_{k,1})$ as well as $(a_{1,\ell})$ is of bounded variation. In the case of bounded convergence, necessary and sufficient conditions on $(a_{k,\ell})$ are as follows: $(a_{k,\ell})$ is of bounded bivariation, $\lim_k (a_{k,\ell} - a_{k,\ell+1}) = 0$ for each fixed $\ell \in \mathbb{N}$, and $\lim_\ell (a_{k,\ell} - a_{k+1,\ell}) = 0$ for each fixed $k \in \mathbb{N}$. In the case of convergence, necessary and sufficient conditions on $(a_{k,\ell})$ are as follows: $(a_{k,\ell})$ is of bounded bivariation, for each fixed $\ell \in \mathbb{N}$, there is $k_\ell \in \mathbb{N}$ such that $a_{k,\ell} = a_{k,\ell+1}$ for all $k \geq k_\ell$, and for each fixed $k \in \mathbb{N}$, there is $\ell_k \in \mathbb{N}$ such that $a_{k,\ell} = a_{k+1,\ell}$ for all $\ell \geq \ell_k$. See [7] for these results.

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Koonduvustunnused kahekordsete ridade jaoks

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On tõestatud ühekordsete ridade hästi tuntud koonduvustunnuste analoogid Pringsheimi mõttes koonduvate kahekordsete ridade jaoks. Muuhulgas on tuletatud absoluutse koonduvuse eri tunnused ja Cauchy korrutise koonduvuse kriteerium. Samuti on täpsustatud mõnede teiste autorite tulemusi antud valdkonnas.