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MATHEMATICS

Comparison of speeds of convergence in Riesz-type families of summability methods

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Abstract. We deal with Riesz-type families (see *Proc. Estonian Acad. Sci. Phys. Math.*, 2002, **51**, 18–34 and *Acta Sci. Math. (Szeged)*, 2004, **70**, 639–657) of summability methods A_α for converging functions and sequences. The methods A_α in a Riesz-type family depend on a continuous parameter α, and are connected through certain generalized integral Nörlund methods. By extending and applying the results of Stadtmüller and Tali (*Anal. Math.*, 2003, **29**, 227–242), we compare speeds of convergence in a Riesz-type family. As expected, the speed of convergence cannot increase if we switch from one summability method to a stronger one. Comparative estimations for speeds are found. In particular, the families of integral Riesz methods, generalized integral Nörlund methods, and Abel- and Borel-type summability methods are considered. Numerical examples are given.

Key words: Riesz-type family of summability methods, Riesz methods, speed of convergence, generalized integral Nörlund methods, Abel-type methods, Borel-type methods.

1. INTRODUCTION AND PRELIMINARIES

Let us consider the functions $x = x(u)$ defined for $u \geq 0$, bounded and measurable in the sense of Lebesgue on every finite interval $[0, u_0]$. Let us denote the set of all these functions by X. Suppose that A is a transformation of functions $x = x(u)$ (or, in particular, of sequences $x = (x_n)$) into functions $Ax = y = y(u) \in X$. If the limit $\lim_{u\to\infty} y(u) = s$ exists, then we say that $x = x(u)$ is convergent to s with respect to the summability method A, and write $x(u) \to s(A)$. If the function $y = y(u)$ is bounded, then we say that x is bounded with respect to the method A, and write $x(u) = O(A)$. We denote by ωA the set of all these functions x, where the transformation A is applied, and by cA and mA the set of all functions x which are, respectively, convergent and bounded with respect to the method A. The summability method A is said to be regular if

$$\lim_{u\to\infty} x(u) = s \Longrightarrow \lim_{u\to\infty} y(u) = s$$

whenever $x \in X$.

The most common summability method for functions x is an integral method A, defined with the help of the transformation

$$y(u) = \int_0^\infty a(u,v)x(v)\mathrm{d}v,$$

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where $a(u,v)$ is a certain function of two variables $u \geq 0$ and $v \geq 0$. We also say that the integral method A is defined by the function $a(u,v)$. An example of the integral summability method is the generalized integral Nörlund method $(N, P(u), Q(u))$, defined with the help of the transformation

$$y(u) = \frac{1}{R(u)} \int_0^u P(u-v)\, Q(v)\, x(v)\, \mathrm{d}v \quad (u > 0),$$

where $P = P(u)$ and $Q = Q(u)$ are nonnegative functions from X such that $R(u) = \int_0^u P(u-v)\, Q(v)\, \mathrm{d}v \neq 0$ for $u > 0$.

For sequences $x = (x_n)$ we do not consider in our paper matrix methods (which are the most common summability methods), but focus ourselves on certain semi-continuous summability methods A, defined by transformations

$$y(u) = \sum_{n=0}^{\infty} a_n(u)\, x_n \quad (u \geq 0),$$

where $a_n(u)$ $(n = 0, 1, 2, \ldots)$ are some functions from X.

As examples on semi-continuous methods the Abel-type methods $A_\alpha = (A, \alpha)$ with $\alpha > -1$ (see [1]) and the Borel-type methods $A_\alpha = (B, \alpha)$ with $\alpha > \alpha_0$ (where α_0 is some fixed number) can be considered (see [2,3]). The Abel-type methods (A, α) are defined by the transformation of $x = (x_n)$ into $y_\alpha = y_\alpha(u)$ with

$$y_\alpha(u) = \frac{1}{(u+1)^{\alpha+1}} \sum_{n=0}^{\infty} A_n^\alpha \left(\frac{u}{u+1} \right)^n x_n, \tag{1.1}$$

where A_n^α are the Cesàro numbers. In particular, if $\alpha = 0$, we have the Abel method $A = (A, 0)$.

The Borel-type methods (B, α) are defined by the transformation

$$y_\alpha(u) = \frac{1}{e^u} \sum_{n=N}^{\infty} \frac{u^{n+\alpha-1}}{\Gamma(n+\alpha)} x_n, \tag{1.2}$$

where $\Gamma(\cdot)$ is the Gamma-function and N is the smallest integer satisfying the inequality $N > \max\{-\alpha_0, -1/2\}$. In particular, if $\alpha = 1$, we have the Borel method $B = (B, 1)$.

One of the basic notions in this paper is the notion of the "speed of convergence". We follow here the definitions based on the definitions for sequences (see [4,5]) and extended for functions in [6,7].

Let $\lambda = \lambda(u)$ be a positive function from X such that $\lambda(u) \to \infty$ as $u \to \infty$. We say that a function $x = x(u)$ is convergent to s with speed λ if the finite limit

$$\lim_{u \to \infty} \lambda(u)\, [x(u) - s]$$

exists. Note that the limit can be zero. If we have

$$\lambda(u)\, [x(u) - s] = O(1)$$

as $u \to \infty$, then x is said to be bounded with speed λ. We use the notations c^λ and m^λ for the sets of all these functions $x = x(u)$ which are convergent to some s with speed λ and bounded with speed λ, respectively. In the obvious manner the notion of speed can be transferred to summability methods. We say that x is convergent or bounded with speed λ with respect to the summability method A if $Ax \in c^\lambda$ or $Ax \in m^\lambda$, respectively.

2. RIESZ-TYPE FAMILIES OF SUMMABILITY METHODS

Here we discuss and extend the notion of a Riesz-type family of summability methods given in papers [6,8].

A. Let us start with some examples.

Example 1. Consider the generalized Nörlund methods $A_\alpha = (N, u^{\alpha-1}, q(u))$, where $\alpha > 0$ and $q = q(u)$ is a positive function from X. These methods are defined with the help of the transformation of x into $A_\alpha x = y_\alpha = y_\alpha(u)$ with

$$y_\alpha(u) = \frac{1}{r_\alpha(u)} \int_0^u (u-v)^{\alpha-1} q(v)x(v)\,\mathrm{d}v \qquad (u > 0),$$

where $r_\alpha = r_\alpha(u) = \int_0^u (u-v)^{\alpha-1} q(v)\,\mathrm{d}v$.

It can be easily shown that any two methods A_γ and A_β with $\beta > \gamma > 0$ are connected through the relation

$$y_\beta(u) = \frac{M_{\gamma,\beta}}{r_\beta(u)} \int_0^u (u-v)^{\beta-\gamma-1} r_\gamma(v) y_\gamma(v)\,\mathrm{d}v \qquad (u > 0), \tag{2.1}$$

and

$$r_\beta(u) = M_{\gamma,\beta} \int_0^u (u-v)^{\beta-\gamma-1} r_\gamma(v)\,\mathrm{d}v \qquad (u > 0), \tag{2.2}$$

where

$$M_{\gamma,\beta} = \frac{\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\beta-\gamma)}. \tag{2.3}$$

Let us prove first relation (2.2), starting from its right side and using the substitutions $v' = v - t$ and $v'' = \frac{v'}{u-t}$:

$$\begin{aligned} M_{\gamma,\beta} \int_0^u (u-v)^{\beta-\gamma-1} r_\gamma(v)\mathrm{d}v &= M_{\gamma,\beta} \int_0^u (u-v)^{\beta-\gamma-1} \left(\int_0^v (v-t)^{\gamma-1} q(t)\mathrm{d}t \right) \mathrm{d}v \\ &= M_{\gamma,\beta} \int_0^u q(t) \left(\int_0^{u-t} (u-t-v')^{\beta-\gamma-1} (v')^{\gamma-1}\mathrm{d}v' \right) \mathrm{d}t \\ &= M_{\gamma,\beta} \int_0^u (u-t)^{\beta-1} q(t) B(\beta-\gamma,\gamma)\mathrm{d}t \\ &= r_\beta(u), \end{aligned}$$

where $B(.,.)$ denotes the Beta-function. The verification of (2.1) follows along the same lines; we just have to replace $r_\gamma(u)$ by $r_\gamma(u) y_\gamma(u)$ and $r_\beta(u)$ by $r_\beta(u) y_\beta(u)$.

In particular, if $q(u) = 1$ $(u \ge 0)$, we have that $r_\alpha(u) = u^\alpha/\alpha$ and methods $(N, u^{\alpha-1}, q(u))$ turn into Riesz methods (R, α) (see [9]), and (2.1) takes the form

$$y_\beta(u) = \frac{M_{\gamma,\beta}}{u^\beta} \int_0^u (u-v)^{\beta-\gamma-1} v^\gamma y_\gamma(v)\,\mathrm{d}v \qquad (u > 0), \tag{2.4}$$

with

$$M_{\gamma,\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\gamma+1)\Gamma(\beta-\gamma)}. \tag{2.5}$$

Note that the same connection formula (2.4) appears for Abel methods (A, β) and (A, γ) defined by (1.1). We have only to exchange places of $y_\gamma(u)$ and $y_\beta(u)$ in it. More precisely, we have the relation (see [1])

$$y_\gamma(u) = \frac{M_{\gamma,\beta}}{u^\beta} \int_0^u (u-v)^{\beta-\gamma-1} v^\gamma y_\beta(v)\,\mathrm{d}v \qquad (u > 0,\ \beta > \gamma > -1), \tag{2.6}$$

where $M_{\gamma,\beta}$ is defined by (2.5).

Example 2. Connection formula (2.6), together with (2.5), appears also if we consider the methods $A_\alpha = (D, \alpha)$ $(\alpha > -1)$, defined with the help of the integral transformation (see [10])

$$y_\alpha(u) = (\alpha + 1)\, u \int_0^\infty \frac{v^\alpha}{(v+u)^{\alpha+2}}\, x(v)\, \mathrm{d}v. \tag{2.7}$$

As there exist many other families with the connection formulas analogous to (2.1) and, in particular, to (2.4), we next consider a more general notion, the notion of a Riesz-type family defined in [6,8], and extend it.

B. Let $\{A_\alpha\}$ be a family of summability methods A_α where[1] $\alpha \underset{(-)}{>} \alpha_1$ and which are defined by transformations of functions $x = x(u) \in \omega A_\alpha \subset X$ into functions $A_\alpha x = y_\alpha = y_\alpha(u) \in X$. Suppose that for any $\beta > \gamma \underset{(-)}{>} \alpha_1$ we have the relation

$$\omega A_\gamma \subset \omega A_\beta \tag{2.8}$$

or

$$\omega A_\beta \subset \omega A_\gamma. \tag{2.9}$$

Definition 1. *A family $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_1)$ is said to be a Riesz-type family if for every $\beta > \gamma \underset{(-)}{>} \alpha_1$*

$\mathscr{A}$) *relation* (2.8) *holds and the methods A_γ and A_β are connected through* (2.1) *or*

$\mathscr{B}$) *relation* (2.9) *holds and the methods A_γ and A_β are connected through the relation*

$$y_\gamma(u) = \frac{M_{\gamma,\beta}}{r_\beta(u)} \int_0^u (u-v)^{\beta-\gamma-1}\, r_\gamma(v)\, y_\beta(v)\, \mathrm{d}v \quad (u > 0), \tag{2.10}$$

where $r_\gamma = r_\gamma(u)$ and $r_\beta = r_\beta(u)$ are some positive functions from X related through (2.2) *and $M_{\gamma,\beta}$ is a constant depending on γ and β.*

In other words, a Riesz-type family is a family where every two methods are connected through the connection formula

$$A_\beta = C_{\gamma,\beta} \circ A_\gamma \quad (\beta > \gamma \underset{(-)}{>} \alpha_1)$$

in case $\mathscr{A}$), and

$$A_\gamma = C_{\gamma,\beta} \circ A_\beta \quad (\beta > \gamma \underset{(-)}{>} \alpha_1)$$

in case $\mathscr{B}$), where $C_{\gamma,\beta}$ is the integral method defined with the help of the function

$$c_{\gamma,\beta}(u,v) = \begin{cases} M_{\gamma,\beta}\,(u-v)^{\beta-\gamma-1}\, r_\gamma(v) \,/\, r_\beta(u) & \text{if } 0 \le v < u, \\ 0 & \text{if } v \ge u. \end{cases}$$

Note that Definition 1 in case $\mathscr{A}$) was given in [6,8]. We see that the methods $(N, u^{\alpha-1}, q(u))$ $(\alpha > 0)$ and (A, α) and (D, α) $(\alpha > -1)$ discussed above form Riesz-type families. The first of them is a Riesz-type family of case $\mathscr{A}$), and the other two are Riesz-type families of case $\mathscr{B}$).

Let us consider some more examples of Riesz-type families.

Example 3. Let $\{A_\alpha\}$ be the family of generalized Nörlund methods $(N, p_\alpha(u), q(u))$ $(\alpha > \alpha_0)$, defined with the help of positive functions $p = p(u) \in X$ and $q = q(u) \in X$ and number α_0 such that

$$r_\alpha(u) = \int_0^u p_\alpha(u-v)\, q(v)\, \mathrm{d}v > 0 \qquad (u > 0,\ \alpha > \alpha_0),$$

[1] The notation $\alpha \underset{(-)}{>} \alpha_1$ means that we consider parameter values $\alpha > \alpha_1$ or $\alpha \ge \alpha_1$ with some fixed number α_1.

where $p_\alpha(u) = \int_0^u (u-v)^{\alpha-1}\, p(v)\,\mathrm{d}v$. It is known that relation (2.1), together with (2.2) and (2.3), holds for any $\beta > \gamma > \alpha_0$ (see [11]), and thus this family is a Riesz-type family of case $\mathscr{A}$).

Example 4. Consider the family $\{A_\alpha\}$ of Borel-type methods $A_\alpha = (B, \alpha, q_n)$ defined in [8]. Let (q_n) be a nonnegative sequence with $q_0 > 0$ such that the power series $\sum q_n u^n$ has the radius of convergence $R = \infty$ and $q_n > 0$ at least for one $n \in \mathbb{N}$. Denote

$$r_\alpha(u) = \sum_{n=1}^{\infty} \frac{n!\, q_n\, u^{n+\alpha-1}}{\Gamma(n+\alpha)}$$

and define the methods (B, α, q_n) $(\alpha > -1/2)$ for converging sequences $x = (x_n)$ with the help of the transformation

$$y_\alpha(u) = \frac{1}{r_\alpha(u)} \sum_{n=1}^{\infty} \frac{n!\, q_n\, u^{n+\alpha-1}}{\Gamma(n+\alpha)}\, x_n \qquad (u > 0).$$

The methods (B, α, q_n) satisfy relations (2.1) and (2.2) with $M_{\gamma,\beta} = 1/\Gamma(\beta-\gamma)$ (see [8]). Thus $\{A_\alpha\}$ is a Riesz-type family of case $\mathscr{A}$). In particular, if $q_n = \frac{1}{n!}$, we get the Borel-type methods $(B, \alpha) = (B, \alpha, 1/n!)$ (see (1.2)) because in this case $r_\alpha(u) \sim e^u$ as $u \to \infty$.

C. We discuss here the property of monotony of a Riesz-type family.

Lemma 1. *Let $\{A_\alpha\}$ $(\alpha \underset{(-)}{>} \alpha_1)$ be a Riesz-type family. The methods $C_{\gamma,\beta}$ are regular for all $\beta > \gamma > \alpha_1$. These methods are regular also for all $\beta > \gamma = \alpha_1$, provided that the condition*

$$\lim_{u\to\infty} \int_0^u r_{\alpha_1}(v)\,\mathrm{d}v = \infty \tag{2.11}$$

holds.

Proof. For the case $\beta > \gamma > \alpha_1$, this result was proved in [1] as Proposition 1. It remains to prove our statement if $\beta > \gamma = \alpha_1$. Because of the relation

$$C_{\alpha_1,\beta} = C_{\delta,\beta} \circ C_{\alpha_1,\delta} \qquad (\beta > \delta > \alpha_1)$$

(which follows from (2.1)) in case $\mathscr{A}$) and the relation

$$C_{\alpha_1,\beta} = C_{\alpha_1,\delta} \circ C_{\delta,\beta} \qquad (\beta > \delta > \alpha_1)$$

(which follows from (2.10)) in case $\mathscr{B}$), it suffices to verify our statement only for $\alpha_1 < \beta < \alpha_1 + 1$. We use Theorem 6 from [9], which gives the sufficient conditions for the regularity of integral methods. Since the methods $C_{\alpha_1,\beta}$ are defined by positive functions and $\int_0^u c_{\alpha_1,\beta}(u,v)\,\mathrm{d}v = 1$ by (2.2), it remains to show that

$$\lim_{u\to\infty} \int_0^{v_0} c_{\alpha_1,\beta}(u,v)\,\mathrm{d}v = 0 \tag{2.12}$$

for every finite $v_0 > 0$. Supposing that $v \le v_0 < u$, we get with the help of (2.11) that

$$\begin{aligned} c_{\alpha_1,\beta}(u,v) &= \frac{M_{\alpha_1,\beta}\,(u-v)^{\beta-\alpha_1-1}\, r_{\alpha_1}(v)}{r_\beta(u)} = O_{v_0}(1)\, \frac{(u-v)^{\beta-\alpha_1-1}}{\int_0^u (u-v)^{\beta-\alpha_1-1}\, r_{\alpha_1}(v)\,\mathrm{d}v} \\ &= O_{v_0}(1)\, \frac{(u-v)^{\beta-\alpha_1-1}}{u^{\beta-\alpha_1-1}\int_0^u r_{\alpha_1}(v)\,\mathrm{d}v} = O_{v_0}(1)\left(1-\frac{v}{u}\right)^{\beta-\alpha_1-1} \frac{1}{\int_0^u r_{\alpha_1}(v)\,\mathrm{d}v} \\ &= O_{v_0}(1)\left(1-\frac{v_0}{u}\right)^{\beta-\alpha_1-1} \frac{1}{\int_0^u r_{\alpha_1}(v)\,\mathrm{d}v} \to 0 \end{aligned}$$

uniformly for $0 < v \le v_0$ as $u \to \infty$. Hence condition (2.12) is satisfied for every $v_0 > 0$. □

Remark 1. As we can see from the previous proof, the transformations $C_{\gamma,\beta}$ ($\beta > \gamma \underset{(-)}{>} \alpha_1$) transform all bounded functions of X into bounded functions of X again.

Proposition 1. *Let* $\{A_\alpha\}$ ($\alpha \underset{(-)}{>} \alpha_1$) *be a Riesz-type family. Then we have for functions* $x = x(u)$ *and numbers* s *and* $\beta > \gamma \underset{(-)}{>} \alpha_1$ *in case* $\mathscr{A}$) *that*

$$x(u) = O\,(A_\gamma) \Longrightarrow x(u) = O\,(A_\beta) \text{ and } \quad x(u) \to s\,(A_\gamma) \Longrightarrow x(u) \to s\,(A_\beta),$$

and in case $\mathscr{B}$) *that*

$$x(u) = O\,(A_\beta) \Longrightarrow x(u) = O\,(A_\gamma) \text{ and } \quad x(u) \to s\,(A_\beta) \longrightarrow x(u) \to s\,(A_\gamma),$$

provided in both cases that (2.11) *is satisfied if* $\gamma = \alpha_1$ *is included.*

Proof. This result follows directly from Definition 1 because the methods $C_{\gamma,\beta}$ are regular by Lemma 1. □

3. COMPARISON OF SPEEDS OF CONVERGENCE IN A RIESZ-TYPE FAMILY

Theorem 1 below describes how the speed of convergence changes if we go from one summability method in the family to a stronger one.

Theorem 1. *Let* $\{A_\alpha\}$ ($\alpha > \alpha_0$) *be a Riesz-type family. Let there be given some positive function* $\lambda = \lambda(u) \to \infty$ *from* X *and some number* $\gamma > \alpha_0$ *such that* $\frac{r_\gamma(u)}{\lambda(u)} \in X$.

(i) *Then we have for functions* $x = x(u)$ *and numbers* s *and* $\beta \geq \gamma$
in case $\mathscr{A}$) *that*

$$\lambda(u)\,[y_\gamma(u) - s] = O(1) \implies \lambda_\beta(u)\,[y_\beta(u) - s] = O(1),$$

and in case $\mathscr{B}$) *that*

$$\lambda(u)\,[y_\beta(u) - s] = O(1) \implies \lambda_\beta(u)\,[y_\gamma(u) - s] = O(1),$$

where the speeds are related through the formulas

$$\lambda_\beta(u) = \frac{r_\beta(u)}{b_\beta(u)} \;\text{ with } b_\beta(u) = M_{\gamma,\beta} \int_0^u (u-v)^{\beta-\gamma-1}\, b_\gamma(v)\,\mathrm{d}v \;\text{ and }\; b_\gamma(u) = \frac{r_\gamma(u)}{\lambda(u)}. \tag{3.1}$$

(ii) *Moreover, we have*
in case $\mathscr{A}$) *that*

$$\lambda(u)\,[y_\gamma(u) - s] \to t \implies \lambda_\beta(u)\,[y_\beta(u) - s] \to t,$$

and in case $\mathscr{B}$) *that*

$$\lambda(u)\,[y_\beta(u) - s] \to t \implies \lambda_\beta(u)\,[y_\gamma(u) - s] \to t,$$

provided in both cases that

$$\lim_{u\to\infty} \int_0^u b_\gamma(v)\,\mathrm{d}v = \infty. \tag{3.2}$$

Proof.

Case $\mathscr{A}$). Set $\alpha_1 = \gamma$ and consider another family of summability methods B_α ($\alpha \geq \gamma$), defined by the transformations of x into $\eta_\alpha = \eta_\alpha(u)$ with

$$\eta_\alpha(u) = \lambda_\alpha(u)\, y_\alpha(u)\,,$$

where $\lambda_\alpha = \lambda_\alpha(u)$ is given according to (3.1). The methods B_α obey the relation

$$\eta_\beta(u) = \frac{M_{\gamma,\beta}}{b_\beta(u)} \int_0^u (u-v)^{\beta-\gamma-1} \, b_\gamma(v) \, \eta_\gamma(v) \, \mathrm{d}v \tag{3.3}$$

and form therefore a Riesz-type family. Notice that we have for $\alpha \geq \gamma$:

$$\lambda_\alpha(u) \, [y_\alpha(u) - s] = O(1) \iff x(u) - s = O\,(B_\alpha), \tag{3.4}$$

$$\lambda_\alpha(u) \, [y_\alpha(u) - s] \to t \iff x(u) - s \to t\,(B_\alpha), \tag{3.5}$$

where $\lambda_\gamma(u) = \lambda(u)$. Now Proposition 1 in case $\mathscr{A}$) (apply it to B_α and $x(u) - s$ instead of A_α and $x(u)$) yields the desired result. Notice that relation (3.3) defines the connection methods $C^*_{\gamma,\beta}$ such that $B_\beta = C^*_{\gamma,\beta} \circ B_\gamma$.

Case $\mathscr{B}$). Define the methods B_β and B_γ by transformations of x into the functions η_β and η_γ, respectively, where $\eta_\beta(u) = \lambda(u) \, y_\beta(u)$ and $\eta_\gamma(u) = \lambda_\beta(u) \, y_\gamma(u)$. Now we have the relation

$$\eta_\gamma(u) = \frac{M_{\gamma,\beta}}{b_\beta(u)} \int_0^u (u-v)^{\beta-\gamma-1} \, b_\gamma(u) \, \eta_\beta(v) \, \mathrm{d}v, \tag{3.6}$$

which yields the desired result due to the regularity of connection methods $C^*_{\gamma,\beta}$ which have in case of (3.6) the same shape as in case of (3.3). □

Remark 2. Under restriction (3.2) the condition $\lambda(u) \to \infty$ implies $\lambda_\beta(u) \to \infty$ in Theorem 1. This follows from the regularity of methods $C_{\gamma,\beta}$ and $C^*_{\gamma,\beta}$ (apply $C^*_{\gamma,\beta}$ to the function $\lambda(u)[x(u) - s] \to t$, where $t \neq 0$ and $C_{\gamma,\beta}$ to the function $x(u)$).

We note that case $\mathscr{A}$) of Theorem 1 can be considered as a generalization of case A) of Theorem 1 of [7], which was proved for matrix case. Certain evaluations of the speed of convergence for matrix Nörlund methods in Banach spaces were proved in a recent paper [12].

Next we will compare the speeds $\lambda = \lambda(u)$ and $\lambda_\beta = \lambda_\beta(u)$ described in Theorem 1 by proving some inequalities.

Let $a = a(u)$ and $b = b(u)$ be two positive functions from X. If there exist positive numbers c_1, c_2, and u_0 such that the condition

$$c_1 \, b(u) \leq a(u) \leq c_2 \, b(u) \tag{3.7}$$

holds for every $u > u_0$, we write

$$a(u) \approx b(u).$$

If the function $b = b(u)$ is nondecreasing and condition (3.7) is satisfied with some positive numbers c_1 and c_2 for any $u > 0$, then we say that the function $a = a(u)$ is almost nondecreasing.

Proposition 2. *Let there be given a Riesz-type family $\{A_\alpha\}$ $(\alpha > \alpha_0)$ and an almost nondecreasing function $\lambda = \lambda(u)$. Suppose that $\lambda_\beta = \lambda_\beta(u)$ $(\beta > \gamma > \alpha_0)$ is defined through* (3.1). *Then for $\beta > \gamma > \alpha_0$ we have*

$$\lambda_\beta(u) \leq M \, \lambda(u) \quad (u > 0),$$

where M is some positive constant independent of u.

Proof. By the relation $r_\gamma(u) = b_\gamma(u) \, \lambda(u)$ and the other formulas (3.1) we have

$$\lambda_\beta(u) = \frac{r_\beta(u)}{b_\beta(u)} = \frac{\int_0^u (u-v)^{\beta-\gamma-1} \, b_\gamma(v) \, \lambda(v) \, \mathrm{d}v}{\int_0^u (u-v)^{\beta-\gamma-1} \, b_\gamma(v) \, \mathrm{d}v} \leq M \, \lambda(u)$$

for any $u > 0$. □

This result says that the speed of convergence cannot be improved by switching to a stronger summability method. It is consistent with the results known for matrix methods (see [4,12]), which say that a regular triangular matrix method cannot improve the speed of convergence (see also Proposition 2 in [7]). However, the speed cannot become much worse if we switch to a stronger method.

Proposition 3. *Let there be given a Riesz-type family* $\{A_\alpha\}$ $(\alpha > \alpha_0)$ *and a positive function* $\lambda = \lambda(u)$. *Suppose that* $\lambda_\beta = \lambda_\beta(u)$ $(\beta > \gamma > \alpha_0)$ *is defined through* (3.1). *If* $b_\gamma(u) = r_\gamma(u)/\lambda(u)$ *is almost nondecreasing, then for* $\beta > \gamma > \alpha_0$ *we have*

$$\lambda_\beta(u) \geq \frac{K\, r_\beta(u)}{r_\gamma(u)\, u^{\beta-\gamma}}\, \lambda(u) \quad (u > 0),$$

where K is some constant independent of u.

Proof. With the help of formulas (3.1) we find that

$$\begin{aligned} \lambda_\beta(u) &= \frac{r_\beta(u)}{b_\beta(u)} = \frac{r_\beta(u)}{M_{\gamma,\beta} \int_0^u (u-v)^{\beta-\gamma-1}\, b_\gamma(v)\, \mathrm{d}v} \\ &\geq \frac{N\, r_\beta(u)}{b_\gamma(u) \int_0^u (u-v)^{\beta-\gamma-1}\, \mathrm{d}v} = \frac{K\, r_\beta(u)}{b_\gamma(u)\, u^{\beta-\gamma}} \\ &= \frac{K\, r_\beta(u)}{r_\gamma(u)\, u^{\beta-\gamma}}\, \lambda(u), \end{aligned}$$

where the coefficients $M_{\gamma,\beta}$ are determined by the given Riesz-type family, and N and K depend on γ and β but not on u. □

Remark 3. If both $\lambda(u)$ and $b_\gamma(u)$ are almost nondecreasing, then for $\beta > \gamma > \alpha_0$ we have by Propositions 2 and 3

$$\frac{K\, r_\beta(u)}{r_\gamma(u)\, u^{\beta-\gamma}}\, \lambda(u) \leq \lambda_\beta(u) \leq M\, \lambda(u) \quad (u > 0),$$

where K and M are positive constants independent of u.

4. EXAMPLES ON THE COMPARISON OF SPEEDS OF CONVERGENCE

Applying Theorem 1, we find here comparative evaluations of speeds of convergence for summability methods in some special Riesz-type families.

Example 5. We consider the family of Riesz methods $A_\alpha = (R, \alpha) = (N, u^{\alpha-1}, 1)$ $(\alpha > 0)$. Let us choose the speed of convergence $\lambda(u) = (u+1)^\rho$ $(\rho > 0)$ and some number $\gamma > 0$.

Suppose that $x = x(u)$ is a function having the given speed of convergence $\lambda(u)$ with respect to the method $A_\gamma = (R, \gamma)$. Determine with the help of Theorem 1 the speed of convergence $\lambda_\beta(u)$ of $x = x(u)$ with respect to the methods $A_\beta = (R, \beta)$ for $\beta > \gamma$.

Using formulas (3.1), we have $\lambda_\beta(u) = r_\beta(u) / b_\beta(u)$ with $r_\beta(u) = u^\beta/\beta$ and

$$b_\beta(u) = M_{\gamma,\beta} \int_0^u (u-v)^{\beta-\gamma-1}\, \frac{r_\gamma(v)}{\lambda(v)}\, \mathrm{d}v = \frac{M_{\gamma,\beta}}{\gamma} \int_0^u (u-v)^{\beta-\gamma-1}\, \frac{v^\gamma}{(v+1)^\rho}\, \mathrm{d}v \quad (u > 0), \tag{4.1}$$

where $M_{\gamma,\beta}$ is the constant defined by (2.3).

It follows directly from (4.1) that

$$b_\beta(u) \geq \frac{M_{\gamma,\beta}}{\gamma 2^\rho} \int_1^{u/2} (u-v)^{\beta-\gamma-1}\, v^{\gamma-\rho}\, \mathrm{d}v \tag{4.2}$$

for every $u > 2$.

a) If $\rho < \gamma + 1$, then (4.1) yields due to Theorem 42 of [9] the equivalence

$$b_\beta(u) \sim \frac{M_{\gamma,\beta}}{\gamma} \int_0^u (u-v)^{\beta-\gamma-1} v^{\gamma-\rho}\, \mathrm{d}v$$

as $x \to \infty$. Calculating the last integral with the help of substitution $t = v/u$, we get:

$$\begin{aligned}\int_0^u (u-v)^{\beta-\gamma-1} v^{\gamma-\rho}\, \mathrm{d}v &= u^{\beta-\rho} \int_0^1 (1-t)^{\beta-\gamma-1} t^{\gamma-\rho}\, \mathrm{d}t \\ &= u^{\beta-\rho} B(\beta-\gamma, \gamma-\rho+1).\end{aligned}$$

Thus we have in this case that

$$b_\beta(u) \sim \frac{M_{\gamma,\beta} B(\beta-\gamma, \gamma-\rho+1)}{\gamma} u^{\beta-\rho}$$

and

$$\lambda_\beta(u) \sim \frac{\Gamma(\gamma+1)\Gamma(\beta-\rho+1)}{\Gamma(\beta+1)\Gamma(\gamma-\rho+1)} u^\rho \tag{4.3}$$

as $u \to \infty$.

Evaluating the functions $b_\beta(u)$ and $\lambda_\beta(u)$, we do not calculate in further estimations (as we did in (4.2) and (4.3)) the exact values of numerical coefficients any more. Moreover, in order to shorten our writings, we do not emphasize further the dependence of these coefficients on parameters γ, β, and ρ with the help of indices in these coefficients.

b) If $\rho = \gamma + 1$, it follows from (4.1) that

$$\begin{aligned} b_\beta(u) &\le L_1 \int_0^{u/2} (u-v)^{\beta-\gamma-1} (v+1)^{-1}\, \mathrm{d}v + L_1 \int_{u/2}^u (u-v)^{\beta-\gamma-1} (v+1)^{-1}\, \mathrm{d}v \\ &\le L_2 u^{\beta-\gamma-1} \int_0^{u/2} \frac{\mathrm{d}v}{v+1} + \frac{L_3}{u} \int_{u/2}^u (u-v)^{\beta-\gamma-1}\, \mathrm{d}v \\ &\le L_4 u^{\beta-\gamma-1} \log u + L_5 u^{\beta-\gamma-1} \log u \\ &= L_6 u^{\beta-\gamma-1} \log u \end{aligned}$$

for every $u > u_0$, where u_0 is some positive number and $L_1, L_2, ..., L_6$ are constants independent of u.

On the other hand, inequality (4.2) gives us that

$$b_\beta(u) \ge M_1 \int_1^{u/2} (u-v)^{\beta-\gamma-1} (v+1)^{-1}\, \mathrm{d}v \ge M_2 u^{\beta-\gamma-1} \int_1^{u/2} \frac{\mathrm{d}v}{v+1} \ge M_3 u^{\beta-\gamma-1} \log u$$

for every $u > u_1$, where u_1 is some positive number bigger than 2, and M_1, M_2, and M_3 are constants independent of u. Thus we have shown that in this case

$$b_\beta(u) \approx u^{\beta-\gamma-1} \log u$$

and

$$\lambda_\beta(u) \approx \frac{u^{\gamma+1}}{\log u}. \tag{4.4}$$

c) If $\rho > \gamma + 1$, then starting from (4.1) and (4.2) and discussing analogously to case b), we come to the relations

$$b_\beta(u) \approx u^{\beta-\gamma-1}$$

and

$$\lambda_\beta(u) \approx u^{\gamma+1}. \tag{4.5}$$

As a result we have proved by (4.3)–(4.5) the following estimations:

$$\lambda_\beta(u) \sim \frac{\Gamma(\gamma+1)\,\Gamma(\beta-\rho+1)}{\Gamma(\beta+1)\,\Gamma(\gamma-\rho+1)}\,\lambda(u) \quad \text{if } \rho < \gamma+1,$$

$$\lambda_\beta(u) \approx \begin{cases} \dfrac{\lambda(u)}{\log u} & \text{if } \rho = \gamma+1, \\ \lambda(u)\,u^{\gamma-\rho+1} & \text{if } \rho > \gamma+1. \end{cases}$$

Notice that case $\mathscr{A}$) of statement (i) of Theorem 1 holds here for any $\beta > \gamma > 0$ and $\rho > 0$. Moreover, if $\rho < \gamma+1$ or $\rho = \gamma+1$, then condition (3.2) is satisfied and also case $\mathscr{A}$) of statement (ii) of Theorem 1 works here.

Example 6. Consider the family of Abel-type methods $A_\alpha = (A,\alpha)$ $(\alpha > -1)$. Suppose that $\lambda(u)$ is the same as in the previous example. Fix some number $\gamma > -1$ and pose the same task as in the previous example to find $\lambda_\beta(u)$ for $\beta > \gamma > -1$.

Here the connection method $C_{\gamma,\beta}$ is the same as for the Riesz methods (compare relations (2.6) and (2.4). So the same situation as in the previous example appears here and we get the same speed $\lambda_\beta(u)$. Thus, case $\mathscr{B}$) of statement (i) of Theorem 1 holds here. Moreover, if $\rho < \gamma+1$ or $\rho = \gamma+1$, then condition (3.2) is satisfied and also case $\mathscr{B}$) of statement (ii) of Theorem 1 works.

The same situation appears if we consider the family of methods $A_\alpha = (D,\alpha)$ $(\alpha > -1)$ (see Example 2).

Example 7. Let us consider the Borel-type methods $A_\alpha = (B,\alpha,1/n!) = (B,\alpha)$ $(\alpha > -1/2)$. Here we have $r_\alpha(u) \sim e^u$ (see Example 4).

Suppose that $\lambda(u) = (u+1)^\rho\, e^u$, fix some $\gamma > -1/2$, and find $\lambda_\beta(u)$ for $\beta > \gamma$ again. Now we get for $\beta > \gamma$ with the help of relations (3.1) that

$$b_\beta(u) \approx \int_0^u (u-v)^{\beta-\gamma-1}\,\frac{e^v}{(v+1)^\rho\, e^v}\,\mathrm{d}v = \int_0^u (u-v)^{\beta-\gamma-1}\,(v+1)^{-\rho}\,\mathrm{d}v.$$

Evaluating the last integral in the same way as in Example 5, we get the following results:

$$b_\beta(u) \;\approx\; \begin{cases} u^{\beta-\gamma-1} & \text{if } \rho > 1, \\ u^{\beta-\gamma-1}\log u & \text{if } \rho = 1, \\ u^{\beta-\gamma-\rho} & \text{if } \rho < 1 \end{cases}$$

and

$$\lambda_\beta(u) = \frac{r_\beta(u)}{b_\beta(u)} \;\approx\; \begin{cases} \dfrac{e^u}{u^{\beta-\gamma-1}} \sim \dfrac{\lambda(u)}{u^{\beta-\gamma+\rho-1}} & \text{if } \rho > 1, \\ \dfrac{e^u}{u^{\beta-\gamma-1}\log u} \sim \dfrac{\lambda(u)}{u^{\beta-\gamma}\log u} & \text{if } \rho = 1, \\ \dfrac{e^u}{u^{\beta-\gamma-\rho}} \sim \dfrac{\lambda(u)}{u^{\beta-\gamma}} & \text{if } \rho < 1. \end{cases}$$

Example 8. Consider here the methods $A_\alpha = (N, u^{\alpha-1}, e^{u^\varphi})$ $(\alpha > 0)$, where $0 < \varphi < 1$ is some fixed number. Suppose that $\lambda(u) = (u+1)^\rho$ $(\rho > 0)$. We have for $\beta \geq \gamma$ that (see [7, p. 236])

$$r_\beta(u) \approx e^{u^\varphi}\, u^{(1-\varphi)\beta}.$$

Now we get with the help of (3.1) that

$$b_\beta(u) \approx e^{u^\varphi} u^{(1-\varphi)\beta-\rho}$$

and

$$\lambda_\beta(u) \approx u^\rho \sim \lambda(u)$$

if $\beta > \gamma$. So, both statements (i) and (ii) in case $\mathscr{A}$) of Theorem 1 apply here with

$$\lambda_\beta(u) \approx \lambda(u).$$

Example 9. Let A_α be the same methods as in the previous example, but suppose that $\lambda(u) = e^{u^\varphi}$ $(0 < \varphi < 1)$. Then we have by (3.1) that $b_\gamma(u) \approx u^{(1-\varphi)\gamma}$ and

$$b_\beta(u) \approx \int_0^u (u-v)^{\beta-\gamma-1} b_\gamma(v)\,\mathrm{d}v \approx u^{\beta-\gamma+(1-\varphi)\gamma}$$

for $\beta > \gamma$. Therefore statements (i) and (ii) of Theorem 1 in case $\mathscr{A}$) are true with

$$\lambda_\beta(u) \approx e^{u^\varphi} u^{\varphi(\gamma-\beta)} = u^{\varphi(\gamma-\beta)} \lambda(u).$$

This series of examples could be continued.

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Summeeruvuskiiruste võrdlemine Rieszi-tüüpi peredes

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On võrreldud funktsioonide (ja jadade) summeeruvust eri summeerimismenetluste korral Rieszi-tüüpi peredes (vt [6] ja [8]). On üldistatud ja rakendatud töös [7] saadud tulemusi jadade summeeruvuskiiruste võrdlemisel. On tõestatud teoreem, mis võimaldab võrrelda summeeruvuskiirusi eri menetluste korral Rieszi-tüüpi peres. On hinnatud summeeruvuskiirusi võrratuste abil. Näidetena on vaadeldud Rieszi ja üldistatud Nörlundi integraalmenetlusi, samuti Abeli- ning Boreli-tüüpi summeerimismenetlusi.