Proceedings of the Estonian Academy of Sciences,
2008, **57**, 1, 3–17
doi: 10.3176/proc.2008.1.01
Available online at www.eap.ee

MATHEMATICS

On the acceleration of convergence by regular matrix methods

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Received 9 October 2007

Abstract. Regular matrix methods that improve and accelerate the convergence of sequences and series are studied. Some problems related to the speed of convergence of sequences and series with respect to matrix methods are discussed. Several theorems on the improvement and acceleration of the convergence are proved. The results obtained are used to increase the order of approximation of Fourier expansions and Zygmund means of Fourier expansions in certain Banach spaces.

Key words: convergence acceleration, matrix methods, Fourier expansions, order of approximation.

1. INTRODUCTION AND PRELIMINARIES

In recent years the most significant results in convergence acceleration have been proved for nonlinear methods of acceleration (see, for example, [1,2]). The present paper deals with regular matrix methods that improve and accelerate the convergence of sequences and series. The research has been inspired by the papers [3–8] where this problem is considered. Some data on the improvement of convergence by regular matrix methods are available also in [9]. Let us describe the main results of the mentioned papers more precisely. For this purpose we need some notions. Let $M = (m_{nk})$ be a matrix with real or complex entries. Throughout the paper we assume that indices and summation indices change from 0 to ∞ if not specified otherwise. A sequence $x = (x_k)$ is said to be *M-summable* if the sequence $Mx = (M_n x)$, where

$$M_n x = \sum_k m_{nk} x_k,$$

is convergent.

We denote the set of all M-summable sequences by c_M. Thus, a matrix M determines the summability method on c_M, which we also denote by M. A method M is called *sequence-to-sequence regular* (shortly, *Sq-Sq regular*) if

$$\lim_n M_n x = \lim_n x_n$$

for each convergent sequence $x = (x_k)$. Classically the following concepts are used to estimate and compare the speeds of convergence of sequences.

Definition 1.1 ([1,5]). *A method M is called accelerating the convergence if the relation*

$$\frac{|M_n x - \lim_n M_n x|}{|x_n - \lim_n x_n|} \to 0 \text{ for } n \to \infty \tag{1.1}$$

holds for every convergent sequence $x = (x_n)$. *If relation* (1.1) *holds for particular* x, *then it is said that* M *accelerates the convergence of this* x. *If* M *accelerates the convergence of* x, *then it is said that* Mx *converges faster than* x.

Definition 1.2 (cf. [5], p. 310). *A matrix method* M *is said to be accelerating with respect to a matrix method* A *if* Mx *converges faster than* Ax *for every* $x \in c_A$. *If* Mx *converges faster than* Ax *for particular* $x \in c_A$, *then* M *is said to be accelerating with respect to* A *for this* x.

Weakened criteria are used to estimate and compare the speeds of convergence of sequences in [3,4] and [6–9]. Let $\lambda = (\lambda_k)$ be a sequence with $0 < \lambda_k \nearrow$.

Definition 1.3 ([3,4]). *A convergent sequence* $x = (x_k)$ *with*

$$\lim_k x_k = \varsigma \text{ and } l_k = \lambda_k (x_k - \varsigma)$$

is called bounded with the speed λ (*shortly*, λ*-bounded*) *if* $l_k = O(1)$ *and convergent with the speed* λ (*shortly*, λ*-convergent*) *if there exists the finite limit* $\lim_k l_k$.

We denote the set of all λ-bounded sequences by m^λ and the set of all λ-convergent sequences by c^λ. For $\lambda_k = O(1)$ we get $c^\lambda = m^\lambda = c$, where c is the set of all convergent sequences. A sequence $x = (x_k)$ is called A^λ*-bounded* or A^λ*-summable* if $Ax \in m^\lambda$ or $Ax \in c^\lambda$, respectively. We denote the set of all A^λ-bounded sequences by m_A^λ and the set of all A^λ-summable sequences by c_A^λ. If λ is a bounded sequence, then $m_A^\lambda = c_A^\lambda = c_A$.

Let $\mu = (\mu_k)$ be another sequence with $0 < \mu_k \nearrow$.

Definition 1.4 ([4,6–8]). *A method* M *is called improving* λ*-convergence or* λ*-boundedness of sequences if, respectively,* $c^\lambda \subseteq c_M^\mu$ *or* $m^\lambda \subseteq m_M^\mu$ *with* $\mu_k/\lambda_k \longrightarrow \infty$. *If* $c \subseteq c_M^\mu$ *or* $c \subseteq m_M^\mu$ *with* $\mu_k \neq O(1)$, *then* M *is called improving the convergence of sequences*.

Definition 1.5 ([9]). *We say that* M *improves* A^λ*-summability or* A^λ*-boundedness if, respectively,* $c_A^\lambda \subseteq c_M^\mu$ *or* $m_A^\lambda \subseteq m_M^\mu$ *with* $\mu_k/\lambda_k \longrightarrow \infty$. *If* $c_A \subseteq c_M^\mu$ *or* $c_A \subseteq m_M^\mu$, *we say that* M *improves* A*-summability.*

It is not difficult to see that if A is the identity method, i.e. $A = I = (\delta_{nk})$, where $\delta_{nn} = 1$ and $\delta_{nk} = 0$ for $n \neq k$, then Definition 1.5 coincides with Definition 1.4.

Kornfeld [5] proved that any Sq-Sq regular method M cannot accelerate the convergence and cannot be accelerating with respect to another Sq-Sq regular method A. Kangro ([3], pp. 139–140) proved that an Sq-Sq regular *triangular* method $M = (m_{nk})$ (i.e. $m_{nk} = 0$ for $k > n$) cannot improve the λ-convergence. In [9] it is proved that any Sq-Sq regular triangular method improves neither the convergence nor the λ-boundedness for an unbounded speed λ. In [9] it is also shown that any triangular method M improves neither A-summability nor A^λ-boundedness for a *normal* method $A = (a_{nk})$ (i.e. A is triangular and $a_{nn} \neq 0$) if $\lim_n M_n(x) = \lim_n A_n x$ for every $x \in c_A$ or $x \in m_A^\lambda$, respectively. Tammeraid [6–8] generalized the concepts of A^λ-summability and A^λ-boundedness, considering instead of a matrix with real or complex entries a matrix whose elements are bounded linear operators from a Banach space X into a Banach space Y. He proved that a triangular Sq-Sq regular method cannot improve the λ-boundedness ([8], pp. 370–371) and the λ-convergence ([7], p. 91).

The aim of the present paper is to inquire into the properties of nontriangular regular matrix methods improving and accelerating the convergence. The question is whether there exist such regular methods that improve the convergence, the λ-boundedness or the A^λ-boundedness. The results of the papers [3–9] show that the answer to this question is always negative for triangular Sq-Sq regular methods. The results giving a positive answer to that question are proved in the present paper for nontriangular regular methods (Propositions 3.1–3.3, Corollary 3.2, Theorem 4.2). Moreover, the convergence acceleration on the subsets of convergent sequences and series is studied. In addition to Sq-Sq regular methods, *series-to-sequence regular* (shortly, *Sr-Sq regular*) methods are used, and it is proved that for some cases these methods have better convergence improving and accelerating properties than Sq-Sq regular methods (compare, for example, Theorem 3.2 and Proposition 3.1). A method M is called Sr-Sq regular if

$$\lim_n M_n x = \sum_k x_k$$

for each $x = (x_k) \in cs$, where

$$cs = \{x = (x_k) \mid (X_n) \in c\};\ X_n = \sum_{k=0}^{n} x_k.$$

It is easy to see that the set of sequences cs is equivalent to the set of convergent series. We note that Sr-Sq regular methods play an important role in the approximation theory (see, for example, [10,11]). In the present paper Sr-Sq regular methods are used for increasing the order of approximation of Fourier expansions and Zygmund means of Fourier expansions in certain Banach spaces.

The paper is organized as follows. In Section 2 some notions and auxiliary results are presented, which are needed later. In Section 3 the improvement of the convergence and λ-boundedness, and convergence acceleration by nontriangular Sq-Sq and Sr-Sq regular methods are studied. Also some examples of Sr-Sq regular methods improving the convergence of series, and some examples of Sq-Sq regular methods improving λ-boundedness of sequences are presented. Besides, the sufficient conditions for a nontriangular method M to be accelerating for all elements from a certain subset of c or cs are specified. In Section 4 it is shown that using some nontriangular regular method M, it is possible to improve A^{λ}-boundedness for some unbounded speed λ if A is an Sr-Sq regular Zygmund method Z^r $(r > 1)$. Also the sufficient conditions for M to be accelerating with respect to Z^r for all elements from a certain subset of c_{Z^r} are found. In Section 5 the obtained results are used for increasing the order of approximation of Fourier expansions and Zygmund means of Fourier expansions in certain Banach spaces.

2. AUXILIARY RESULTS

Throughout this paper we assume that $\lambda = (\lambda_k)$ and $\mu = (\mu_k)$ are sequences with $0 < \lambda_k, \mu_k \nearrow \infty$; A is a normal matrix with its inverse matrix $A^{-1} = (\eta_{kl})$; $B = (b_{nk})$ is a triangular matrix, and $M = (m_{nk})$ is an arbitrary matrix. We say that M transforms m_A^{λ} or c_A into m_B^{μ} if the transformation $y = Mx$ exists and $y \in m_B^{\mu}$ for every $x \in m_A^{\lambda}$ or $x \in c_A$, respectively.

The equalities

$$\sum_{k=0}^{r} m_{nk} x_k = \sum_{l=0}^{r} H_{nl}^r t_l \quad (n = 0, 1, \ldots),$$

with

$$H_{nl}^r = \begin{cases} \sum\limits_{k=l}^{r} m_{nk}\eta_{kl} & (l \le r), \\ 0 & (l > r), \end{cases}$$

and $t_l = A_l x$ hold for each $x \in c_A$. Therefore we get the following results by Theorem 20.2 of [12] (see also [4], pp. 138–139) and Theorem 2.3.7 of [13].

Lemma 2.1 (cf. [14], p. 55). *The matrix transformation $y = Mx$ exists for each $x \in m_A^{\lambda}$ if and only if*

$$there\,exist\,finite\,limits\ \lim_r H_{nl}^r = H_{nl}, \tag{2.1}$$

$$there\,exist\,finite\,limits\ \lim_r \sum_{l=0}^{r} H_{nl}^r, \tag{2.2}$$

$$\sum_l \frac{|H_{nl}|}{\lambda_l} = O_n(1), \tag{2.3}$$

$$\lim_r \sum_{l=0}^{r} \frac{\left|H_{nl}^r - H_{nl}\right|}{\lambda_l} = 0. \tag{2.4}$$

Lemma 2.2. *The matrix transformation $y = Mx$ exists for each $x \in c_A$ if and only if conditions* (2.1), (2.2) *are fulfilled and*

$$\sum_{l=0}^{r} |H_{nl}^r| = O_n(1). \tag{2.5}$$

Let $G = (g_{nk}) = BM$ and

$$\gamma_{nl}^r = \begin{cases} \sum\limits_{k=l}^{r} g_{nk}\eta_{kl} & (l \le r), \\ 0 & (l > r). \end{cases}$$

Then we can formulate

Lemma 2.3 (see [14], pp. 55–57). *A matrix M transforms m_A^λ into m_B^μ if and only if conditions* (2.1)–(2.4) *are fulfilled and*

$$(\rho_n) \in m^\mu \text{ for } \rho_n = \lim_r \sum_{l=0}^{r} \gamma_{nl}^r, \tag{2.6}$$

$$\text{there exist finite limits } \lim_n \gamma_{nl} = \gamma_l \text{ for } \gamma_{nl} = \lim_r \gamma_{nl}^r, \tag{2.7}$$

$$\sum_l \frac{|\gamma_l|}{\lambda_l} < \infty, \tag{2.8}$$

$$\mu_n \sum_l \frac{|\gamma_{nl} - \gamma_l|}{\lambda_l} = O(1). \tag{2.9}$$

We say that methods A and B are *M-consistent* on m_A^λ or on c_A if the transformation Mx exists and

$$\lim_n B_n(Mx) = \lim_n A_n x$$

for each $x \in m_A^\lambda$ or $x \in c_A$, respectively. If B is the identity matrix, i.e. $B = I$, then $B_n(Mx) = M_n x$ for each $x \in m_A^\lambda$ or $x \in c_A$. Hence M-consistency of A and I on m_A^λ or on c_A coincides with the usual consistency of A and M respectively on m_A^λ or on c_A. By Lemmas 2 and 4 of [9] and Lemma 2.2 we immediately get the following results.

Lemma 2.4. *A matrix M transforms c_A into m_B^μ if and only if conditions* (2.1), (2.2), (2.5)–(2.7) *hold and*

$$\sum_l |\gamma_l| < \infty, \tag{2.10}$$

$$\mu_n \sum_l |\gamma_{nl} - \gamma_l| = O(1). \tag{2.11}$$

Lemma 2.5. *Matrix methods A and B are M-consistent on c_A if and only if conditions* (2.1), (2.2), (2.5) *hold and* $\lim_n \rho_n = 1$, $\gamma_l = 0$ *and* $\sum_l |\gamma_{nl}| = O(1)$.

3. ACCELERATION OF CONVERGENCE AND IMPROVEMENT OF CONVERGENCE AND λ-BOUNDEDNESS WITH REGULAR MATRIX METHODS

First we consider the relation between the improvement of λ-boundedness and the acceleration of convergence in m^λ.

Theorem 3.1. *If M improves λ-boundedness of sequences, then M accelerates the convergence of all sequences from the subset $\widehat{m}^\lambda$ of m^λ, defined as follows*:

$$\widehat{m}^\lambda = \{x = (x_n) \in m^\lambda \mid \lambda_n \left| x_n - \lim_n x_n \right| > m;\ m > 0\}.$$

Proof. As M improves λ-boundedness of sequences, there exists $\mu = (\mu_k)$, $\mu_k/\lambda_k \longrightarrow \infty$ so that $\widehat{m}^\lambda \subset m^\lambda \subseteq m^\mu_M$. Therefore for every $x = (x_n) \in \widehat{m}^\lambda$ we get

$$\frac{\mu_n |M_n x - \lim_n M_n x|}{\lambda_n |x_n - \lim_n x_n|} = O(1).$$

Hence relation (1.1) holds for every $x \in \widehat{m}^\lambda$. Thus M accelerates the convergence of all sequences from $\widehat{m}^\lambda$.

A method M is said to be *conull* if $c \subseteq c_M$ and

$$\lim_n \sum_k m_{nk} = \sum_k \lim_n m_{nk}.$$

It is well known that a conull method cannot be Sq-Sq regular (see [13], p. 49). Now we prove the following auxiliary result.

Lemma 3.1. *If A and B are M-consistent on c_A and M transforms c_A into m^μ_B, then $\Gamma = (\gamma_{nl})$ is a conull matrix.*

Proof. By Lemma 2.5 we get $\gamma_l = 0$. Hence

$$\lim_n \sum_l \gamma_{nl} = 0 \text{ and } \sum_l |\gamma_{nl}| = O(1)$$

by condition (2.11) of Lemma 2.4, because $0 < \mu_k \nearrow \infty$. Therefore $\Gamma = (\gamma_{nl})$ is an Sq-Sq conservative method by Theorem 2.3.7 of [13] and

$$\lim_n \sum_l \gamma_{nl} = \sum_l \gamma_l.$$

Thus Γ is a conull method.

As $\Gamma = M$ for $A = B = I$ and any conull method cannot be Sq-Sq regular, with the help of Lemma 3.1 we get the following result.

Theorem 3.2. *Any Sq-Sq regular method cannot improve the convergence.*

As $c \subseteq c_A$ for an Sq-Sq regular method A, Theorem 3.2 implies

Theorem 3.3. *Any Sq-Sq regular method M cannot improve A^λ-summability of any other Sq-Sq regular method A.*

Note that it is possible to prove Theorems 3.2 and 3.3 with the help of Theorem 1 of [4].

Now we inquire into the properties of nontriangular Sr-Sq regular methods improving and accelerating the convergence. For this purpose we first introduce some necessary notions. A series is said to be λ-*bounded* if the sequence of partial sums of this series is λ-bounded. Let

$$bs^{\lambda} = \{x = (x_k) \mid (X_n) \in m^{\lambda}\};\ X_n = \sum_{k=0}^{n} x_k.$$

It is easy to see that $bs^{\lambda} \subseteq cs$.

Definition 3.1. *We say that M accelerates the convergence of a series $\sum_k x_k$ if the sequence Mx (where $x = (x_k)$) converges faster than the sequence of partial sums (X_n) of this series. If Mx converges faster than (X_n) for every $x \in cs$, then we say that M accelerates the convergence of series.*

Definition 3.2. *If $bs^{\lambda} \subseteq m_M^{\mu}$ with $\mu_k/\lambda_k \longrightarrow \infty$, we say that M improves λ-boundedness of series. If $cs \subseteq m_M^{\mu}$ with $\mu_k \neq O(1)$, we say that M improves the convergence of sequences.*

From Definitions 3.1 and 3.2 and Theorem 3.1 we immediately get

Corollary 3.1. *If M improves λ-boundedness of series, then M accelerates the convergence of all sequences from the subset $\widehat{bs}^{\lambda}$ of bs^{λ}, defined as follows:*

$$\widehat{bs}^{\lambda} = \{x = (x_n) \mid (X_n) \in \widehat{m}^{\lambda}\}.$$

The assertion of Theorem 3.2 cannot be extended to Sr-Sq regular methods. Indeed, it is not difficult to see that the Sr-Sq regular method $M = (m_{nk})$, where $m_{nk} = 1$ for all n and k, improves the convergence of series. For getting more complicated examples we first prove

Lemma 3.2. *Let $M = (m_{nk})$ be such an Sr-Sq regular method where $m_{n0} = 1$. Then $bs^{\lambda} \subseteq m_M^{\mu}$ if and only if*

$$\mu_n \sum_l \frac{|\Delta m_{nl}|}{\lambda_l} = O(1) \tag{3.1}$$

and $cs \subseteq m_M^{\mu}$ if and only if

$$\mu_n \sum_l |\Delta m_{nl}| = O(1). \tag{3.2}$$

Proof. It is sufficient to show for $bs^{\lambda} \subseteq m_M^{\mu}$ that condition (3.1) is equivalent to the conditions of Lemmas 2.1 and 2.3 if $B = I$ and $A = \Sigma = (a_{nk})$, where

$$a_{nk} = 1 \text{ if } k \leq n \text{ and } a_{nk} = 0 \text{ if } k > n.$$

Similarly it is sufficient to show for $cs \subseteq m_M^{\mu}$ that condition (3.2) is equivalent to the conditions of Lemmas 2.2 and 2.4 if $A = \Sigma$ and $B = I$. For the proof of the above-mentioned equivalences we first note that

$$\gamma_{nl}^r = H_{nl}^r,\ \ \Gamma = (\gamma_{nl}) = (H_{nl})$$

for $A = \Sigma$ and $B = I$, where the inverse matrix $\Sigma^{-1} = (\eta_{nk})$ of Σ is defined by the equalities

$$\eta_{nn} = 1,\ \eta_{n,n-1} = -1 \text{ and } \eta_{nk} = 0 \text{ if } k > n \text{ or } k < n-1.$$

Therefore we have $\gamma_{nl}^0 = H_{nl}^0 = m_{n0}$ and

$$\gamma_{nl}^r = H_{nl}^r = \begin{cases} \Delta m_{nl} & (l \leq r-1), \\ m_{nr} & (l = r), \\ 0 & (l > r) \end{cases}$$

for $r \geq 1$ and

$$\sum_{l=0}^{r} \gamma_{nl}^{r} = \sum_{l=0}^{r} H_{nl}^{r} = m_{n0},\ \rho_n = m_{n0},\ \gamma_{nl} = H_{nl} = \Delta m_{nl}.$$

It immediately follows from these relations that conditions (2.1), (2.2), (2.6), and (2.7) are fulfilled.

We also get that the equalities

$$\sum_{l=0}^{r} \frac{\left|H_{nl}^{r} - H_{nl}\right|}{\lambda_l} = \frac{|m_{n,r+1}|}{\lambda_r}$$

and

$$\sum_{l=0}^{r} |H_{nl}^{r}| = \sum_{l=0}^{r-1} |\Delta m_{nl}| + |m_{nr}|$$

hold. Moreover, it follows from the Sr-Sq regularity of M that

$$\sum_{l} |\Delta m_{nl}| = O(1),\ \lim_{n} m_{nk} = 1$$

(see Proposition 14 of [15]) and the finite limits $\lim_r m_{nr}$ exist (see [16], pp. 199–200). Hence conditions (2.4), (2.5) hold and $\gamma_l = \lim_n \Delta m_{nl} = 0$. Therefore conditions (2.8), (2.10) also hold and conditions (2.9), (2.11), respectively, take the forms (3.1) and (3.2). As now the validity of condition (2.3) follows from (3.1), the proof is complete.

Let us define $M = (m_{nk})$ as follows:

$$m_{nk} = \begin{cases} 1 & (k=0), \\ 1 - \dfrac{k^s}{(n+1)^{s+t}} & (k \geq 1), \end{cases} \tag{3.3}$$

where $s < 0$ and $s+t > 0$. As $\lim_n m_{nk} = 1$ and

$$\sum_{k=0}^{N} |\Delta m_{nk}| = \frac{2 - (N+1)^s}{(n+1)^{s+t}} = O(1), \tag{3.4}$$

M is an Sr-Sq regular method by Proposition 14 of [15]. Also $\lim_k m_{nk} = 1 \neq 0$. Consequently, the M defined in this way is equivalent to Σ, i.e. $c_M = cs$ (see [16], pp. 199–200).

Proposition 3.1. *The Sr-Sq regular method $M = (m_{nk})$ defined by* (3.3), *where $s < 0$ and $s+t > 0$*, *improves the convergence of series.*

Proof. It is sufficient to show by Lemma 3.2 that condition (3.2) holds for some unbounded sequence μ. We define $\mu = (\mu_k)$ by the equalities

$$\mu_k = (k+1)^{\beta};\ \beta > 0. \tag{3.5}$$

As from relation (3.4) we get

$$\sum_{k=0}^{\infty} |\Delta m_{nk}| = \frac{2}{(n+1)^{s+t}},$$

it follows that

$$\mu_n \sum_{k=0}^{\infty} |\Delta m_{nk}| = 2(n+1)^{\beta - s - t} = O(1)$$

if and only if $\beta \leq s+t$. Thus condition (3.2) is valid and consequently, $cs \subseteq m_M^{\mu}$ for $\beta = s+t$.

Further, we notice that condition (3.1) follows from (3.2) for every unbounded sequence λ. Hence from the proof of Proposition 3.1 we imply the following result with the help of Corollary 3.1 and Lemma 3.2.

Corollary 3.2. *The Sr-Sq regular method* $M = (m_{nk})$ *defined by* (3.3), *where* $s < 0$ *and* $s+t > 0$, *improves* λ*-boundedness of series and accelerates the convergence of all sequences from* $\widehat{bs}^{\lambda}$ *for* $\lambda = (\lambda_k)$, *defined by equalities*

$$\lambda_k = (k+1)^{\alpha}; \ \alpha > 0 \tag{3.6}$$

if $\alpha < s+t$.

Let now $M = (m_{nk})$ be defined by the relation

$$m_{nk} = \begin{cases} 1 - \dfrac{k}{(n+1)^{cn+\beta+\sigma}} & (k \leq n+N), \\ 0 & (k > n+N), \end{cases} \tag{3.7}$$

where N is a fixed positive integer and $c,\ \beta,\ \sigma > 0$. Using Proposition 14 of [15], it is not difficult to check that the method defined in this way is Sr-Sq regular.

Proposition 3.2. *Let* $M = (m_{nk})$ *be defined by* (3.7), *where* N *is a fixed positive integer and* $c,\ \beta,\ \sigma > 0$. *Then* M *improves* λ*-boundedness of series and accelerates the convergence of all sequences from* $\widehat{bs}^{\lambda}$ *for* $\lambda = (\lambda_k)$, *defined by the equalities* $\lambda_k = (k+1)^{ck}$ *if* $\beta \leq c$ *and* $\sigma \geq 1$.

Proof. Let $\mu = (\mu_k)$ be defined with the help of equalities

$$\mu_k = (k+1)^{ck+\beta}.$$

As $\mu_k/\lambda_k \longrightarrow \infty$, it is sufficient to show by Definition 3.2 and Corollary 3.1 that $bs^{\lambda} \subseteq m_M^{\mu}$. For this purpose we prove that condition (3.1) is fulfilled. Let us write

$$T = \mu_n \sum_l \frac{|\Delta m_{nl}|}{\lambda_l} = T_1 + T_2,$$

where

$$T_1 = \mu_n \sum_{l=0}^{n} \frac{|\Delta m_{nl}|}{\lambda_l} \text{ and } T_2 = \mu_n \sum_{l=n+1}^{\infty} \frac{|\Delta m_{nl}|}{\lambda_l}.$$

As the sequence λ is monotonically increasing and

$$\Delta m_{nl} = \begin{cases} \dfrac{1}{(n+1)^{cn+\beta+\sigma}} & (l \leq n+N-1), \\ 1 - \dfrac{N+n}{(n+1)^{cn+\beta+\sigma}} & (l = n+N), \end{cases}$$

we have

$$T_1 \leq \mu_n \sum_{l=0}^{n} |\Delta m_{nl}| = (n+1)^{1-\sigma} = O(1),$$

since $\sigma \geq 1$, and

$$T_2 \leq \frac{\mu_n}{\lambda_{n+1}} \sum_{l=n+1}^{\infty} |\Delta m_{nl}| = \frac{(n+1)^{cn+\beta}}{(n+2)^{c(n+1)}} \left[\frac{N-1}{(n+1)^{cn+\beta+\sigma}} + \left| 1 - \frac{N+1}{(n+1)^{cn+\beta+\sigma}} \right| \right]$$

$$= O(1)\,(n+1)^{\beta-c} = O(1)\,,$$

since $\beta \leq c$. Consequently, $T = O(1)$, i.e condition (3.1) is satisfied. As M is Sr-Sq regular and $m_{n0} = 1$, $bs^{\lambda} \subseteq m_M^{\mu}$ by Lemma 3.2.

Now we show that there exist nontriangular Sq-Sq regular methods, improving λ-boundedness for an unbounded speed λ. Let $M = (m_{nk})$ be defined by the relation

$$m_{nl} = \begin{cases} \dfrac{l}{s_n} & \left(\lfloor n^b \rfloor \leq l \leq \left\lfloor (n+1)^b \right\rfloor\right), \\ 0 & \left(l < \lfloor n^b \rfloor \text{ or } l > \left\lfloor (n+1)^b \right\rfloor\right), \end{cases} \tag{3.8}$$

where

$$s_n = \sum_{l=\lfloor n^b \rfloor}^{\lfloor (n+1)^b \rfloor} l,$$

$b > 1$, and the symbol $\lfloor x \rfloor$ denotes the integer part of number x. According to Theorem 2.3.7 of [13], the M defined in this way is Sq-Sq regular.

Proposition 3.3. *Let $M = (m_{nk})$ be defined by* (3.8), *where $b > 1$. Then M improves λ-boundedness and accelerates the convergence of all sequences from $\widehat{m}^{\lambda}$ for $\lambda = (\lambda_k)$ defined by equalities* (3.6).

Proof. It is sufficient to prove by Definition 1.4 and Theorem 3.1 that the conditions of Lemma 2.3 are fulfilled for $A = B = I$ and for some $\mu = (\mu_k)$ satisfying the property $\mu_k/\lambda_k \longrightarrow \infty$. We define such $\mu = (\mu_k)$ by the relation

$$\mu_k = (k+1)^{\alpha+\sigma};\ \ \sigma > 0$$

and notice that the transformation $y = Mx$ exists for each $x \in c$. Hence conditions (2.1)–(2.4) are fulfilled. As

$$\gamma_{nl}^{r} = H_{nl}^{r} = \begin{cases} m_{nl} & (l \leq r), \\ 0 & (l > r) \end{cases}$$

and $\gamma_{nl} = H_{nl} = m_{nl}$, we have $\gamma_l = 0$ and $\rho_n = 1$. Consequently, conditions (2.6)–(2.8) are satisfied. Further, we can write

$$V = \mu_n \sum_l \frac{|\gamma_{nl} - \gamma_l|}{\lambda_l} \leq \frac{\mu_n}{\lambda_{\lfloor n^b \rfloor}} = \frac{(n+1)^{\alpha+\sigma}}{(\lfloor n^b \rfloor + 1)^{\alpha}} = O(1)\,(n+1)^{\alpha+\sigma-b\alpha}\,.$$

Hence $V = O(1)$, i.e., condition (2.9) is valid if $\sigma \leq (b-1)\,\alpha$. Such positive σ exists for $b > 1$. This completes the proof.

The Sq-Sq regular methods improving λ-boundedness are rather specific. We show that the following result holds.

Proposition 3.4. *Let $M = (m_{nk})$ be an Sq-Sq regular method, and the property $\mu_k/\lambda_{k+N} \longrightarrow \infty$ be fulfilled for $\lambda = (\lambda_k)$, $\mu = (\mu_k)$ and for all $N = 0, 1, \ldots$. If there exist $\varepsilon > 0$ and a nonnegative integer I so that*

$$\sum_{l=0}^{n+I} |m_{nl}| \geq \varepsilon$$

for all n, then $m^{\lambda} \not\subseteq m_M^{\mu}$.

Proof. As $\gamma_{nl} = m_{nl}$ and $\gamma_l = 0$ (see the proof of Proposition 3.3),

$$\mu_n \sum_l \frac{|\gamma_{nl} - \gamma_l|}{\lambda_l} = \mu_n \sum_l \frac{|m_{nl}|}{\lambda_l} \geq \frac{\mu_n}{\lambda_{n+I}} \sum_{l=0}^{n+I} |m_{nl}| \geq \frac{\mu_n}{\lambda_{n+I}} \varepsilon \longrightarrow \infty.$$

Thus condition (2.9) of Lemma 2.3 is not satisfied and therefore $m^\lambda \not\subseteq m_M^\mu$.

4. IMPROVEMENT OF A^λ-BOUNDEDNESS USING NONTRIANGULAR REGULAR MATRIX METHODS

First we explain the relationship between Definitions 1.2 and 1.5.

Theorem 4.1. *If M improves A^λ-boundedness, then M is accelerating with respect to A for all sequences from the subset $\widehat{m}_A^\lambda$ of m_A^λ, defined as follows*:

$$\widehat{m}_A^\lambda = \{x = (x_n) \in m_A^\lambda \mid \lambda_n \left| A_n x - \lim_n A_n x \right| > m;\ m > 0\}.$$

Proof is similar to the proof of Theorem 3.1.

It is proved in [9] that any triangular method M cannot improve A^λ-boundedness for an unbounded speed λ and a normal method A if M is consistent with A on m_A^λ. We show that this assertion cannot be extended to nontriangular methods M. Let us prove that a nontriangular Sr-Sq regular method M improving A^λ-boundedness exists for some normal Sr-Sq regular method A. For this purpose we consider the case where A is a Riesz method. Let (p_n) be a sequence of nonzero complex numbers, $P_n = p_0 + \ldots + p_n \neq 0$, $P_{-1} = 0$ and let $P = (R, p_n) = (a_{nk})$ be a Riesz method generated by (p_n), i.e. (see [17], p. 113)

$$a_{nk} = \begin{cases} 1 - P_{k-1}/P_n & (k \leq n), \\ 0 & (k > n). \end{cases}$$

It is easy to see that P is a normal method.

Lemma 4.1 (see [14], pp. 59–61). *Let P be a Riesz method satisfying the properties*

$$bs^\lambda \subseteq m_P^\lambda,\ P_n = O(P_{n-1}),\ \frac{p_n}{P_n} = O\left(\frac{p_{n-1}}{P_{n-1}}\right). \tag{4.1}$$

A matrix $M = (m_{nk})$ transforms m_P^λ into m_B^μ if and only if the following conditions hold:

$$\sum_l \frac{1}{\lambda_l} \left| P_l \Delta \frac{\Delta m_{nl}}{p_l} \right| = O_n(1) \ \ for\ \Delta m_{nl} = \Delta_l m_{nl} = m_{nl} - m_{n,l+1}, \tag{4.2}$$

$$\lim_l \frac{P_l m_{nl}}{p_l \lambda_l} = 0, \tag{4.3}$$

$$(1,0,0,\ldots) \in m_G^\mu, \tag{4.4}$$

$$there\ exist\ the\ finite\ limits\ \lim_n g_{nl} = g_l, \tag{4.5}$$

$$\sum_l \frac{1}{\lambda_l} \left| P_l \Delta \frac{\Delta g_l}{p_l} \right| < \infty, \tag{4.6}$$

$$\mu_n \sum_l \frac{1}{\lambda_l} \left| P_l \Delta \frac{\Delta (g_{nl} - g_l)}{p_l} \right| = O(1). \tag{4.7}$$

Now we consider (R, p_n) in the special case where p_n is defined by the relation $p_n = (n+1)^r - n^r$ $(r > 1)$. The Riesz method defined in this way is called the Zygmund method and is denoted by Z^r (see [17], p. 112). Thus $Z^r = (a_{nk})$ is defined by the relation

$$a_{nk} = \begin{cases} 1 - \left(\dfrac{k}{n+1}\right)^r & (k \le n), \\ 0 & (k > n). \end{cases}$$

It is not difficult to verify that Z^r is an Sr-Sq regular method. We also have $cs \not\subseteq m^{\lambda}_{Z^r}$ for each unbounded sequence λ. Indeed, let $\bar{Z}^r = (\Delta a_{nk})$ for $Z^r = (a_{nk})$. Then (see [17], pp. 51–52)

$$Z^r_n x = \bar{Z}^r_n X \tag{4.8}$$

for every $x = (x_k) \in c_{Z^r}$, where $X = (X_k)$ is the sequence of partial sums of series $\sum_k x_k$. Hence $\bar{Z}^r$ is an Sq-Sq regular method, because Z^r is Sr-Sq regular. As $\bar{Z}^r$ cannot improve the convergence of sequences by Corollary 3 of [9], Z^r cannot improve the convergence of series by (4.8), i.e., $cs \not\subseteq m^{\lambda}_{Z^r}$ for each unbounded speed λ.

Using Lemma 4.1 for $B = I$, $M = \Sigma$, and $P = Z^r$, it is possible to prove that $m^{\lambda}_{Z^r} \subsetneqq cs$ if $\lambda = (\lambda_k)$ is defined by the relation $\lambda_k = (k+1)^{\alpha}$, $\alpha > 1$. In addition, $M = (m_{nk})$ defined by (3.3), where $s < 0$ and $s + t > 0$, is Sr-Sq regular. Therefore we immediately get

Proposition 4.1. *Let $M = (m_{nk})$ be defined by* (3.3), *where $s < 0$ and $s + t > 0$, and $\lambda = (\lambda_k)$ by the relation $\lambda_k = (k+1)^{\alpha}$, $\alpha > 1$. Then Z^r $(r > 1)$ and M are consistent on $m^{\lambda}_{Z^r}$.*

Now we prove the main result of this section.

Theorem 4.2. *The Sr-Sq regular method M defined by* (3.3), *where $s < 0$ and $s + t > 1$, improves $(Z^r)^{\lambda}$-boundedness and is accelerating with respect to Z^r for all sequences from $\widehat{m}^{\lambda}_{Z^r}$ if $\lambda = (\lambda_k)$ is defined by equalities* (3.6), *where $1 < \alpha < r$ and $\alpha < s + t$.*

Proof. It is sufficient to show by Definition 1.5 and Theorem 4.1 that $m^{\lambda}_{Z^r} \subseteq m^{\mu}_{M}$ for some speed $\mu = (\mu_k)$, satisfying the property $\mu_k / \lambda_k \longrightarrow \infty$. To show it, we prove that the conditions of Lemma 4.1 are fulfilled for $P = Z^r$, $B = (\delta_{nk})$, for $M = (m_{nk})$ defined by (3.3), and for $\mu = (\mu_k)$ defined by the relation

$$\mu_k = (k+1)^{s+t}.$$

First we note that conditions (4.1) are satisfied (see [14], p. 62). Further, we can write

$$L = \sum_l \frac{1}{\lambda_l} \left| P_l \Delta \frac{\Delta m_{nl}}{p_l} \right| = L_1 + L_2,$$

where

$$L_1 = \frac{1}{(n+1)^{s+t}} \left| 1 - \frac{2^s - 1}{2^r - 1} \right|$$

and

$$L_2 = \frac{1}{(n+1)^{s+t}} \sum_{l=1}^{\infty} (l+1)^{r-\alpha} \left| \frac{(l+1)^s - l^s}{(l+1)^r - l^r} - \frac{(l+2)^s - (l+1)^s}{(l+2)^r - (l+1)^r} \right|.$$

We subsequently get with the help of the mean-value theorem of Cauchy that

$$L_2 = \frac{|s|}{r(n+1)^{s+t}} \sum_{l=1}^{\infty} (l+1)^{r-\alpha} \left|(l+\theta_l)^{s-r} - (l+1+\theta_{l+1})^{s-r}\right|$$
$$= \frac{|s(r-s)|}{r(n+1)^{s+t}} \sum_{l=1}^{\infty} (l+1)^{r-\alpha} (1+\theta_{l+1}-\theta_l) \left(l+\theta_l+\theta_l^1\right)^{s-r-1}$$
$$= \frac{|s(r-s)|}{r(n+1)^{s+t}} \sum_{l=1}^{\infty} \left(\frac{l+1}{1+\theta_l+\theta_l^1}\right)^r (1+\theta_{l+1}-\theta_l) \left(l+\theta_l+\theta_l^1\right)^{s-1} (l+1)^{-\alpha},$$

where $0 < \theta_l$, $\theta_{l+1} < 1$, and $0 < \theta_l^1 < 2$. As

$$\left(\frac{l+1}{1+\theta_l+\theta_l^1}\right)^r = O(1) \text{ and } 1+\theta_{l+1}-\theta_l < 2,$$

we have

$$L_2 = O(1)(n+1)^{-s-t} \sum_{l=1}^{\infty} (l+1)^{s-\alpha-1} = O_n(1), \tag{4.9}$$

because $s < \alpha$. Therefore $L = O_n(1)$, i.e. condition (4.2) is satisfied.

We can write with the help of the mean-value theorem of Lagrange that

$$\frac{P_l m_{nl}}{p_l \lambda_l} < \frac{(l+1)^{r-\alpha}}{(l+1)^r - l^r} = \frac{(l+1)^{r-\alpha}}{r(l+\theta_l)^{r-1}} = O(1)(l+1)^{1-\alpha} = o(1),$$

since $0 \le m_{nk} < 1$ and $\alpha > 1$. Thus condition (4.3) is satisfied.

As now $g_{nl} = m_{nl}$ and $g_l = 1$, conditions (4.4)–(4.6) are fulfilled. Further, we write

$$T = \mu_n \sum_l \frac{1}{\lambda_l} \left| P_l \Delta \frac{\Delta(g_{nl}-g_l)}{p_l} \right| = \mu_n L = \mu_n L_1 + \mu_n L_2.$$

As $\mu_n L_1 = O(1)$ and relation (4.9) implies

$$\mu_n L_2 = O(1) \sum_{l=1}^{\infty} (l+1)^{s-\alpha-1} = O(1),$$

since $s < \alpha$, we get $T = O(1)$, i.e., condition (4.7) is fulfilled. Consequently, $m^{\lambda}_{Z^r} \subseteq m^{\mu}_{M}$ by Lemma 4.1. This completes the proof.

5. SOME REMARKS ON INCREASING THE ORDER OF APPROXIMATION OF FOURIER EXPANSIONS BY REGULAR NONTRIANGULAR MATRIX METHODS

Let X be a Banach space with norm $\| \circ \|$, and $c(X)$, $cs(X)$, and $c_A(X)$ be the spaces of convergent sequences, convergent series, and A-summable sequences, respectively. Moreover, let

$$m^{\lambda}(X) = \{x = (x_k) \mid x_k \in X, \exists \lim x_k = \xi, \lambda_k \| x_k - \xi \| = O(1)\},$$

$$bs^{\lambda}(X) = \{x = (x_k) \mid x_k \in X, (X_n) \in m^{\lambda}(X), \text{ where } X_n = \sum_{k=0}^{n} x_k\},$$

$$m_A^{\lambda}(X) = \{x = (x_k) \mid x_k \in X, \exists \lim_n A_n x = \xi, \lambda_n \| A_n x - \xi \| = O(1)\}.$$

Remark 5.1. All results of this paper are valid if scalar-valued sequences or sequence sets are replaced by corresponding X-valued sequences or sequence spaces (see [14], pp. 58–59; [4], p. 139).

Considering Remark 5.1, we can use the results of our paper for increasing the order of approximation of Fourier expansions and Z^r-means of Fourier expansions in Banach spaces. We assume that a total sequence of mutually orthogonal continuous projections (T_k) $(k = 0, 1, \ldots)$ on X exists, i.e., T_k is a bounded linear operator of X into itself, $T_k x = 0$ for all k implies $x = 0$, and $T_j T_k = \delta_{jk} T_k$. Then we may associate formal Fourier expansion

$$x \sim \sum_k T_k x$$

to each x from X. It is known (see [11], pp. 74–75, 85–86) that the sequence of projections (T_k) exists if, for example, $X = C_{2\pi}$ is the set of all 2π-periodic functions, which are uniformly continuous and bounded on $\mathbf{R}$, $X = L^p_{2\pi}$ $(1 \le p < \infty)$ is the set of all 2π-periodic functions, Lebesgue integrable to the pth power over $(-\pi, \pi)$ or $X = L^p(\mathbf{R})$ $(1 \le p < \infty)$ is the set of all functions, Lebesgue integrable to the pth power over $\mathbf{R}$.

Let $M = (m_{nk})$ be defined by (3.3), where $s < 0$ and $s + t > 0$. Then we put

$$M_n x = T_0 x + \sum_{k=1}^{\infty} \left[1 - \frac{k^s}{(n+1)^{s+t}}\right] T_k x \tag{5.1}$$

for every $x \in X$ if the series in (5.1) are convergent. Using Remark 5.1, we immediately get the following result from Corollary 3.2.

Corollary 5.1. *Let M_n be defined by* (5.1) *and $x_0 \in X$. If the estimation*

$$m < (n+1)^{\alpha} \| \sum_{k=0}^{n} T_k x_0 - x_0 \| < K$$

holds for some numbers $m, K > 0$, and for $0 < \alpha < s + t$, then

$$(n+1)^{s+t} \| M_n x_0 - x_0 \| = O(1), \tag{5.2}$$

i.e., M-means increase the order of approximation of Fourier expansion of x_0.

Let M be defined by (3.7), where N is a positive integer and c, β, $\sigma > 0$. Then we set

$$M_n x = \sum_{k=0}^{n+N} \left[1 - \frac{k}{(n+1)^{cn+\beta+\sigma}}\right] T_k x \tag{5.3}$$

for every $x \in X$. Using Remark 5.1, we immediately get the following result from Proposition 3.2.

Corollary 5.2. *Let M_n be defined by* (5.3) *and $x_0 \in X$. If the estimation*

$$m < (n+1)^{cn} \| \sum_{k=0}^{n} T_k x_0 - x_0 \| < K$$

holds for $c > 0$ and for some numbers $m, K > 0$, then

$$(n+1)^{cn+\beta} \| M_n x_0 - x_0 \| = O(1)$$

for $0 < \beta \le c$ and $\sigma > 1$, i.e., M-means increase the order of approximation of Fourier expansion of x_0.

Now we write

$$Z_n^r x = \sum_{k=0}^{n} \left[1 - \left(\frac{k}{n+1} \right)^r \right] T_k x \tag{5.4}$$

for Zygmund method Z^r and for every $x \in X$. Using Remark 5.1, Proposition 4.1, and Theorem 4.2, we immediately get

Corollary 5.3. *Let M_n and Z_n^r be defined by* (5.1) *and* (5.4), *respectively*, *and $x_0 \in X$. If the estimation*

$$m < (n+1)^{\alpha} \parallel Z_n^r x_0 - x_0 \parallel < K$$

holds for $\alpha \in (1, r)$ and for some numbers $m, K > 0$, then estimation (5.2) *for $s + t > \alpha$ and $s < 0$ also holds*, *i.e.*, *M-means increase the order of approximation of Z^r-means of x_0.*

Note that several comparison theorems for the orders of approximation of Fourier expansions, similar to Corollary 5.3, were proved in [14,18,19]. However, in all above-mentioned results the order of approximation of Fourier expansions by M-means was not higher than the corresponding order of approximation by Zygmund means.

ACKNOWLEDGEMENT

I am grateful to Professor A. Tali for valuable advice.

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Koonduvuse kiirendamisest regulaarsete maatriksmenetlustega

Ants Aasma

On uuritud jadade ja ridade koonduvuse kiirendamist ning parandamist mittekolmnurksete regulaarsete maatriksmenetlustega. Varem on tõestatud, et kolmnurkne regulaarne maatriksmenetlus ei saa parandada ei koonduvust, λ-koonduvust (λ on monotoonselt kasvav positiivne tõkestamata jada) ega ka λ-tõkestatust. Nüüd on näidatud, et ka mittekolmnurkse maatriksiga defineeritud regulaarne jada-jada-teisendus ei saa parandada koonduvust, kuid võib siiski parandada λ-tõkestatust. Veel on näidatud, et leidub mittekolmnurkse maatriksiga defineeritud regulaarne rida-jada-teisendus, mis parandab koonduvust või A^{λ}-tõkestatust, kus A on regulaarne rida-jada-teisendusega antud Zygmundi menetlus. Veel on uuritud koonduvuse kiirendamist teatavatel koonduvate ridade ja jadade alamhulkadel.