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Tarski's system of geometry and betweenness geometry with the group of movements

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Abstract. Recently, in a paper by Tarski and Givant (*Bull. Symbolic Logic*, 1999, **5**, 175–214), Tarski's system of geometry was revived. The system originated in Tarski's lectures of 1926–27, but was published in the 1950s–60s and in 1983. On the other hand, the author's papers of 2005–07 revived the betweenness geometry, initiated by the Estonian scientists Sarv, Nuut, and Humal in the 1930s, and by the author in 1964. It is established here that Tarski's system of geometry is essentially the same as Euclidean continuous betweenness geometry with a group of movements.

Key words: Tarski's system of geometry, betweenness geometry, group of movements.

1. INTRODUCTION

The recent paper [1] by Alfred Tarski (1902–83) and Steven Givant can be considered as revival of Tarski's system of geometry. Let us cite ([1], pp. 175, 176): "In his 1926–27 lectures at the University of Warsaw, Alfred Tarski gave an axiomatic development of elementary Euclidean geometry ... . [...] Substantial simplifications in Tarski's axiom system and the development of geometry based on them were obtained by Tarski and his students during the period 1955–65. All of these various results were described in Tarski [2–4] and Gupta [5]." "[Section 2] outlines the evolution of Tarski's set of axioms from the original 1926–27 version to the final versions used by Szmielew and Tarski in their unpublished manuscript and by Schwabhäuser–Szmielew–Tarski [6]."

In Tarski's system of axioms the only primitive geometrical objects are *points*: a, b, c, ... There are two primitive geometrical (that is non-logical) notions: the ternary relation B of "soft betweenness" and quaternary relation $\equiv$ of "equidistance" or "congruence of segments". The axioms are: the reflexivity, transitivity, and identity axioms for equidistance; the axiom of segment construction $\exists x(B(qax) \wedge ax \equiv bc)$ a.o.; reflexivity, symmetry, inner and outer transitivity axioms for betweenness; the axiom of continuity, and some others.

In 1904, Veblen [7] initiated "betweenness geometry" with the same primitive objects – *points*, and the only primitive notion – *strict betweenness*; the name *betweenness geometry* was given afterwards by Hashimoto [8].

This standpoint was developed further in Estonia, first by Nuut [9] in 1929 (for dimension one, as a geometrical foundation of real numbers). In 1931 Sarv [10] proposed a self-dependent axiomatics for the betweenness relation in the arbitrary dimension n, extending Veblen's approach so that all axioms of connection, including also those concerning lines, planes, etc., became consequences. This self-dependent axiomatics was simplified and then perfected by Nuut [11] and Tudeberg (from 1936 Humal) [12]. As a result, an extremely simple axiomatics was worked out for n-dimensional geometry using only two basic concepts: "point" and "between".

The author of the present paper developed in [13] a comprehensive theory of *betweenness geometry*, based on this axiomatics (see also [14]). In [13] the notions "collineation" and "flag" are defined in betweenness geometry, and also the notion "group of collineations" is introduced by appropriate axioms. Using a complementary axiom, this group is turned into the "group of motions". These axioms say that for two flags there exists one and only collineation in this group, which transports one flag into the other.

The purpose of the present paper is to show that the axiomatics in [13] gives the foundation of *absolute geometry*, the common part of Euclidean and non-Euclidean hyperbolic (i.e. Lobachevski–Bolyai) geometry, and that by adding a form of Euclid's axiom, one obtains Tarski's system of geometry.

2. TARSKI'S SYSTEM OF GEOMETRY

Recall that the original form of this system was constructed in 1926–27. It appeared in [4], which was submitted for publication in 1940, but appeared only in 1967 in a restricted number of copies. This paper (which is really a short monograph) is reproduced on pp. 289–346 of *Collected Papers* [15], volume 4.

All the axioms are formulated in terms of two primitive notions, the ternary relation of soft betweenness, B, and the quaternary relation of equidistance, $\equiv$, among points of a geometrical space. The original set consists of 20 axioms for 2-dimensional Euclidean geometry. The possibility of modifying the dimension axioms in order to obtain an axiom set for n-dimensional geometry is briefly mentioned.

The next version of the axiom set appeared in [2]. A rather substantial simplification of the axiom set was obtained in 1956–57 as a result of joint efforts by Eva Kallin, Scott Taylor, and Tarski, and discussed by Tarski in his lecture course on the foundations of geometry given at the University of California, Berkeley. It appeared in print in [3].

The last simplification obtained so far is due to Gupta [5]. It is shown in this work that some axioms can be derived from the remaining axioms. As a result, the axiom set consists of the following 11 axioms (see [1], also [6], 1.2):

Reflexivity, Transitivity, and Identity Axioms for Equidistance:

Ax.1. $ab \equiv ba$, Ax.2. $(ab \equiv pq) \wedge (ab \equiv rs) \rightarrow pq \equiv rs$, Ax.3 $ab \equiv cc \rightarrow a = b$.

Axiom of Segment Construction:

Ax.4 $\exists x(B(qax) \wedge (ax \equiv bc))$.

Five-Segment Axiom:

Ax.5 $$[(a \neq b) \wedge B(abc) \wedge B(a'b'c') \wedge (ab \equiv a'b') \wedge (bc \equiv b'c') \wedge (ad \equiv a'd') \wedge (bd \equiv b'd')] \rightarrow (cd \equiv c'd').$$

Inner Transitivity Axiom for Betweenness:

Ax.6 $B(abd) \wedge B(bcd) \rightarrow B(abc)$.

Inner Form of Pasch Axiom:

Ax.7 $B(apc) \wedge B(bqc) \rightarrow \exists x[B(qxa) \wedge B(pxb)]$.

Lower n-Dimensional Axiom for $n = 3, 4, \ldots$:

Ax.8 $^{(n)}$ $\exists a \exists b \exists c \exists p_1 \exists p_2 \ldots \exists p_{n-1}$

$$\Big[\bigwedge_{1 \leq i < j < n} p_i \neq p_j \wedge \bigwedge_{i=2}^{n-1}(ap_1 \equiv ap_i) \wedge \bigwedge_{i=2}^{n-1}(bp_1 \equiv bp_i) \wedge \bigwedge_{i=2}^{n-1}(cp_1 \equiv cp_i) \wedge [\neg B(abc) \wedge \neg B(bca) \wedge \neg B(cab)]\Big].$$

Upper n-Dimensional Axiom for $n = 2, 3, \ldots$:

Ax.9$^{(n)}$

$$\Big[\bigwedge_{1 \leq i < j < n} p_i \neq p_j \wedge \bigwedge_{i=2}^{n-1}(ap_1 \equiv ap_i) \wedge \bigwedge_{i=2}^{n-1}(bp_1 \equiv bp_i) \wedge \bigwedge_{i=2}^{n-1}(cp_1 \equiv cp_i)\Big] \rightarrow [B(abc) \vee B(bca) \vee B(cab)].$$

Axiom of Continuity:

Ax.10 $$\exists a \forall x \forall y[(x \in X) \wedge (y \in Y) \rightarrow B(axy)] \rightarrow \exists b \forall x \forall y[(x \in X) \wedge (y \in Y) \rightarrow B(xby)].$$

A Form of Euclid's Axiom:

Ax.11 $B(adt) \wedge B(bdc) \wedge (a \neq d) \rightarrow \exists x \exists y[B(abx) \wedge B(acy) \wedge B(ytx)]$.

This axiom set is denoted by $EG^{(n)}$ in [1]. In [6] the lower and upper dimension axioms differ from those in $EG^{(n)}$; this axiom set is denoted by $EH^{(n)}$.

In [1] it is noted that Tarski's system of foundations of geometry has a number of distinctive features, in which it differs from most, if not all, systems of foundation of Euclidean geometry that are known from the literature. Of the earlier systems, probably the two closest in spirit to the present one are those of Pieri [16] and Veblen [7].

3. CONTINUOUS BETWEENNESS GEOMETRY WITH MOVEMENTS

Veblen's approach, based only on the axioms for betweenness, was developed further in 1930–64 by the Estonian scholars Nuut, Sarv, Humal, and Lumiste (see [9–13]). The most complete studies, [10] and [13], were published in Estonian, and are therefore not widely available. Betweenness geometry has been revived in the author's papers [17,18].

The set of axioms for betweenness geometry is as follows [17]:

$$\mathbf{B1}\text{: } (a \neq b) \Rightarrow \exists c, (abc); \ \ \mathbf{B2}\text{: } (abc) = (cba);$$

$$\mathbf{B3}\text{: } (abc) \Rightarrow \neg(acb); \ \ \mathbf{B4}\text{: } \langle abc\rangle \wedge [abd] \Rightarrow [cda]; \ \ \mathbf{B5}\text{: } (a \neq b) \Rightarrow \exists c, \neg[abc];$$

$$\mathbf{B6}\text{: } \neg[abc] \wedge (abd) \wedge (bec) \Rightarrow \exists f, ((afc) \wedge (def)),$$

where $(abc) = B(abc) \wedge (a \neq b \neq c \neq a)$ means that b is *strictly between* a and c, $\langle abc\rangle = (abc) \vee (bca) \vee (cab)$ means that the triplet (a, b, c) is *correct*, and $[abc] = \langle abc\rangle \vee (a = b) \vee (b = c) \vee (c = a)$ means that (a, b, c) is *collinear*.

Betweenness geometry as a system of consequences from these axioms is developed in [10,13,14,17,18]. In [18] coordinates were introduced, algebraic extension to ordered projective geometry was given, and collineations were investigated.

In particular, the subset $\{x|(axb)\}$ is called an *interval ab* with ends a and b. If in an interval one has (axy), then it is said that x precedes y. This turns the interval into an *ordered* point-set.

If a, b, c are non-collinear, then they are said to be *vertices*, the intervals bc, ca, ab *sides* (opposite to a, b, c, respectively) of the *triangle* $\triangle abc$, which is considered as the union of all of them.

In [13] it is proved (Theorem 13) that

For a triangle $\triangle abc$, *the subset* $\{x|\exists y, (byc) \wedge (axy)\}$ *does not depend on the reordering of vertices* a, b, c.

It is natural to call this subset the *interior* of the triangle $\triangle abc$. Here any permutation of a, b, c is admissible.

A betweenness geometry is said to be *continuous* (see [13]) if the following axiom is satisfied.

Axiom of Continuity:

If the points of an interval ab *are divided into two classes so that every point* x *of the first class precedes every point* y *of the second class*, *then there exists a point* z *which is either the last point of the first class*, *or the first point of the second class*,

or, by means of only the betweenness relation (taking along also some concepts of the set theory):

$\mathbf{B7}$: $\{(ab = X \cup Y) \wedge [(x \in X) \wedge (y \in Y) \rightarrow (axy)]\}$
$\rightarrow \exists\{[(z \in X) \wedge (axz)] \vee [(z \in Y) \wedge (azy)]\}$.

Among collineations movements can be introduced in betweenness geometry, following [10] and [13].

First the configuration of a *flag* must be defined. Let us start with the *line* $L_{ab} = \{x|[xab]\}$, with $a \neq b$, the *interval* $ab = \{x|(axb)\}$, and the *half-line* $(ab = \{x|(axb) \vee (x = b) \vee (abx)\}$ of the line L_{ab}, containing b, with initial point a. In [13] and [17] it is proved that if two different points c, d belong to a line L_{ab}, then $L_{cd} = L_{ab}$, i.e. the line is uniquely defined by any two of its different points.

Next the *plane* will be defined by $P_{abc} = Q_a \cup Q_b \cup Q_c$ with non-collinear a, b, c, where $Q_a = L_{ab} \cup L_{ac} \bigcup_{x \in bc} L_{ax}$. It is obvious that the plane P_{abc} does not depend on the reordering of the points a, b, c. In [13] it is proved (Theorem 18) that if three non-collinear points d, e, f belong to the plane P_{abc}, then $P_{def} = P_{abc}$, i.e. the plane is uniquely defined by any three of its non-collinear points. Moreover, it is established (Theorem 19) that if two different points of a line L_{ab} belong to a plane P_{abc}, then all points of this line belong to this plane.

It is said that two points x, y of the plane are on the same side of this line if there is no point of this line between x and y, but they are on different sides of this line if there exists z of this line so that (xzy). In [13] it is proved (Theorem 22) that a line L_{ab} belonging to the plane decomposes all remaining points of this plane into two classes so that every two points of the same class are on the same side of this line, but every two points of different classes are on different sides of this line. These classes are said to be *half-planes*, and this line is considered as their common *boundary*. Here $\{x|\exists y[(y \in L_{ab}) \wedge (cyx)]$ is the half-plane not containing c. The other half-plane of P_{abc}, with the same boundary L_{ab} (i.e. containing c), will be denoted by $(ab|c$.

A betweenness geometry is said to be *two-dimensional* (or *plane geometry*) if there exist three non-collinear points a, b, c and all other points x belong to P_{abc}. For this geometry the *flag* $F = F(abc)$ is defined as the triple $F(abc) = (a, (ab, (ab|c)$.

Any one-to-one map f of a betweenness plane onto itself is said to be a *collineation* if $(abc) \Rightarrow (f(a)f(b)f(c))$, i.e. if the betweenness relation remains valid by f. Then also $[abc] \Rightarrow [f(a)f(b)f(c)]$, i.e. every collinear point-triplet maps into a collinear point-triplet. Hence every line maps into a line; from this stems the term "collineation". It is also clear that every flag maps into a flag, and that all collineations of a betweenness plane form a group.

Now a new axiom for betweenness plane geometry can be formulated.

Axiom of Movements:

$\mathbf{B8}$: *For any two flags F and F' there exists one and only one collineation f so that $F' = f(F)$.*

In this situation f is called the *movement* which transports F into F'. It is clear that all movements form a subgroup of the group of all collineations.

The same axiom and its conclusion can be formulated also for higher dimensions. For instance, for dimension three this is as follows.

Four points a, b, c, d of a plane are called *coplanar*, otherwise not coplanar (or *tetrahedral*). The betweenness geometry is said to be *three-dimensional* if there exist four tetrahedral points a, b, c, d, but all other four points are either tetrahedral, or coplanar. It is said also that then one has the geometry of a 3-*space* S_{abcd}.

In [13] it is proved 1) that the 3-space is uniquely defined by any of its tetrahedral points (Theorem 29), 2) that if two planes in a 3-space have a common point, then they have a common line (which contains all their common points, if these planes are different) (Theorem 30), and 3) that a plane P_{abc} in a 3-space decomposes all remaining points of this 3-space into two classes, so that every two points of the same class are such that there exist no points of the plane between them, but for the points of different classes such a point exists (Theorem 31). These two classes are called the *half*-3-*spaces* with the common boundary P_{abc}. If this half-3-space contains d, then it will be denoted by $(abc|d$.

The *flag* $F = F(abcd)$ in a 3-space is defined as the quadruple

$$F(abcd) = F(a, (ab, (ab|c, (abc|d\,).$$

Now the *Axiom of Movement* $\mathbf{B8}$ can be formulated also for the 3-space, and gives the same conclusion.

In the same manner this axiom and its conclusion can be extended also for higher than 3 dimensions.

4. COMPARISON OF THESE TWO SYSTEMS OF GEOMETRY

Let us first try to show that almost all axioms of Tarski's system are valid in continuous betweenness geometry with movements. Here we need a conclusion

$$(*) \qquad B(abd) \wedge (d = a) \rightarrow (a = b)$$

in this system, which is established in [5] (see [1], p. 190). (Note that in [6] this (*) is taken as axiom Ax.6, and axiom Ax.6 above is then proved as a consequence 3.5(1).)

The axioms Ax.6 and Ax.7, formulated by only the soft betweenness relation $B(abc) = (abc) \vee (a = b) \vee (b = c) \vee (c = a)$, can be verified in the following way.

First let us consider Ax.6, which is now

$$[(abd) \vee (a = b) \vee (b = d) \vee (d = a)] \wedge [(bcd) \vee (b = c) \vee (c = d) \vee (d = b)]$$
$$\rightarrow [(abc) \vee (a = b) \vee (b = c) \vee (c = a)].$$

Here $(abd) \wedge (bcd) \rightarrow (abc)$ follows from [17], Lemma 8, (9), if we interchange a and d, and use $\mathbf{B2}$, so that this (9) gives $(bcd) \wedge (abd) \rightarrow (abc)$.

For the other possibilities Ax.6 is also valid; for instance, $(abd) \wedge (c = d) \rightarrow (abc)$, but the other possibilities are either obvious (because, in particular, $(d = a)$ implies $(a = b)$, due to (*)), or lead to contradictions.

Let us consider Ax.7:

$$[(apc) \vee (a = p) \vee (p = c) \vee (c = a)] \wedge [(qcb) \vee (q = c) \vee (c = b) \vee (b = q)]$$

$$\rightarrow \exists x[(axq) \vee (a = x) \vee (x = q) \vee (q = a)] \wedge [(bpx) \vee (b = p) \vee (p = x) \vee (x = b)].$$

Here $(apc) \wedge (qcb) \rightarrow \exists x(axq) \wedge (bpx)$ follows directly from $\mathbf{B6}$. Further, $(apc) \wedge (q = c)$ implies (apq), and hence for $x = p$ implies that $(axq) \wedge (p = x)$. Next, $(apc) \wedge (c = b)$ gives for $x = a$ that $(a = x) \wedge (xpb)$, where due to $\mathbf{B2}$, $(xpb) = (bpx)$, but $(apc) \wedge (b = q)$ implies for $x = b$ that $(x = q) \wedge (x = b)$.

If in the first premise one takes $(a = p)$, then the conclusion is true by $x = a = p$. The premise $(p = c) \wedge (qcb)$ gives for $x = q$ that $(x = q) \wedge (bpq)$, where due to $\mathbf{B2}$ and $p = c$ here $(bpq) = (qpb) = (qcb)$. For the premise $(c = a) \wedge (qcb)$, (*) must be used, due to which the first premise of Ax.7 implies $a = p$; now the consequence is satisfied by $x = a = p$.

For $(p = c) \wedge (q = c)$, $x = c = p = q$ fits; for $(p = c) \wedge (c = b)$ there is $b = p$, and here $x = q$ fits; for $(p = c) \wedge (b = q)$, $x = q$ fits.

Finally, for $(c = a) \wedge (q = c)$, (*) gives again that $a = p$ and now $x = a = c = p = q$ fits; for $(c = a) \wedge (c = b)$, $x = a = b = c$ fits; for $(c = a) \wedge (b = q)$, again $a = p$ must be used, due to (*), and therefore here $x = a = p$ fits. This finishes the verification of Ax.7 in betweenness geometry.

Further let us consider the axioms for equidistance. First the meaning of equidistance must be defined.

In the betweenness geometry with movements two point-pairs a, b and p, q are said to be *equidistant*, and are then denoted $ab \equiv pq$, if there exists a movement which transports ab into pq.

Then the inverse movement transports pq into ab, so that also $pq \equiv ab$. If $ab \equiv rs$ due to another movement, then these movements together transport pq into rs, hence $pq \equiv rs$. This shows that Ax.2 is here valid. Also Ax.3 is valid because every movement is one-to-one map.

The verification of Ax.1 is more complicated: $ab \equiv ba$. This can be made, following [13], by including the Axiom of Continuity $\mathbf{B7}$.

Let us consider the movement f, defined by $f(F(abc)) = F'(bac)$. Then $f(a) = b$ and in order to verify Ax.1, we need to show that $f(b) = a$.

Denoting $f(b) = a_1$, one has three possibilities: (aa_1b), (a_1ab), $a_1 = a$. For the first two of them it is needed to show that each leads to a contradiction.

Let us start with the first possibility: (aa_1b), and denote $f(a_1) = b_1$. Here (aa_1b) gives after f that $(bb_1a_1) = (a_1b_1b)$, due to $\mathbf{B2}$. Our aim is to show that here is a contradiction.

Using once more the movement f, we see that the flag $F(abc)$ moves by $g = f \circ f$ into $F_1(a_1bc)$. This shows that b_1 belongs to the half-line $(a_1b$, thus $(b_1 = b) \vee (a_1b_1b) \vee (a_1bb_1)$. Here the last component contradicts, due to $\mathbf{B3}$, (a_1b_1b) above.

The first component is also impossible, because then the movement g would preserve the flag $F^*(bac)$ and must be, due to $\mathbf{B8}$, the unit movement, but this would be a contradiction.

To accomplish the verification of Ax.1, one has to show that the middle component (a_1b_1b) also leads to a contradiction. Here the Axiom of Continuity $\mathbf{B7}$ is needed.

Denoting $g(a_1) = a_2$ and $g(b_1) = b_2$, one may conclude from $(aa_1b) \wedge (a_1b_1b)$ that $(a_1a_2b_1) \wedge (a_2b_2b_1)$, because g is a collineation. Now the conclusions (8) and (9) of [13] can be used; they give that (aa_1a_2) holds. Further, denoting $g(a_2) = a_3$ and $g(b_2) = b_3$, one may conclude that $(a_2a_3b_2) \wedge (a_3b_3b_2)$, which implies (aa_2a_3) as above. This process can be continued. As a result, one has that in ab there is an infinite sequence $\{a_1, a_2, a_3, ..., a_k, ...\}$ so that $(aa_{k-1}a_k)$ for all $k = 2, 3, \dots$.

Now one can divide the interval ab into $ab = X \cup Y$ so that X consists of the points x, each of which precedes some point a_k of the above sequence, and Y of the points y, each of which follows all points of this sequence. Here obviously (axy). Due to $\mathbf{B7}$, $\exists\{[(z \in X) \wedge (axz)] \vee [(z \in Y) \wedge (azy)]\}$. But (axz) is impossible because $(z \in X)$ means that there exists a_k so that (aza_k), where (aa_ka_{k+1}) and thus $a_k \in X$; taking now $a_k = x$, one has (azx) which contradicts (axz), due to $\mathbf{B3}$.

Hence (azy) and thus $z \in Y$. Denoting now $g(z) = z'$, one has the possibilities $(az'z) \vee (azz') \vee (z' = z)$. It remains to show that each of them leads to a contradiction.

For $(az'z)$ it is impossible that $z' \in Y$, because then there would be (azz') which contradicts $(az'z)$, due to $\mathbf{B7}$. Hence $z' \in X$ and there exists a_n so that $(az'a_n)$, thus for some k one has $(a_{k-1}z'a_k)$. Considering now the inverse movement g^{-1}, one has $g^{-1}(a_n) = a_{n-1}$, $g^{-1}(z') = z$, and so $(a_{k-1}z'a_k)$ leads to $(a_{k-2}za_{k-1})$, which contradicts $z \in Y$.

For the second possibility (azz'), let us denote $g^{-1}(z) = z^*$ and show that here (z^*zb). Indeed, one has first, due to [13], Lemma 8, that $(aa_1z) \wedge (azz') \rightarrow (a_1zz')$, then $(azb) \wedge (g(a) = a_1) \wedge (g(z) = z') \wedge (g(b) = b_1) \rightarrow (a_1z'b_1)$, and due to the same Lemma 8 $(a_1zz') \wedge (a_1z'b_1) \rightarrow (zz'b_1)$, thus $(zz'b_1) \wedge (g^{-1}(z) = z^*) \wedge (g^{-1}(z') = z) \wedge (g^{-1}(b_1) = b) \rightarrow (z^*zb)$.

This means that z^* belongs to the half-line $(za$, hence $(az^*z) \vee (z^*az) \vee (z^* = a)$. Here (az^*z) says that $z^* \in X$ and this, as above, leads to a contradiction. Also $(z^*az) \vee (z^* = a)$ leads to a contradiction, because after using g one would have $(za_1z') \vee (z = a_1)$, which is impossible.

It remains to consider the third possibility $z' = z$. Then g would preserve the point z, the half-line $(za$, and the half-plane $(za|c$, thus the flag $F^*(zac)$. Hence g would be the unit movement, but this contradicts $(g(a) = a_1) \wedge (aa_1b)$.

The result is that all possibilities above, except $a_1 = a$, lead to contradictions. Hence $f(b) = a$, but this verifies that f transports ab into ba, and that Ax.1 is valid.

The movement which transports ab into ba is said to be the *reversion* of the interval ab.

Similarly, one may introduce the reversion of an angle $\angle abc$. Let us consider the movement f which transports the flag $F(b, (ba, (ba|c)$ into the flag $F'(b, (bc, (bc|a)$. Here $ba \equiv bc$, and f will be the reversion of the interval ac. For the angle $\angle abc$ this f is called its *reversion*.

Next let us consider the axioms Ax.4 and Ax.5., which are dealing with segments. For Ax.4 one can take a point p so that (qap), and then a point d which does not belong to the lines L_{qa} and L_{bc}. Considering now the flags $F(a, (ap, (ap|d)$ and $F'(b, (bc, (bc|d)$, and using the movement f defined by $f(F') = F$, one can take $x = f(c)$, which makes Ax.4 valid.

For Ax.5 one can use the movement f^* which transports the flag $F(a, (ab, (ab|d)$ into the flag $F'(a', (a'b', (a'b'|d^*)$, where d and d^* belong to different half-planes with the boundary $L_{a'b'}$. Then, due to $ab \equiv a'b'$, there is $f^*(b) = b'$, and, due to $bc \equiv b'c'$, there is $f^*(c) = c'$. Moreover, f^* transports d into a point d'' so that $a'd'' \equiv ad \equiv a'd'$ and $b'd'' \equiv bd \equiv b'd'$.

Let us consider the reversion g of the angle $\angle d'a'd''$. It transports d' into a point of the half-line $(a'd''$, which due to $da \equiv d'a'$, $da \equiv d''a'$ coincides with d''. (Here Theorem 45 of [13] must be used, which states that in a half-line $(a'p$ there exists one and only one point b' such that $a'b' \equiv ab$, where ab is a given interval.) Since g is involutory, it transports d'' back to d', so that g is also the reversion of $d'd''$. The same argument shows that the reversion of the angle $\angle d'b'd''$ is also the reversion of $d'd''$, thus is g. Hence g interchanges d' and d'', preserving a', b', and c'. It follows that $f^* \circ g^{-1}$ transports a, b, c, d into a', b', c', d', respectively, hence $cd \equiv c'd'$, as is needed in Ax.5.

The axioms Ax.10 and Ax.11 are formulated by means of only the soft betweenness relation $B(abc)$. In Ax.10 also the concepts of the set theory are used and it is obvious that Ax.10 follows from $\mathbf{B7}$.

To obtain Ax.11, one must add one more axiom to the axioms $\mathbf{B1}$–$\mathbf{B8}$ of the continuous betweenness geometry with movements.

A Form of Euclid's Axiom:

$$\mathbf{B9}\text{: } (adt) \wedge (bdc) \wedge (a \neq d) \to \exists x \exists y[(abx) \wedge (acy) \wedge (ytx)].$$

So one obtains the *Euclidean continuous betweenness geometry with movements*.

Finally it remains to consider the n-dimensional axioms Ax.8$^{(n)}$ and Ax.9$^{(n)}$.

In betweenness geometry the dimension is introduced by a definition. The simplest way for this is as follows.

In [17] it is established that a betweenness geometry is the same as an ordered join geometry, as developed in [19]. There for a subset S of points its linear hull is defined as follows. A subset S is called *convex* if $S \supset z, y$ implies $S \supset zy$. A convex set S for which $S \supset z, y$ implies $S \supset (zy$ is called a *linear set*. The least linear set which contains a given set S is called the *linear hull* of S and denoted by $< S >$.

If $S = \{a_1, ..., a_{n+1}\}$ is a set of $n + 1$ different points, then these are said to be *linearly independent* if no fewer than $n + 1$ of them generate this S (in the sense that S is their linear hull). Then they form a *basis* of $< S >$ and n is called the *dimension* of $< S >$, which is then called an *n-dimensional space*.

Note that Sarv in [10] introduces the $(n{+}1)$-*simplex* recursively: The interval ab is a 2-simplex, the triangle $\triangle abc$ is a 3-simplex, a *tetrahedron*, defined by 4 points which are not coplanar, is a 4-simplex, ... , an $(n + 1)$-simplex is defined by $n + 1$ points which are not in an $(n{-}1)$-dimensional space. Here the n-dimensional space is the set of points of all lines which connect the points of a face of an $(n + 1)$-simplex, as an m-simplex, with the points of the opposite face, as an $(n - m - 1)$-simplex; $0 \leq m \geq n - 1$.

In Tarski's original system of 1926–27 the axioms of dimension were given only for dimension 2 and they coincide with Sarv's corresponding definitions (see [6], pp. 22–23).

For higher dimension n, Ax.$8^{(n)}$ and Ax.$9^{(n)}$ are reformulated by means of only the betweenness relation in [6], p. 119, and then they coincide also with Sarv's definition.

The result of the above discussion can be formulated as a statement:

In the Euclidean continuous betweenness geometry with the group of movements all axioms of Tarski's system are valid.

Since in [10] and [6] the development in both geometries led to the Cartesian space over the Pythagorian ordered field, a more precise statement can be formulated:

The Euclidean continuous betweenness geometry with the group of movements coincides with Tarski's system of geometry.

Note that if in Tarski's system Ax.11 (A Form of Euclid's Axiom) is replaced by the Axiom of Hyperbolic Parallels, then this system will give rise to hyperbolic geometry. A variant of its realization is presented by W. Schwabhäuser in Part II of [6]. The same can be made, of course, also in the continuous betweenness geometry with the group of movements.

A complete presentation of hyperbolic geometry by means of analytic methods is given in Nuut [20]. Nuut's investigations into hyperbolic universe geometry are analysed in [21,22].

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Tarski geomeetria süsteem ja liikumisrühmaga vahelsusgeomeetria

Ülo Lumiste

Ühes hiljutises, 1999. aasta artiklis on uuesti elustatud Tarski geomeetria süsteem, mis on alguse saanud Tarski loengutest aastail 1926–1927, avaldatud aastail 1950–1960 ja seejärel 1983. aastal. Teisalt on autori artiklites aastail 2005–2007 uuesti elustatud vahelsusgeomeetria, mis on alguse saanud eesti teadlaste J. Sarve, J. Nuudi ja A. Humala 1930. aastate töödest ning leidnud käsitlemist autori publikatsioonides 1964. aastal. Käesolevas artiklis on tõestatud, et Tarski geomeetria süsteem on tegelikult sama mis eukleidiline liikumisrühmaga pidev vahelsusgeomeetria.