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Galbed algebras and their sectional representation

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Abstract. In this paper we prove that the unitization $A \times \mathbb{K}$ of a topological algebra A is (α_n)-galbed if and only if A is (α_n)-galbed. We also find sufficient conditions under which a unital strongly galbed algebra can be represented as a subalgebra of some section algebra.

Key words: galbed algebras, sectional representation, representations of topological algebras.

1. INTRODUCTION

Let $\mathbb{C}$ be the field of complex numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{K}$ either $\mathbb{C}$ or $\mathbb{R}$, and $\mathbb{N} = \{0, 1, 2, ...\}$. From the definition of a topological vector space E it follows that for each neighbourhood O of zero in E, a fixed $n \in \mathbb{N}$ and fixed numbers $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ there exists a neigbourhood U of zero in E such that

$$\sum_{k=0}^{n} \lambda_k U \subset O.$$

By a *topological algebra* we mean a topological vector space A over $\mathbb{K}$, which is also an associative algebra over $\mathbb{K}$ such that the multiplication in A is separately continuous[1]. As usual, l^0 denotes the set of all sequences (α_n) of elements of $\mathbb{K}$, in which there is only a finite number of elements which are different from zero. By l^1 we denote the set of all sequences (α_n) of elements of $\mathbb{K}$ for which the series

$$\sum_{k=0}^{\infty} |\, \alpha_k \,|$$

[1] It means that for every $a \in A$ and every neighbourhood O of zero in A there exists a neighbourhood U of zero in A such that $aU, Ua \subset O$.

converges. Let $l := l^1 \backslash l^0$. A topological algebra (or a topological vector space) A is called a *galbed algebra* (a *galbed space*) if there exists a sequence $(\alpha_n) \in l$ such that for every neighbourhood O of zero in A there exists a neighbourhood U of zero in A such that

$$\left\{ \sum_{k=0}^{n} \alpha_k a_k : a_0, \ldots, a_n \in U \right\} \subset O$$

for each $n \in \mathbb{N}$. If we have already specified such $(\alpha_n) \in l$ which makes A a galbed algebra (a galbed space), then we call A an (α_n)-*galbed algebra* (an (α_n)-*galbed space*). Moreover, if $\alpha_0 \neq 0$ and

$$\alpha := \inf_{n>0} |\alpha_n|^{1/n} > 0,$$

then A is called a *strongly galbed algebra* (a *strongly galbed space*). A well-known example of a strongly galbed algebra (space) is an *exponentially galbed algebra* (an *exponentially galbed space*) A in which for every neighbourhood O of zero in A there exists a neighbourhood U of zero in A such that

$$\left\{ \sum_{k=0}^{n} \frac{a_k}{2^k} : a_0, \ldots, a_n \in U \right\} \subset O$$

for each $n \in \mathbb{N}$.

The terms "galbed space", "exponentially galbed space", and "exponentially galbed algebra" were introduced by Turpin (see [1–5]). The terms "galbed algebra" and "strongly galbed algebra" were introduced by Mati Abel and Mart Abel (see [6–12]). The present paper is based on Veiko Lehto's bachelor's thesis [13]. The need for writing this paper arises from the fact that [13] was written in Estonian, but the results in [13] might be also of interest as well as useful for readers, whose Estonian is not good enough to follow the proofs. In the present paper several results of Mart Abel and Mati Abel, known for exponentially galbed algebras, are generalized for the class of galbed algebras (in part 1) or for the class of strongly galbed algebras (in part 2).

2. GALBED ALGEBRAS AND THEIR UNITIZATION

2.1. Strongly galbed algebras with bounded elements

For every algebra A let $Z(A)$ denote the centre of A. An element $a \in A$ is *bounded* (see [14]) in A if there exists a number $\lambda \in \mathbb{C} \backslash \{0\}$ such that the set

$$\left\{ \left(\frac{a}{\lambda} \right)^n : n \in \mathbb{N} \right\}$$

is bounded in A.

Proposition 1. *Let $(\alpha_n) \in l$ and I be a two-sided ideal in a topological algebra A. If A is an (α_n)-galbed algebra with bounded elements, then both A/I and $Z(A/I)$ are (α_n)-galbed algebras with bounded elements.*

Proof. Let $(\alpha_n) \in l$ and I be a two-sided ideal in a topological algebra (A, τ). Let A be an (α_n)-galbed algebra with bounded elements, $\pi : A \to A/I$ the canonical homomorphism, and O' a neighbourhood of zero in $(A/I, \tau_I)$, where τ_I stands for the quotient topology defined by the topology τ of A. Since the quotient map π is continuous, $O = \pi^{-1}(O')$ is a neighbourhood of zero in (A, τ).

Since A is (α_n)-galbed, there exists a neighbourhood U of zero in A such that

$$\left\{ \sum_{k=0}^{n} \alpha_k a_k : a_0, \ldots, a_n \in U \right\} \subset O$$

for each $n \in \mathbb{N}$. Take $V' = \pi(U)$, $n \in \mathbb{N}$ and arbitrary elements $x_0, \ldots, x_n \in V'$. Then there exist elements $a_0, \ldots, a_n \in U$ such that $x_i = \pi(a_i)$ for every $i \in \{0, \ldots, n\}$. Notice that

$$\sum_{k=0}^{n} \alpha_k x_k = \sum_{k=0}^{n} \alpha_k \pi(a_k) = \pi\left(\sum_{k=0}^{n} \alpha_k a_k \right).$$

It is clear that

$$\pi\left(\left\{ \sum_{k=0}^{n} \alpha_k a_k : a_0, \ldots, a_n \in U \right\} \right) \subset \pi(O) = \pi(\pi^{-1}(O')) = O'$$

for every $n \in \mathbb{N}$. Hence,

$$\left\{ \sum_{k=0}^{n} \alpha_k x_k : x_0, \ldots, x_n \in V' \right\} = \pi\left(\left\{ \sum_{k=0}^{n} \alpha_k a_k : a_0, \ldots, a_n \in U \right\} \right) \subset O'$$

for every $n \in \mathbb{N}$. Hence, A/I is an (α_n)-galbed algebra.

Let O'' be any neighbourhood of zero in $(Z(A/I), \tau_Z)$, where $\tau_Z = \{T' \cap Z(A/I) : T' \in \tau_I\}$ is the subspace topology on $Z(A/I)$, generated by τ_I. Then there exists a neighbourhood O' of zero in A/I such that $O'' = O' \cap Z(A/I)$. Since A/I is (α_n)-galbed, there exists a neigbourhood V' of zero in A/I such that

$$\left\{ \sum_{k=0}^{n} \alpha_k x_k : x_0, \ldots, x_n \in V' \right\} \subset O'$$

for every $n \in \mathbb{N}$. Fix an arbitrary $n \in \mathbb{N}$. Take $V'' = V' \cap Z(A/I)$ and arbitrary elements $y_0, \ldots, y_n \in V''$. Then

$$\sum_{k=0}^{n} \alpha_k y_k \in Z(A/I)$$

and

$$\left\{ \sum_{k=0}^{n} \alpha_k y_k : y_0, \ldots, y_n \in V'' \right\} \subset \left\{ \sum_{k=0}^{n} \alpha_k x_k : x_0, \ldots, x_n \in V' \right\} \subset O'.$$

Since $n \in \mathbb{N}$ was arbitrary, we get

$$\left\{ \sum_{k=0}^{n} \alpha_k y_k : y_0, \ldots, y_n \in V'' \right\} \subset O' \cap Z(A/I) = O''$$

for every $n \in \mathbb{N}$. Hence, $Z(A/I)$ is an (α_n)-galbed algebra.

Analogously to the second part of the proof of Theorem 2.1 in [6], we get that all elements in A/I and in $Z(A/I)$ are bounded.

Proposition 2. *Let A be a topological algebra and $(\alpha_n) \in l$. The unitization $A \times \mathbb{K}$ of A in the product topology is an (α_n)-galbed algebra if and only if A is an (α_n)-galbed algebra.*

Proof. Let $(\alpha_n) \in l$, A be a topological algebra and O a neighbourhood of zero in $A \times \mathbb{K}$. Then there exist neighbourhoods of zero U_1 in A and V_1 in $\mathbb{K}$ such that $U_1 \times V_1 \subset O$. If A is (α_n)-galbed, then there exist a neighbourhood U_2 of zero in A and a number $\epsilon > 0$ such that

$$\left\{ \sum_{k=0}^{n} \alpha_k a_k : a_0, \ldots, a_n \in U_2 \right\} \subset U_1$$

for every $n \in \mathbb{N}$ and $V_2 = \{\lambda \in \mathbb{K} : |\lambda| < \epsilon\} \subset V_1$. Since $(\alpha_n) \in l$, we have

$$0 \leq \sum_{k=0}^{n} |\alpha_k| < \sum_{k=0}^{\infty} |\alpha_k| < \infty$$

for every $n \in \mathbb{N}$. Let

$$\beta := \sum_{k=0}^{\infty} |\alpha_k| \quad \text{and} \quad V_3 := \frac{1}{\beta} V_2.$$

Now, $U_2 \times V_3$ is a neighbourhood of zero in $A \times \mathbb{K}$. Fix an arbitrary $n \in \mathbb{N}$ and elements $(\alpha_0, \lambda_0), \ldots, (\alpha_n, \lambda_n) \in U_2 \times V_3$. We have

$$\sum_{k=0}^{n} \alpha_k a_k \in U_1$$

and
$$\left|\sum_{k=0}^{n}\alpha_k\lambda_k\right| \leq \sum_{k=0}^{n}|\,\alpha_k\lambda_k\,| \leq \max_{i\in\{0,\ldots,n\}}|\,\lambda_i\,|\sum_{k=0}^{n}|\,\alpha_k\,| < \frac{\epsilon}{\beta}\beta = \epsilon.$$

Hence,

$$\sum_{k=0}^{n}\alpha_k(a_k,\lambda_k) = \left(\sum_{k=0}^{n}\alpha_k a_k, \sum_{k=0}^{n}\alpha_k\lambda_k\right) \in U_1 \times V_2 \subset O.$$

Since $n \in \mathbb{N}$ was arbitrary,

$$\left\{\sum_{k=0}^{n}\alpha_k(a_k,\lambda_k) : (a_0,\lambda_0),\ldots,(a_n,\lambda_n) \in U_2 \times V_3\right\} \subset O$$

for every $n \in \mathbb{N}$, which means that $A \times \mathbb{K}$ is an (α_n)-galbed algebra.

Conversely, let $A \times \mathbb{K}$ be an (α_n)-galbed algebra for some $(\alpha_n) \in l$. Let O be a neighbourhood of zero in A. Take any neighbourhood V of zero in $\mathbb{K}$. Then $O \times V$ is a neighbourhood of zero in $A \times \mathbb{K}$. Hence, there exists a neighbourhood W of zero in $A \times \mathbb{K}$ such that

$$\left\{\sum_{k=0}^{n}\alpha_k(a_k,\lambda_k) : (a_0,\lambda_0),\ldots,(a_n,\lambda_n) \in W\right\} \subset O \times V$$

for every $n \in \mathbb{N}$. Now, there exist a neighbourhood U of zero in A and a neighbourhood V_1 of zero in $\mathbb{K}$ such that $U \times V_1 \subset W$. Fix $n \in \mathbb{N}$ and $(a_0,\lambda_0),\ldots,(a_n,\lambda_n) \in U \times V_1$. Then

$$\left(\sum_{k=0}^{n}\alpha_k a_k, \sum_{k=0}^{n}\alpha_k\lambda_k\right) = \sum_{k=0}^{n}\alpha_k(a_k,\lambda_k) \in O \times V.$$

Hence,
$$\left\{\sum_{k=0}^{n}\alpha_k a_k : a_0,\ldots,a_n \in U\right\} \subset O$$

for every $n \in \mathbb{N}$, which means that A is an (α_n)-galbed algebra.

Proposition 3. *Let A be a topological algebra and $(\alpha_n) \in l$ a sequence such that $\alpha_k \neq 0$ for each $k \in \mathbb{N}$. The unitization $A \times \mathbb{K}$ of A in the product topology is an (α_n)-galbed algebra with bounded elements if and only if A is an (α_n)-galbed algebra with bounded elements.*

Proof. Let $(\alpha_n) \in l$ be a sequence such that $\alpha_k \neq 0$ for every $k \in \mathbb{N}$.

Let A be an (α_n)-galbed algebra with bounded elements. By Proposition 2, $A\times\mathbb{K}$ is also an (α_n)-galbed algebra. Let (a_0,λ_0) be an arbitrary element of $A\times\mathbb{K}$ and W an arbitrary neighbourhood of zero in $A \times \mathbb{K}$. Then there exist a number

$\mu_W > 0$, a balanced neighbourhood O of zero in $A \times \mathbb{K}$, a neighbourhood U of zero in A, and a balanced neighbourhood V of zero in K such that[2] $e_{A\times\mathbb{K}} \subset \mu_W O$ and $U \times V \subset O \subset W$. As A is (α_n)-galbed, there exists a balanced neighbourhood U_1 of zero in A such that

$$\left\{\sum_{k=0}^{n} \alpha_k c_k : c_0, \ldots, c_n \in U_1\right\} \subset U$$

for every $n \in \mathbb{N}$. Since all elements of A are bounded in A, there exist numbers $\lambda_{a_0} \in \mathbb{C}\backslash\{0\}$, $\mu_1 \geq 1$, and $\gamma > 0$ such that

$$\left\{\left(\frac{a_0}{\lambda_{a_0}}\right)^n : n \in \mathbb{N}\right\} \subset \mu_1 U_1 \quad \text{and} \quad \{\lambda : |\lambda| \leq \gamma\} \subset V.$$

Fix arbitrary $m \in \mathbb{N}\backslash\{0\}$ and put

$$\beta := \min\left\{\min_{k\in\{1,\ldots,m\}}\{|\alpha_k|\}, \gamma\right\},$$

$$\mu_2 := \max\{|\lambda_{a_0}|, |\lambda_0|\} \quad \text{and} \quad \mu_0 := \frac{2}{\sqrt[m]{\beta}}\,\mu_2.$$

Let $k \in \{1, \ldots, m\} \subset \mathbb{N}$. Since U_1 is a balanced neighbourhood,

$$\frac{1}{\alpha_k}\binom{m}{k}\left(\frac{a_0}{\mu_0}\right)^k\left(\frac{\lambda_0}{\mu_0}\right)^{m-k} = \frac{\binom{m}{k}\beta}{2^m\alpha_k}\left(\frac{\lambda_{a_0}}{\mu_2}\right)^k\left(\frac{\lambda_0}{\mu_2}\right)^{m-k}\left(\frac{a_0}{\lambda_{a_0}}\right)^k,$$

and

$$\left|\frac{\binom{m}{k}\beta}{2^m\alpha_k}\right|, \left|\left(\frac{\lambda_{a_0}}{\mu_2}\right)^k\right|, \left|\left(\frac{\lambda_0}{\mu_2}\right)^{m-k}\right| \leq 1,$$

we get

$$\frac{1}{\alpha_k}\binom{m}{k}\left(\frac{a_0}{\mu_0}\right)^k\left(\frac{\lambda_0}{\mu_0}\right)^{m-k} \in \mu_1 U_1$$

for each $k \in \{1, \ldots, m\}$. Therefore,

$$\sum_{k=1}^{m}\alpha_k\left(\frac{1}{\alpha_k}\binom{m}{k}\left(\frac{a_0}{\mu_0}\right)^k\left(\frac{\lambda_0}{\mu_0}\right)^{m-k}\right) = \sum_{k=1}^{m}\binom{m}{k}\left(\frac{a_0}{\mu_0}\right)^k\left(\frac{\lambda_0}{\mu_0}\right)^{m-k} \in \mu_1 U.$$

Notice that

$$\left|\left(\frac{\lambda_0}{\mu_0}\right)^m\right| = \left|\left(\frac{\lambda_0}{\mu_2}\right)^m\frac{\beta}{2^m}\right| = \left|\left(\frac{\lambda_0}{\mu_2}\right)\right|^m\left|\frac{\beta}{2^m}\right| \leq 1 \cdot \frac{\gamma}{2^m} < \gamma.$$

[2] Here $e_{A\times\mathbb{K}} = (\theta_A, 1)$ is the unit element of $A \times \mathbb{K}$.

Since V is balanced, we also have

$$\Big(\frac{\lambda_0}{\mu_o}\Big)^m \in V = \mu_1\Big(\frac{1}{\mu_1}V\Big) \subset \mu_1 V.$$

Therefore,

$$\left(\frac{(a_0,\lambda_0)}{\mu_0}\right)^m = \left(\sum_{k=1}^{m}\binom{m}{k}\Big(\frac{a_0}{\mu_0}\Big)^k\Big(\frac{\lambda_0}{\mu_0}\Big)^{m-k}, \Big(\frac{\lambda_0}{\mu_0}\Big)^m\right) \in \mu_1 O.$$

Since $m \in \mathbb{N}\backslash\{0\}$ was arbitrary, we get

$$\left(\frac{(a_0,\lambda_0)}{\mu_0}\right)^m \in \mu_1 O$$

for every $m \in \mathbb{N}\backslash\{0\}$. Remember that

$$\left(\frac{(a_0,\lambda_0)}{\mu_0}\right)^0 = e_{A\times\mathbb{K}} \subset \mu_W O.$$

Take $\mu := \max\{\mu_1, \mu_W\}$. Since O is balanced,

$$\left(\frac{(a_0,\lambda_0)}{\mu_0}\right)^n \in \mu O$$

for every $n \in \mathbb{N}$, which means that all elements of $A \times \mathbb{K}$ are bounded.

Conversely, suppose that $A \times \mathbb{K}$ is an (α_n)-galbed algebra with bounded elements. By Proposition 2, A is also an (α_n)-galbed algebra.

Let a_0 be an arbitrary element of A and U an arbitrary neighbourhood of zero in A. Fix any neighbourhood V_0 in $\mathbb{K}$. Then $U \times V_0$ is a neighbourhood of zero in $A \times \mathbb{K}$ and $(a_0, 0) \in A \times \mathbb{K}$. Since all elements of $A \times \mathbb{K}$ are bounded, there exist numbers $\lambda \in \mathbb{C}\backslash\{0\}$ and $\mu := \mu_{U,V_0} > 0$ such that

$$\left\{\Big(\Big(\frac{a_0}{\lambda}\Big)^n, 0\Big) : n \in \mathbb{N}\right\} = \left\{\left(\frac{(a_0,0)}{\lambda}\right)^n : n \in \mathbb{N}\right\} \subset \mu(U \times V_0) = \mu U \times \mu V_0.$$

Hence,

$$\Big\{\Big(\frac{a_0}{\lambda_0}\Big)^n : n \in \mathbb{N}\Big\} \subset \mu U,$$

which means that a_0 is bounded in A.

Corollary 1. *Let A be a topological algebra. The unitization $A \times \mathbb{K}$ of A in the product topology is a strongly galbed algebra with bounded elements if and only if A is a strongly galbed algebra with bounded elements.*

Proof. Let A be a strongly galbed algebra with bounded elements. Then there exists a sequence $(\alpha_n) \in l$ with $\alpha_0 \neq 0$ and

$$\alpha := \inf_{n>0} |\alpha_n|^{1/n} > 0$$

such that A is (α_n)-galbed. Notice that $\alpha_k \neq 0$ for every $k \in \mathbb{N}$. Hence, $A \times \mathbb{K}$ is an (α_n)-galbed algebra with bounded elements, by Proposition 3. Therefore, $A \times \mathbb{K}$ is also a strongly galbed algebra with bounded elements.

Conversely, let $A \times \mathbb{K}$ be a strongly galbed algebra with bounded elements. Then there exists a similar sequence $(\alpha_n) \in l$ as above. Notice that $\alpha_k \neq 0$ for every $k \in \mathbb{N}$. Hence, A is an (α_n)-galbed algebra with bounded elements, by Proposition 3. Therefore, A is also a strongly galbed algebra with bounded elements.

2.2. Galbed algebras with bounded elements

Let $(\alpha_n) \in l$. Then there are infinitely many elements different from zero in (α_n). Let P_n be the number of nonzero elements among $\alpha_0, \ldots, \alpha_n$. We will construct a sequence (i_n) of indexes as follows:

$$i_0 := \min\{k : \alpha_k \neq 0\},$$

$$i_1 := \min\{k : k > i_0, \alpha_k \neq 0\},$$

$$i_2 := \min\{k : k > i_1, \alpha_k \neq 0\},$$

$$\cdots$$

$$i_j := \min\{k : k > i_{j-1}, \alpha_k \neq 0\}$$

for every $j \in \mathbb{N}\backslash\{0\}$. Now we construct the sequence (β_n) by $\beta_k := \alpha_{i_k}$ for every $k \in \mathbb{N}$. Then the sequence (β_n) consists of all nonzero elements of (α_n) and for every $m \in \mathbb{N}$,

$$\sum_{k=0}^{m} \alpha_k a_k = \sum_{k=0}^{P_m-1} \beta_k b_k,$$

where $b_k = a_{i_k}$ for every $k \in \mathbb{N}$.

Definition 1. *We call the sequence $(\beta_n) \in l$, constructed from $(\alpha_n) \in l$ as above, the* underlying sequence *of (α_n).*

Proposition 4. *Let A be a topological algebra, $(\alpha_n) \in l$, and (β_n) be the underlying sequence of (α_n). Then A is an (α_n)-galbed algebra if and only if A is a (β_n)-galbed algebra.*

Proof. Let A be an (α_n)-galbed algebra and O a neighbourhood of zero in A. Then there exists a neighbourhood U of zero in A such that

$$\left\{\sum_{k=0}^{n} \alpha_k a_k : a_0, \ldots, a_n \in U\right\} \subset O$$

for every $n \in \mathbb{N}$. Let (β_n) be the underlying sequence of (α_n). Define $P_{-1} := 0$ and a map $f : \mathbb{N} \to \mathbb{N}$ as follows: for every $k \in \mathbb{N}$ find $m = m(k) \in \mathbb{N}$ such that $P_m = k + 1$ and $P_{m-1} = k$. Take $f(k) := m$ and $b_k := a_{f(k)}$ for every $k \in \mathbb{N}$. Then

$$\sum_{k=0}^{n} \beta_k b_k = \sum_{k=0}^{f(n)} \alpha_k a_k$$

for every $n \in \mathbb{N}$. Fix an arbitrary $l \in \mathbb{N}$ and consider the sets

$$H_1 := \left\{\sum_{k=0}^{f(l)} \alpha_k a_k : a_0, \ldots, a_{f(l)} \in U\right\}$$

and

$$H_2 := \left\{\sum_{k=0}^{l} \beta_k b_k : b_0, \ldots, b_l \in U\right\}.$$

For every element of H_2 we have

$$\sum_{k=0}^{l} \beta_k b_k = \sum_{k=0}^{f(l)} \alpha_k a_k \in H_1.$$

Hence, $H_2 \subset H_1$. As $l \in \mathbb{N}$ was arbitrary, we have

$$\left\{\sum_{k=0}^{n} \beta_k b_k : b_0, \ldots, b_n \in U\right\} \subset H_1 \subset O$$

for every $n \in \mathbb{N}$. Hence, A is a (β_n)-galbed algebra.

Conversely, let A be a (β_n)-galbed algebra, where (β_n) is the underlying sequence of (α_n). Moreover, let O be any neighbourhood of zero in A. Then there exists a neighbourhood U of zero in A such that

$$\left\{\sum_{k=0}^{n} \beta_k b_k : b_0, \ldots, b_n \in U\right\} \subset O$$

for each $n \in \mathbb{N}$. Fix an arbitrary $m \in \mathbb{N}$ and consider the sets

$$G_1 := \left\{\sum_{k=0}^{m} \alpha_k a_k : a_0, \ldots, a_m \in U\right\},$$

$$G_2 := \left\{ \sum_{k=0}^{P_m - 1} \beta_k b_k : b_0, \ldots, b_{P_m - 1} \in U \right\}$$

with $\beta_k = \alpha_{i_k}$ as above. Take $a_j := b_k$ for $j = i_k$ and $a_j := \theta_A$ otherwise. Since for every element of G_1 we have

$$\sum_{k=0}^{m} \alpha_k a_k = \sum_{k=0}^{P_m - 1} \beta_k b_k \in G_2,$$

$G_1 \subset G_2$. As $m \in \mathbb{N}$ is arbitrary, we have

$$\left\{ \sum_{k=0}^{n} \alpha_k a_k : a_0, \ldots, a_n \in U \right\} \subset G_2 \subset O$$

for each $n \in \mathbb{N}$, which means that A is an (α_n)-galbed algebra.

Next, we give two simple corollaries.

Corollary 2. *Let A be a topological algebra, $(\alpha_n) \in l$, and (β_n) be the underlying sequence of (α_n). Then $A \times \mathbb{K}$ is an (α_n)-galbed algebra if and only if $A \times \mathbb{K}$ is a (β_n)-galbed algebra.*

Proof. The result follows immediately if we put $A \times \mathbb{K}$ instead of A in Proposition 4.

Corollary 3. *Let A be a topological algebra, $(\alpha_n) \in l$, and (β_n) be the underlying sequence of (α_n). Then A is a (β_n)-galbed algebra with bounded elements if and only if $A \times \mathbb{K}$ is a (β_n)-galbed algebra with bounded elements.*

Proof. Since $\beta_k \neq 0$ for every $k \in \mathbb{N}$, the desired result follows directly if we substitute (β_n) for (α_n) in Proposition 3.

Now we are ready for our first main result.

Theorem 1. *Let A be a topological algebra and $(\alpha_n) \in l$. Then A is an (α_n)-galbed algebra with bounded elements if and only if $A \times \mathbb{K}$ is an (α_n)-galbed algebra with bounded elements.*

Proof. Let A be a topological algebra, $(\alpha_n) \in l$, and (β_n) be the underlying sequence of (α_n). By Proposition 4, A is an (α_n)-galbed algebra (with bounded elements) if and only if A is a (β_n)-galbed algebra (with bounded elements). By Corollary 3, A is a (β_n)-galbed algebra with bounded elements if and only if $A \times \mathbb{K}$ is a (β_n)-galbed algebra with bounded elements. By Corollary 2, $A \times \mathbb{K}$ is a (β_n)-galbed algebra (with bounded elements) if and only if $A \times \mathbb{K}$ is an (α_n)-galbed algebra (with bounded elements). Hence, A is an (α_n)-galbed algebra with bounded elements if and only if $A \times \mathbb{K}$ is an (α_n)-galbed algebra with bounded elements.

3. SECTIONAL REPRESENTATION OF UNITAL STRONGLY GALBED ALGEBRAS

In this section we generalize part c) of Theorem 3.13 in [6] from exponentially galbed algebras to strongly galbed algebras. For this purpose we need to generalize some other results. Throughout this section, we consider only topological algebras over $\mathbb{C}$ (i.e., $\mathbb{K} = \mathbb{C}$ in the definition of a topological algebra). Remember that in unital algebras every ideal is regular.

A topological algebra A is *σ-complete* if every Cauchy net of elements of A converges in A. A topological algebra A is *topologically primitive* if there exists a closed maximal regular left (or right) ideal M of A such that $\{a \in A : aA \subset M\} = \{\theta_A\}$ ($\{a \in A : Aa \subset M\} = \{\theta_A\}$, respectively). Let M be a maximal regular left (or right) ideal of A. Then the two-sided ideal $P = \{a \in A : aA \in M\}$ ($P = \{a \in A : Aa \in M\}$, respectively) is a *primitive ideal of A, defined by M*.

Proposition 5. *Let A be a unital σ-complete topologically primitive strongly galbed Hausdorff algebra* (*over* $\mathbb{C}$) *with bounded elements and P a primitive ideal*, *defined by a closed maximal left* (*or right*) *ideal of A. Then $Z(A/P)$ is topologically isomorphic to* $\mathbb{C}$.

Proof. Since P is a primitive ideal defined by a closed maximal left (or right) ideal of A, P is closed. Now, A/P is topologically primitive by [15], Proposition 9, and Hausdorff by [16], Proposition 5. As in the proof of Theorem 2 in [16], it is easy to show that A/P is σ-complete. As A/P is strongly galbed algebra with bounded elements by Proposition 1, $Z(A/P)$ is topologically isomorphic to $\mathbb{C}$ by Theorem 3.1 of [7].

The set of all regular two-sided ideals of a topological algebra A, which are maximal as left or right ideals, is denoted by $m(A)$. Let now A be a unital topological algebra and B a closed subalgebra of $Z(A)$, containing the unital element e_A of A. An ideal $M \in m(B)$ is an *extendible ideal* of B if

$$I(M) := \mathrm{cl}_A\left\{\sum_{k=1}^{n} a_k m_k : n \in \mathbb{N}, a_1, \ldots, a_n \in A, m_1, \ldots, m_n \in M\right\} \neq A.$$

The set of all extendible ideals of B is denoted by $m_e(B)$. For general definitions see [6], pp. 18–19, or [13], pp. 4–5.

Proposition 6. *Let A be a unital σ-complete topologically primitive strongly galbed Hausdorff algebra with bounded elements*, *M a closed maximal left* (*right or two-sided*) *ideal of A*, *and B a closed subalgebra of $Z(A)$*, *containing the unital element e_A of A. Then*
1) *every $b \in B$ defines a number $\lambda \in \mathbb{C}$ such that $b - \lambda e_a \in M$*;
2) $M \cap B \in m_e(B)$.

Proof. The proof is similar to the proof of Proposition 3.1 in [6], because we need only the fact that A/P is topologically isomorphic to $\mathbb{C}$ for a primitive ideal P defined by M. For detailed proof see the proof of Proposition 2.8 in [13].

For every $M \in m_e(B)$ and $a \in A$ let $A_M := A/I(M)$, let $\kappa_M : A \to A_M$ be the canonical homomorphism and $a\hat{}\colon m_e(B) \to A_M$ a map, defined by $a\hat{}(M) := \kappa_M(a)$. Let I be a closed maximal left (right or two-sided) ideal of A and J be a closed maximal left (right or two-sided) ideal of A_M. Then $\kappa_M(I) = \{a\hat{}(M) : a \in I\}$ and $\kappa_M^{-1}(J) = \{a \in A : a\hat{}(M) \in J\}$.

Proposition 7. *Let A be a unital σ-complete topologically primitive strongly galbed Hausdorff algebra with bounded elements*, *I a closed maximal left* (*right or two-sided*) *ideal of A*, *and B a closed subalgebra of $Z(A)$*, *containing the unital element e_A of A. Then there exists $M \in m_e(B)$ and a closed maximal left* (*right or two-sided*, *respectively*) *ideal J of A_M such that $I = \kappa_M^{-1}(J)$.*

Proof. Let I be a closed maximal left (right or two-sided) ideal of A. By Proposition 6, part 2), $M := I \cap B \in m_e(B)$. Let $J := \kappa_M(I)$. Then J is a closed left (right or two-sided) ideal of A_M and $\kappa_M^{-1}(J)$ is a closed maximal left (right or two-sided, respectively) ideal of A by Lemma 3.5 of [6], pp. 52–54. Moreover, $I \subset \kappa_M^{-1}(J)$. Since I is maximal left (right or two-sided, respectively) ideal of A, $I = \kappa_M^{-1}(J)$.

The complex (or the triple) (B, π, X), where B and X are topological spaces and $\pi : B \to X$ is a continuous open surjection, is called a *fibre bundle*. A map $f : X \to B$ is said to be a *section* of the fibre bundle (B, π, X) or shortly, a *section* of π if $\pi[f(x)] = x$ for all $x \in X$. The set of all continuous sections of π is denoted by $\Gamma(\pi)$. If all the fibres $B_x := \{b \in B : \pi(b) = x\}$ of the fibre bundle (B, π, X) are topological algebras, it is possible to define algebraic operations and topology on $\Gamma(\pi)$ so that $\Gamma(\pi)$ becomes a topological algebra (see [6], p. 22, or [13], p. 8). This topological algebra is called a *section algebra*.

Let A and B be topological algebras. Every continuous homomorphism $f : A \to B$ is a *representation* of A in B. If A is the section algebra $\Gamma(\pi)$ for some fibre bundle (B, π, X), then f is a *sectional representation* of A.

For every representation $f : A \to B$ we can consider B as a left (or right) A-module if we define the module multiplication $\cdot_f$ in B by $a \cdot_f b := f(a)b$ ($b \cdot_f a := f(a)b$, respectively) for each $a \in A$ and $b \in B$. If B and $\{\theta_B\}$ are the only A-submodules of B, we call f an *irreducible representation*.

A topological algebra A over $\mathbb{C}$ is a *Gelfand–Mazur algebra* if the quotient algebra A/M is topologically isomorphic to $\mathbb{C}$ for every $M \in m(A)$. The intersection of the kernels of all irreducible representations of a topological algebra A is called the *topological radical* of A and is denoted by rad$A$. It is known that if A is a Gelfand–Mazur algebra, then rad$A$ is the intersection of all closed maximal regular left (or right) ideals of A. Moreover, every strongly galbed algebra with bounded elements is a Gelfand–Mazur algebra by Theorem 4.2 of [12].

Proposition 8. *Let A be a unital σ-complete topologically primitive strongly galbed Hausdorff algebra with bounded elements*, I *a closed maximal left* (*right or two-sided*) *ideal of* A, *and* B *a closed subalgebra of* $Z(A)$, *containing the unital element* e_A *of* A. *Then*

$$\mathrm{rad}A = \bigcap\{\kappa_M^{-1}(\mathrm{rad}A_M) : M \in m_e(B)\}.$$

Proof. The proof is similar to the proof of Proposition 3.7 in [6].

Let A be a unital strongly galbed algebra (over $\mathbb{C}$) with bounded elements and B a subalgebra of $Z(A)$, for which $m_e(B) \neq \emptyset$. As in Proposition 1, we can show that B is a strongly galbed algebra with bounded elements, thus a Gelfand–Mazur algebra. Hence, for every $M \in m_e(B)$ there exists a nontrivial homomorphism $\phi_M : B \to \mathbb{C}$ such that $\ker\phi_M = M$. Let

$$\Delta := \bigcup_{M \in m_e(B)} A_M.$$

Then, for every $a \in A$, the map $a\hat{}$, defined above, is a map from $m_e(B)$ into Δ. Let $\pi : \Delta \to m_e(B)$ be a map which assigns to every $d \in \Delta$ the ideal $M \in m_e(B)$ for which $d \in A_M$. In [6], pp. 60–61, it is shown that such π is a well-defined open and continuous map and that $A_{M_1} \cap A_{m_2} \neq \emptyset$ if and only if $M_1 = M_2$.

On $m_e(B)$ we consider the Gelfand topology τ; a subbase of neighbourhoods of $M_0 \in m_e(B)$ consists of sets

$$O(M_0, \epsilon, b) := \{M \in m_e(B) :|(\phi_M - \phi_{M_0}|< \epsilon\},$$

where $\epsilon > 0$ and $b \in B$ vary. On A_M we consider the quotient topology τ_M and on Δ the topology $\tau_\Delta := \{\pi^{-1}(U) : U \in \tau\}$. Then A_M is a topological algebra, $(\Delta, \pi, m_e(B))$ is a fibre bundle, and $a\hat{} \in \Gamma(\pi)$ for every $a \in A$.

Let $\Upsilon : A \to \Gamma(\pi)$ be a map, defined by $\Upsilon(a) := a\hat{}$ for every $a \in A$. It is known (see [6], pp. 60–61) that Υ is a continuous map and hence, a sectional representation of A.

Proposition 9. *Let A be a unital σ-complete topologically primitive topologically semisimple strongly galbed Hausdorff algebra with bounded elements. Then the map Υ is one-to-one.*

Proof. Using Proposition 8, the proof becomes similar to the proof of Proposition 3.12 in [6].

Finally, we give our second main result of this paper.

Theorem 2. *Let A be a unital σ-complete topologically primitive topologically semisimple strongly galbed Hausdorff algebra* (*over* $\mathbb{C}$) *with bounded elements. Then A can be considered as a subalgebra of the section algebra* $\Gamma(\pi)$.

Proof. Since Υ is a one-to-one sectional representation by Proposition 9, A can be considered as a subalgebra of $\Gamma(\pi)$.

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Gälbalgebrad ja nende lõikeesitus

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On näidatud, et topoloogiline algebra A on (α_n)-gälb algebra siis ja ainult siis, kui algebrale A ühiku juurdetoomisel tekkinud algebra on (α_n)-gälb algebra. Teise põhitulemusena on leitud piisavad tingimused selleks, et ühikuga tugevalt gälb algebra oleks vaadeldav teatava lõikekujutuste algebra alamalgebrana. Artikli aluseks on töö [13].