Estonian Journal of Engineering, 2008, **14**, 3, 220–231 doi: 10.3176/eng.2008.3.03

Exact travelling wave solutions in strongly inhomogeneous media

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Received 22 February 2008, in revised form 12 May 2008

Abstract. Approximate travelling wave solutions to linear, one-dimensional wave equations with varying coefficients (the case of an inhomogeneous medium) are usually found using asymptotic procedures such as the WKB approach. For certain conditions put on the coefficients, this procedure leads to exact solutions. We show that such exact travelling wave solutions exist for a limited class of strongly inhomogeneous media and prove the existence and uniqueness of such waves. Using the obtained solutions, the solution of the relevant Cauchy problem is expressed in elementary functions. This approach enables a detailed and straightforward analysis of the processes of wave transformation and reflection in a specific type of strongly inhomogeneous media.

Key words: wave equations, inhomogeneous medium, travelling waves, exact solutions, WKB approach.

1. INTRODUCTION

The solutions of the type $u(x,t) = f(x-t)$, where x is a spatial coordinate and t is time, are usually called travelling waves. The analysis of these solutions has become a specific and rapidly developing subject of non-linear mathematics and wave physics [1–6]. Their existence usually means that the set of underlying equations is invariant with respect to the shift of the coordinate x and time t. In the one-dimensional case, the initial partial differential equations (PDEs) can be reduced to a set of ordinary differential equations (ODEs). Qualitative methods of the oscillation theory can be applied to find the travelling wave solutions of the resulting ODEs and to investigate their properties [7–9]. In particular, travelling wave solutions (such as different kinds of solitons, cnoidal and shock waves, etc.) can be found in explicit form for well-known equations of non-linear physics, such as the Burgers, Korteweg-de Vries, Gardner and Klein–Gordon equations [10,11].

Generally an exact travelling wave solution does not exist if the medium is inhomogeneous along the direction of wave propagation. The inhomogeneity is reflected in the mathematical problems through variable coefficients of the governing equations. If amplitudes or phases of the solutions can be assumed as slowly varying quantities, asymptotic approaches such as the WKB approximation can be applied to find approximate wave solutions [12–16].

There are a few examples [17,18] when asymptotic WKB solutions for certain variable coefficients are exact solutions. Such solutions may be interpreted as travelling waves in inhomogeneous media. Although the fact of their existence has been mentioned several decades ago [17,18], their interpretation as travelling waves as well as their physical relevance is still under discussion in the physical literature. Ginzburg [17] argues that any solution in the form of the ansatz $u = A\exp i(\omega t - \Psi)$, where Ψ is the function of the position, is a travelling wave. On the other hand, Brekhovskikh [18] claims that all solutions to wave equations can be, in principle, presented in this form, yet not all such solutions are travelling waves. To the knowledge of the authors, very little is known about their properties and even the set of such solutions has not been rigorously established. We address the problem of the existence and the properties of such exact analytical solutions of the generic one-dimensional wave equation with variable coefficients. In other words, we attempt a rigorous description of all possible exact travelling wave solutions in strongly inhomogeneous one-dimensional (1D) media. The key advance from this study is the proof that such solutions exist only for a very limited class of (strongly) inhomogeneous media. It is shown that all existing solutions in the form $u(x,t) = A(x) f(t - \overline{\Psi}(x))$ to the 1D wave equation in such media are actually travelling (but not necessarily monochromatic) waves.

The paper is organized as follows. Different types of the wave equations with variable coefficients are discussed in Section 2. The method for finding travelling wave solutions within an asymptotic approach is presented in Section 3. In Section 4 we describe a rigorous, constructive proof of the uniqueness of these solutions. The dynamics of such solutions is illustrated, based on a particular solution of the Cauchy problem, in Section 5. Conclusions are summarized in Section 6.

2. WAVE EQUATION

The generic 1D wave equation with variable coefficients can be presented in three different forms:

$$\frac{\partial^2 u}{\partial t^2} - c^2(x)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1}$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left[c^2(x)\frac{\partial u}{\partial x}\right] = 0, \qquad (2)$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 [c^2(x)u]}{\partial x^2} = 0, \qquad (3)$$

where $u(x,t)$ is the wave function and $c(x)$ is an arbitrary continuous or discontinuous function having the meaning of the local wave celerity. The scope and conditions applied to the function $c(x),$ and appropriate boundary conditions for Eqs. (1)–(3) may widely vary depending on the particular physical problem. This will be discussed later. Generally, the wave function is supposed to be bounded in space (except possibly at the boundary points) but it is not necessarily smooth in the class of generalized functions, as is customary when solving hyperbolic equations. Obviously, Eq. (3) can be reduced to Eq. (1) by the simple change of variables $U(x,t) = c^2(x)u(x,t);$ therefore only Eqs. (1) and (2) will be analysed below.

3. ASYMPTOTIC AND EXACT SOLUTIONS

First we sketch the basic steps of a commonly used method of mathematical physics for the determination and analysis of travelling wave solutions in weakly inhomogeneous media using Eq. (1). This method is based on the presentation of the solution of Eq. (1) in the following form

$$u(x,t) = A(x)\exp i[\omega t - \Psi(x)], \qquad (4)$$

where $A(x)$ and $\Psi(x)$ are unknown real functions (being the wave amplitude and phase, respectively), and where ω is the (angular) wave frequency, usually determined within the solution procedure of Eq. (1). After substituting ansatz (4) into Eq. (1), this complex equation is equivalent to two real equations:

$$2k\frac{\mathrm{d}A}{\mathrm{d}x} + \frac{\mathrm{d}k}{\mathrm{d}x}A = \frac{\mathrm{d}}{\mathrm{d}x}(kA^2) = 0, \qquad (5)$$

$$[c^2k^2 - \omega^2]A - c^2\frac{\mathrm{d}^2 A}{\mathrm{d}x^2} = 0, \qquad (6)$$

where $k(x)$ is the local wave number

$$k(x) = \frac{\mathrm{d}\Psi}{\mathrm{d}x}. \qquad (7)$$

Equation (5) can be integrated directly

$$k(x)A^2(x) = \text{const}. \tag{8}$$

As a result, a second-order ordinary differential equation (6) for the unknown function $A(x)$ is obtained. Generally, this equation is not simpler than Eq. (1). As the dependence of the solution of Eq. (1) on the properties of the medium is reflected in the coefficient $c^2(x)$, further simplification of Eqs. (1) and (6) is possible if $c^2(x)$ exhibits certain favourable properties.

In many cases of practical interest, $c^2(x)$ is a slowly varying function of the x coordinate. The potential for simplification of the problem by the use of slow changes of the coefficient $c^2(x)$ is exploited in various asymptotic approaches. In the WKB approximation that is often used in physics it is assumed that this coefficient can be presented as $c(x) \equiv c(\varepsilon x)$, where $\varepsilon << 1$. From Eqs. (5)–(8) it follows that in this case $A(x)$ and $k(x)$ are also slowly varying functions of the x coordinate. A direct conjecture from this assumption is that the second term in Eq. (6) is of the order of ε^2 and can be neglected. Then Eq. (6) is purely algebraic and defines the local dispersion relation between the wave frequency and the local wave number:

$$k(x) = \pm \frac{\omega}{c(x)}. \tag{9}$$

The different signs in Eq. (9) correspond to the respective directions of wave propagation along the x axis. The development of the relevant asymptotic procedure and the limits of applicability of the WKB approximation are described in detail in [12–15].

This method can also be used for finding exact solutions to Eq. (1). Basically, Eq. (6) can be solved numerically for arbitrary function $c(x)$. Although the solution $A(x)$ will still depend on the integration constant in Eq. (8), it is easy to specify it for a numerical solution. Further, the corresponding solution of Eq. (1) can be found from expression (4) in a straightforward manner. The resulting solution (4) can be called a travelling wave in an arbitrarily inhomogeneous media [17,18]. In general, such solutions describe the complicated physical process of wave transformation in inhomogeneous media and optionally also wave–medium or wave–wave interactions.

Of specific interest are the cases of parameters of the medium, when Eq. (6) can be solved explicitly in elementary functions. This is possible, for example, if the wave amplitude is a linear function of the coordinate x. Without the loss of generality, we can assume that in such cases

$$A(x) = x. \tag{10}$$

This assumption is equivalent to splitting of Eq. (6) into two equations, Eq. (9) and

$$\frac{d^2 A}{dx^2} = 0. \tag{11}$$

In this case, Eqs. (8), (9) and (11) unambiguously define all the properties of the solution (4). It follows that solutions, satisfying Eqs. (10) and (11), exist only if the function $c(x)$ has the form

$$c(x) = x^2. \tag{12}$$

Then

$$k(x) = \frac{\omega}{x^2}, \quad \Psi(x) = -\frac{\omega}{x}, \tag{13}$$

where signs of $k(x)$ and $\Psi(x)$ correspond to the wave propagation to the right. This particular solution in its final form can be without the loss of generality presented as

$$u(x,t) = x \exp\left[i\omega \left(t + \frac{1}{x} \right) \right]. \tag{14}$$

As mentioned above, the exact set of conditions for the existence of such a solution, its physical meaning and interpretation as a travelling wave, as well as its potential applications have been described vaguely and partially ambiguously in classical studies [[17,18]]. The interpretation of the solution (14) is also complicated by the fact that it is defined on the semi-axis $0 < x < \infty$, at the boundaries of which either the amplitude $A(x)$ or the phase $\Psi(x)$ tend to infinity.

Solution (14) can be interpreted as an elementary travelling wave in the medium considered. It is straightforward to demonstrate that any function

$$u(x,t) = xU\left(t + \frac{1}{x} \right), \tag{15}$$

where $U(\zeta)$ is an arbitrary function, found from initial or boundary conditions, is a particular solution to Eq. (1) provided $c(x) = x^2$, and can be represented as a Fourier series of particular (elementary) solutions given by (14). The solution presented in Eq. (15) can be thus interpreted as a generalization of the (elementary) travelling waves (14).

In a similar manner, travelling wave solutions of Eq. (2) can also be found. They exist, for example, when the function $c(x)$ has the form:

$$c(x) = x^{2/3}, \tag{16}$$

and can be presented in the general form

$$u(x,t) = \frac{1}{x^{1/3}} V(t - 3x^{1/3}), \tag{17}$$

where $V(\zeta)$ is an arbitrary function. This solution is also defined on the semi-axis $0 < x < \infty$. Its amplitude or phase also tends to infinity at the boundaries $x = 0$ and $x = \infty$.

Therefore, a family of exact travelling wave solutions of wave equation (1) or (2) for specific variations of the coefficients (12) and (16) can be found with the use of the underlying ideas stemming from the WKB approximation. This holds for certain types of inhomogeneous media. In what follows we shall analyse the conditions of their existence, and the properties of the corresponding waves.

4. REDUCTION TO THE WAVE EQUATION WITH CONSTANT COEFFICIENTS

It follows from the form of Eqs. (15) and (17) that the functions U and V are solutions of some wave equations with constant coefficients. Therefore a change of variables, reducing a wave equation with variable coefficients to a wave equation with constant coefficients, should exist. As above, we perform the analysis for Eq. (1); the generalization of the procedure to Eq. (2) is straightforward.

The appearance of Eqs. (15) and (17) suggests that the general form of this change of variables is

$$u(x,t) = B(x) U[t, \tau(x)], \tag{18}$$

where one has to define the functions $B(x)$ and $\tau(x)$. After substituting Eq. (18) into Eq. (1), the resulting equation has constant coefficients if and only if the following conditions are satisfied:

$$c(x) \frac{\mathrm{d}\tau}{\mathrm{d}x} = \mathrm{const}, \tag{19}$$

$$\frac{\mathrm{d}^2 B}{\mathrm{d}x^2} = 0, \tag{20}$$

$$B(x) \frac{\mathrm{d}^2 \tau}{\mathrm{d}x^2} + 2 \frac{\mathrm{d}B}{\mathrm{d}x} \frac{\mathrm{d}\tau}{\mathrm{d}x} = 0.$$

In other words, Eqs. (19)–(21) uniquely define the class of inhomogeneous media, for which exact travelling wave solutions for Eq. (1) exist. Notice that if one chooses $A = B$, $\Psi(x) \sim \tau(x)$, then Eqs. (8), (9) and (11) that are used in Section 3, are identically satisfied, provided Eqs. (19)–(21) hold.

Equations (19)–(21) can easily be solved. It is obvious that from $B(x) = ax + b$ (similarly for $c(x)$ and $\tau(x)$) the coefficient a is redundant (since it only shows the non-normalized amplitude) and the coefficient b defines the boundary of the semi-axis, on which the solution exists. Therefore the unique family of their solutions is

$$c(x) = x^2, \quad B(x) = x, \quad \tau(x) = -\frac{1}{x}. \tag{22}$$

It is easy to show that the change of variables, represented by Eq. (18), where $B(x)$ and $\tau(x)$ are defined by Eqs. (22), reduces Eq. (1) to the generic wave equation with constant coefficients

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial \tau^2} = 0. \tag{23}$$

According to solutions given by Eqs. (14) and (15), Eq. (23) is determined on the semi-axis $-\infty < x < 0$.

The above derivation demonstrates that the described reduction of Eq. (1) to the wave equation with constant coefficients with the use of ansatz (4) is possible if and only if functions defined in Eq. (22) coincide with functions in Eqs. (10) and (13). This is possible if and only if $c(x) = x^2$. Consequently, we have proved the uniqueness of the obtained family of exact travelling wave solutions in inhomogeneous media. Notice that the derivation does not rely on the property of weak inhomogeneity (understood as a slow dependence of the medium on the x coordinate) or on the assumption of slow variation of the amplitude or phase of the solution.

An analogous procedure can be performed for Eq. (2). It is easy to demonstrate that a solution in the form of Eq. (17) converts Eq. (2) into Eq. (23) with constant coefficients if and only if

$$c(x) = x^{2/3}, \quad A(x) = x^{-1/3}, \quad \tau(x) = 3x^{1/3}, \tag{24}$$

whereas Eq. (23) again is determined on the semi-axis $-\infty < x < 0$.

Thus the generic wave equation Eq. (1) with variable coefficients can be reduced to the wave equation with constant coefficients if and only if $c(x) = x^2$. Similarly, wave equation (2) can be reduced to Eq. (23) if and only if $c(x) = x^{2/3}$. The resulting Eq. (23), common for both cases, supports travelling wave solutions of fairly general shape propagating in opposite directions. Generally Eq. (23) should be solved on a semi-axis. A benefit from the procedure decribed above is that Eq. (23) has the same type everywhere, whereas Eqs. (1) and (2) change their type at the point $x = 0$ where they are not hyperbolic.

5. CAUCHY PROBLEM WITH FINITE LENGTH INITIAL CONDITIONS

To complete the formal description of travelling waves in the discussed cases, we analyse certain details of mathematical formulation of the correct boundary conditions at both points of singularity. Let us consider a generic Cauchy problem for an initial disturbance with a finite length for Eq. (1) with $c(x) = x^2$. As a simplest example, we choose the following initial conditions for Eq. (23):

$$U(\tau, t=0) = 1(\tau + l^{-1})1(-\tau - L^{-1}), \quad \frac{\partial U}{\partial t}(\tau, t=0) = 0, \tag{25}$$

where $1(\tau)$ is the Heaviside step function and $L > l$ defines the wave area, that is, the borders of the interval on the x axis where $U(\tau, 0) > 0$. A solution of Eq. (1) for small values of time $t < 1/L$, when wave fronts are far from the points of singularity, represents a superposition of two trapezoidal impulses, propagating in opposite directions (Fig. 1). A particular solution of this kind is

$$u(x,t) = \frac{x}{2}[1(-x^{-1} + l^{-1} + t)1(x^{-1} - L^{-1} - t) + 1(-x^{-1} + l^{-1} - t)1(x^{-1} - L^{-1} + t)], \tag{26}$$

The first additive in square brackets of Eq. (26) corresponds to a wave moving to the left towards the point $x = 0$ and the second to a wave moving to the right towards the infinity.

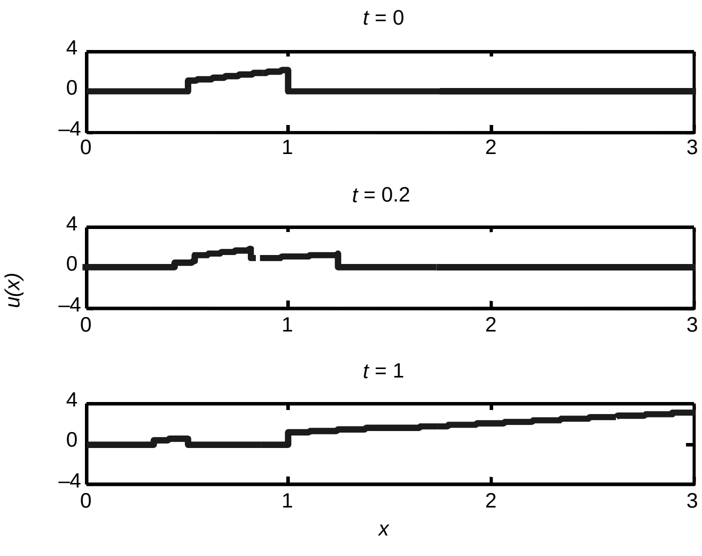

Fig. 1. Scheme of the solution $u(x)$ of Eq. (1) for $t < 1/L$.

The position of the wave front of the left-going wave x_l moves towards the origin as

$$x_l = \frac{l}{1+lt}, \tag{27}$$

and never reaches this point. Therefore there is no need to set a boundary condition at the point $x=0$ as it does not take part in the formation of the wave field. The wave front of the right-going wave at x_r moves to the infinity as

$$x_r = \frac{L}{1-Lt}, \tag{28}$$

and reaches infinity by a finite time $1/L$. The amplitude of this wave front at this moment becomes infinite. Further wave propagation depends on the type of boundary conditions at the point of singularity. Notice that the singularity point $x=\infty$ in Eq. (1) corresponds to the origin $x=0$ in Eq. (23). If boundary conditions at the point $\tau=0$ are set in the form of radiation conditions

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial \tau} = 0, \tag{29}$$

then the right-going wave propagates completely out of the domain after the time moment $t=1/l$. After that, only the left-going impulse continues moving to the origin $x=0,$ whereas its amplitude and duration gradually decrease. Physically, such boundary conditions simulate the breaking of a large-amplitude wave at the coast and its dissipation at the infinity.

If one requires the wave amplitude to be bounded at the infinity, the wave is reflected back from the infinity with an opposite sign of the propagation direction. In this case, for large times $t>1/L$ the following function, corresponding to the "anti-mirror reflection" of the originally right-going wave (Fig. 2) and appears in solution (26):

$$u'(x,t) = -\frac{x}{2} 1(x^{-1} + l^{-1} - t) 1(-x^{-1} - L^{-1} + t). \tag{30}$$

After $t=1/l$, the right-going wave disappears and only two waves of different signs of elevation remain in the system. They both move towards the origin $x=0,$ but never reach this point. The wave field in this stage is described by the following expression:

$$u(x,t) = \frac{x}{2}[1(-x^{-1} + l^{-1} + t)1(x^{-1} - L^{-1} - t) - 1(x^{-1} + l^{-1} - t)1(-x^{-1} - L^{-1} + t)]. \tag{31}$$

The wave field, described by Eq. (2) with $c(x) = x^{2/3}$ can be analysed in a similar way. Qualitatively, such a wave field will be similar to the described case with only the locations of the singularity points interchanged. Some of the waves

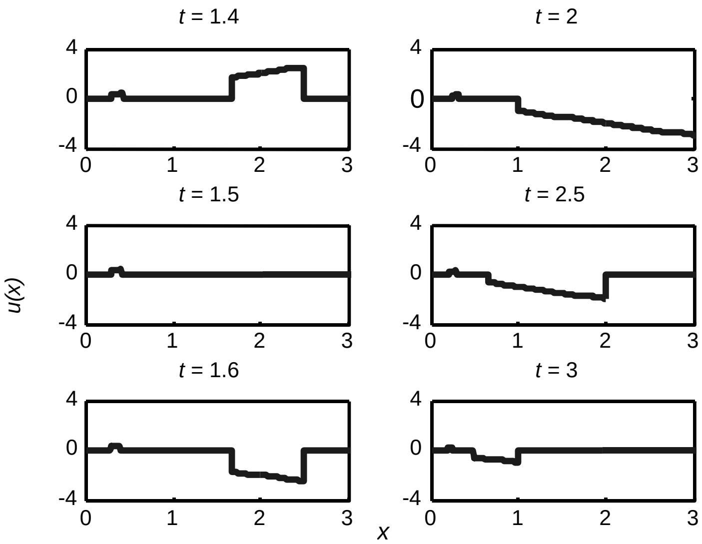

Fig. 2. Scheme of the solution $u(x)$ of Eq. (1) for $t > 1/L$ supporting the wave reflection from infinity.

move to the infinity, but never reach it, and their amplitudes gradually decrease. The waves propagating in the opposite direction reach the origin $x = 0$ by a finite time and their amplitudes increase. Although both Eqs. (1) and (2) can be transformed to the same wave equation (23) with constant coefficients, the described procedure cannot be used for relating the obtained travelling wave solutions of the original equations. The reason is that such solutions only exist for completely different types of the medium. This feature, however, does not exclude the possibility of relating these equations by means of reducing them to other wave equations with constant coefficients.

6. CONCLUSIONS

Two approaches of finding exact travelling wave solutions in a one-dimensional wave equation with variable coefficients have been applied. The first approach uses the ideas of the WKB approximation and concentrates on the case when the asymptotic solution becomes exact. A change of variables, reducing a wave equation with variable coefficients to a wave equation with constant coefficients, is used in the second approach. These solutions and changes of variables exist only for a particular variation of the coefficients.

The presented constructive proof completely solves both the existence and uniqueness problems of these solutions, equivalently, the problem of finding the complete set of travelling waves having a closed analytical form in a inhomogeneous one-dimensional medium. Some examples of solving of the

Cauchy problem in strongly inhomogeneous media are presented. Obtained travelling wave solutions can be applied in oceanography to study the wave transformation above complicated bottom relief, which can be presented as superposition of small sections, for which the wave celerity changes as $c(x) \sim x^2$ or $c(x) \sim x^{4/3}$.

ACKNOWLEDGEMENTS

This research was supported particularly by grants from INTAS (06-1000013-9236, 06-1000014-6046), RFBR (08-05-72011, 08-05-00069, 08-05-91850), Marie Curie network SEAMOCS (MRTN-CT-2005-019374), Estonian Science Foundation (grant No. 7413), EEA grant EMP41 and Scientific School of Corresponding Member of the Russian Academy of Sciences V. A. Zverev. We would like to thank two anonymous reviewers for their comments and Kevin Parnell for his help in preparing the final version of the paper.

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Täpsed lainelahendid tugevalt mittehomogeenses keskkonnas

Ira Didenkulova, Efim Pelinovsky ja Tarmo Soomere

Ühemõõtmelise muutuvate kordajatega lineaarse lainevõrrandi ligikaudsed lainelahendid (lained mittehomogeenses keskkonnas) leitakse tavaliselt asümptootiliste meetoditega, sageli WKB-meetodiga. Teatavatel juhtudel annab WKB-meetod täpse lainelahendi. On näidatud, et taolised täpsed lainelahendid eksisteerivad piiratud mittehomogeensete keskkondade klassi puhul. On tõestatud taoliste täpsete lahendite eksisteerimine ja ühesus, konstrueeritud vastavate Cauchy ülesannete lahendid ning analüüsitud üksiklainete levimist ja peegeldumist.