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Carlson's inequality and interpolation

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Abstract. In this survey, we explain and discuss some recent results concerning the close connection between Carlson type inequalities and interpolation theory. In particular, we point out that a fairly general Carlson type inequality can be used to extend the usefulness of the Gustavsson–Peetre $\langle\cdot\rangle_\varphi$ interpolation method.

Key words: inequalities, Carlson's inequality, interpolation, Peetre J functional, $\pm$ method, Brudnĭ–Kruglyak construction.

1. INTRODUCTION

In 1934, it was proved by Carlson [1] that if $a = \{a_k\}_{k=1}^{\infty}$ is a nonzero sequence of nonnegative numbers, then

$$\left(\sum_{k=1}^{\infty} a_k\right)^4 < \pi^2 \sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} k^2 a_k^2, \tag{1}$$

and the number π^2 is the smallest possible constant. Since then, inequalities of this type have attracted great interest among mathematicians, and even today many research papers have this theme as their central subject.

In modern language, the inequality (1) may be written as

$$\|a\|_1 < \sqrt{\pi}\, \|a\|_{l_2}^{1/2} \, \|a\|_{l_2(k^2)}^{1/2},$$

where $l_2(k^2)$ denotes a weighted l_2 space. Thus, the inequality (1) can be thought of as a special case of the inequality

$$\|a\|_X \leq C \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^{\theta}, \tag{2}$$

where C is some positive constant, X, A_0, and A_1 are normed spaces, and the parameter θ satisfies $0 < \theta < 1$.

Many papers have been written on inequalities of the type (2), henceforth referred to as *Carlson type inequalities*, and several applications have been found (see [2] and the references therein). The purpose of this paper is to give an overview of how Carlson type inequalities connect to interpolation theory. In Section 2, we describe how interpolation theory can be applied to achieve certain inequalities of this type. In Section 3, we go the other way, illustrating how Carlson type inequalities can be applied to achieve embeddings of real interpolation spaces in the weighted Lebesgue spaces. In Section 4, we explain how a certain Carlson type inequality, which was used in connection with interpolation of Orlicz spaces by the Gustavsson–Peetre $\langle\cdot\rangle_\varphi$ method, can be modified so as to extend the class of functions φ that can be used.

2. CARLSON TYPE INEQUALITIES VIA INTERPOLATION

Let $(\Omega, d\mu)$ be any measure space. Suppose that weight functions $w \geq 0$, $w_0 > 0$, and $w_1 > 0$ are defined on Ω, and that the parameters $p, p_0, p_1 \in (0, \infty]$ and $\theta \in (0,1)$ are given. Let X be the weighted Lebesgue space $L_p(\Omega, w^p\, d\mu)$, and let $A_i = L_{p_i}(\Omega, w_i^{p_i}\, d\mu)$, $i = 0, 1$. The main result in this section reads (cf. [3]):

Theorem 1. *Assume that* $0 < p, p_0, p_1 \leq \infty$, $0 < \theta < 1$, *and*

$$\frac{1}{q} = \frac{1}{p} - \frac{1-\theta}{p_0} - \frac{\theta}{p_1} \geq 0.$$

For $k \in \mathbb{Z}$, *define the sets* Ω_k *by*

$$\Omega_k = \left\{ \omega \in \Omega; 2^k \leq \frac{w_0(\omega)}{w_1(\omega)} < 2^{k+1} \right\}.$$

Suppose that for some constant B *it holds that*

$$\mu(\Omega_k) \leq B, \quad k \in \mathbb{Z}. \tag{3}$$

Suppose, moreover, that for some $s \in [q, \infty]$ *we have*

$$\frac{w}{w_0^{1-\theta} w_1^{\theta}} \in L_s(\Omega, d\mu). \tag{4}$$

Then there exists a constant C *such that*

$$\|f\|_X \leq C \|f\|_{A_0}^{1-\theta} \|f\|_{A_1}^{\theta}. \tag{5}$$

Remark 1. The condition (3) is not needed in the case where $q = \infty$ or $s = q$. However, for the case $q < s \leq \infty$, the role of the measure is crucial and not fully understood.

The proof of the above theorem is divided into three steps, which we will briefly describe below (see [3] for details).

Step 1. We first assume that (4) holds for $s = \infty$. We then divide the space Ω into two parts; Ω^0 where the quotient w_0/w_1 is small, and Ω^1 where it is large, and the integral appearing on the left-hand side of the inequality is correspondingly split into a sum of two integrals. Via Hölder's inequality, the hypotheses (3) and (4) can then be used to get upper bounds for the coefficients of the two integrals in terms of geometric series. We then specify what should be meant as "small" and "large", respectively, thus specifying what the sets Ω^0 and Ω^1 really are. In this way, an additive inequality is transformed into the multiplicative inequality (5). It turns out that the constant C can be chosen as

$$C = C_0 \left\| \frac{w}{w_0^{1-\theta} w_1^{\theta}} \right\|_{L_\infty(\Omega, d\mu)} .$$

Step 2. Next, we assume that (4) holds with $s = q$. An application of Hölder's inequality with parameters

$$\frac{p}{q}, \quad \frac{(1-\theta)p}{p_0}, \quad \frac{\theta p}{p_1}$$

then yields the desired inequality with

$$C = \left\| \frac{w}{w_0^{1-\theta} w_1^{\theta}} \right\|_{L_q(\Omega, d\mu)} .$$

Step 3. By the two previous steps, we have the two inequalities

$$\|fw\|_p \leq C_0 \left\| \frac{w}{w_0^{1-\theta} w_1^{\theta}} \right\|_\infty \|fw_0\|_{p_0}^{1-\theta} \|fw_1\|_{p_1}^{\theta} \tag{6}$$

and

$$\|fw\|_p \leq \left\| \frac{w}{w_0^{1-\theta} w_1^{\theta}} \right\|_q \|fw_0\|_{p_0}^{1-\theta} \|fw_1\|_{p_1}^{\theta} . \tag{7}$$

Now, fix the function f and the weights w_0 and w_1. Define the linear (multiplication) operator T on the vector space of measurable functions on Ω by

$$TW = (f w_0^{1-\theta} w_1^{\theta}) W.$$

For any choice of W, we may choose w such that

$$|W| = \frac{w}{w_0^{1-\theta} w_1^{\theta}}, \quad \text{a.e.}$$

Thus the inequalities (6) and (7) state that

$$T : L_\infty(\Omega, d\mu) \to L_p(\Omega, d\mu)$$

with norm at most

$$C_0 \left\| f w_0 \right\|_{p_0}^{1-\theta} \left\| f w_1 \right\|_{p_1}^{\theta}$$

and

$$T : L_q(\Omega, d\mu) \to L_p(\Omega, d\mu)$$

with norm at most

$$\left\| f w_0 \right\|_{p_0}^{1-\theta} \left\| f w_1 \right\|_{p_1}^{\theta},$$

respectively. If (4) is now assumed to hold for some $s \in (q, \infty)$, put

$$\eta = 1 - \frac{q}{s},$$

so that

$$\frac{1}{s} = \frac{1-\theta}{q} + \frac{\eta}{\infty}.$$

It thus follows by the Riesz–Thorin Interpolation Theorem that

$$T : L_s(\Omega, d\mu) \to L_p(\Omega, d\mu)$$

with norm not exceeding

$$C_0^{q/s} \left\| f w_0 \right\|_{p_0}^{1-\theta} \left\| f w_1 \right\|_{p_1}^{\theta}.$$

In other words,

$$\left\| f w \right\|_p \le C_0^{q/s} \left\| \frac{w}{w_0^{1-\theta} w_1^{\theta}} \right\|_s \left\| f w_0 \right\|_{p_0}^{1-\theta} \left\| f w_1 \right\|_{p_1}^{\theta},$$

which is (5).

3. EMBEDDINGS VIA CARLSON TYPE INEQUALITIES

If A_0 and A_1 are compatible Banach spaces, an intermediate space X is said to be of class $\mathcal{C}_J(\theta; A_0, A_1)$ if there is a constant C such that

$$\|f\|_X \leq Ct^{-\theta} J(t, f; A_0, A_1), \quad f \in \Delta(A_0, A_1),$$

where J is the Peetre J functional. This condition is known to be equivalent to the Carlson type inequality (5), and also to the (continuous) embedding

$$(A_0, A_1)_{\theta,1} \subseteq X, \tag{8}$$

where the space on the left-hand side is a real interpolation space. The scale of real interpolation spaces is increasing in the second parameter; thus

$$(A_0, A_1)_{\theta,1} \subseteq (A_0, A_1)_{\theta,r}$$

for any $r > 1$.

Suppose that X, A_0, and A_1 are as in Section 2. Theorem 1 thus states that the given conditions on the weights imply the embedding (8). It turns out, however, that the same conditions are sufficient to achieve a stronger embedding.

Theorem 2. *Suppose that w, w_0, w_1, p, p_0, p_1, and θ satisfy the hypotheses of Theorem* 1. *Then*

$$(A_0, A_1)_{\theta,p} \subseteq X.$$

A detailed proof of this, which uses the Reiteration Theorem for real interpolation spaces, can be found in [3].

4. A BLOCK INEQUALITY

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave function, and let $\langle\cdot\rangle_\varphi$ be the corresponding interpolation method (this method, introduced by Peetre, is sometimes referred to as the $\pm$ method. See Gustavsson and Peetre [4] for details). Let $\mathcal{P}^\pm$ be the class of φ for which

$$\lim_{t\to 0^+} \sup_{s>0} \frac{\varphi(st)}{\varphi(s)} = \lim_{t\to\infty} \sup_{t>0} \frac{\varphi(st)}{t\varphi(st)} = 0,$$

and let $\mathcal{P}^0$ consist of those φ for which

$$\lim_{t\to 0^+} \varphi(t) = \lim_{t\to\infty} \frac{\varphi(t)}{t} = 0.$$

We associate to φ the function

$$\psi(s,t) = \begin{cases} t\varphi(\frac{s}{t}), & s,t > 0, \\ 0, & s = 0 \text{ or } t = 0. \end{cases}$$

Consider the inequality

$$\sum_k a_k \le C\psi\Big(\Big\| \frac{a_k}{\varphi(2^k)} \Big\|_{l_p}, \Big\| 2^k \frac{a_k}{\varphi(2^k)} \Big\|_{l_q} \Big). \tag{9}$$

This inequality was used by Gustavsson and Peetre [4] to identify the spaces arising when the $\langle\cdot\rangle_\varphi$ method is applied to a couple of Orlicz spaces, when $\varphi \in \mathcal{P}^{\pm}$. As was shown by Kruglyak et al. [5], the restriction on φ can be overcome if the inequality is modified as follows:

$$\sum_k a_k \le C\psi\Big(\Big\| \Big\{ \sum_{2^k \in \chi_n} \frac{a_k}{\varphi(2^k)} \Big\}_n \Big\|_{l_p}, \Big\| \Big\{ \sum_{2^k \in \chi_l} 2^k \frac{a_k}{\varphi(2^k)} \Big\}_n \Big\|_{l_q} \Big). \tag{10}$$

Here, the χ_m are intervals, whose endpoints are points from the special sequence associated to φ, which was used by Brudnyĭ and Kruglyak when solving the K-divisibility problem (see [6] and the references therein). In its simplest form, the K-divisibility property says that if there are concave functions $\varphi_0, \varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$K(t, a; A_0, A_1) \le \varphi_0 + \varphi_1,$$

where K is the Peetre K-functional, then there exists a decomposition $a = a_0 + a_1$, where $a_i \in A_i$, $i = 0, 1$, and a constant $\gamma > 0$ for which

$$K(t, x_i; A_0, A_1) \le \gamma\varphi_i, \quad i = 0, 1.$$

Note that the inequality (9) is obtained from the inequality (10) if the intervals χ_n consist of only one point each: $\chi_m = [2^m, 2^m]$. The main result of this section is the following.

Theorem 3.
(a) *In order that there exists a constant C such that* (9) *holds it is necessary and sufficient that $\varphi \in P^{\pm}$.*
(b) *In order that there exists a constant C such that* (10) *holds it is necessary and sufficient that $\varphi \in \mathcal{P}^0$.*

Remark 2. The sufficiency part of (a) was proved by Gustavsson and Peetre [4]. The remaining parts were proved by Kruglyak et al. [5].

Remark 3. As the inequality (10) allows a larger class of φ, Kruglyak et al. were able to use their block version to extend the characterization of the Gustavsson–Peetre $\langle\cdot\rangle_\varphi$ functor (see [5] for details).

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Carlsoni mittevõrdus ja interpolatsioon

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On käsitletud tihedat ühendust Carlsoni-tüüpi mittevõrduse ja interpolatsiooniteooria vahel. Eriti on juhitud tähelepanu asjaolule, et üht kaunis üldist Carlsoni-tüüpi mittevõrdust võib kasutada Gustavssoni–Peetre $\langle\cdot\rangle_\varphi$ interpolatsioonimeetodi kasulikkuse üldistamise juures.