Proc. Estonian Acad. Sci. Phys. Math., 2006, **55**, 1, 43–52

Equivalence of linear control systems on time scales

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Received 17 November 2005

Abstract. The notions of transfer matrix, transfer equivalence, and input-output equivalence for linear control systems on time scales are introduced. These concepts generalize the corresponding continuous- and discrete-time versions. Necessary and sufficient conditions for transfer and input-output equivalence are presented. As the main tool, an extension of the Laplace transform for functions defined on a time scale is used.

Key words: time scale, generalized Laplace transform, transfer matrix, input-output equivalence, transfer equivalence.

1. INTRODUCTION

The transfer and input-output equivalence for control systems has been studied both in continuous- and discrete-time cases. Especially for linear systems the definitions, properties, and results are very similar or even identical (see, for example, [1–3]).

The language of time scales, created in 1988 by Stefan Hilger [4], seems to be an ideal tool for unifying the theories of continuous- and discrete-time systems. One of the main concepts of this tool is the delta derivative, which is a generalization of the ordinary (time) derivative. In case the time scale is the real line, we get the ordinary derivative. In case the time scale is a sequence of integers, the delta derivative of a function is the difference of its values at subsequent points. Thus, differential as well as difference equations are naturally accommodated into the theory. An inverse operation to differentiation, i.e. integration, has also been extended into the time scale domain.

The transfer matrices of continuous- and discrete-time control systems are defined via the Laplace transform and the $\mathcal{Z}$-transform of the input-output equation, respectively. The use of a time scale allows uniting both transforms into one concept – the Laplace transform on a time scale. However, besides standard discrete- and continuous-time systems, this approach allows study of several other cases such as, for example, time scales based on Cantor set, harmonic numbers, etc. Another approach to unify the Laplace transform and $\mathcal{Z}$-transform, restricted to the cases $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z}$, was introduced in [5].

The main goal of this paper is to study the transfer and input-output equivalence for linear control systems, described by the set of input-output polynomial (in delta derivative operation) equations. The paper is organized as follows. In section 2 we recall the calculus on time scales. Section 3 presents the concept of Laplace transform on a time scale. In Section 4 we demonstrate that two discrete-time systems defined in terms of the forward shift operator are classically transfer equivalent if and only if their representations in terms of the forward difference operator are transfer equivalent. Section 5 defines transfer and input-output equivalence for linear control systems on a time scale and presents necessary and sufficient conditions for both cases. Section 6 presents some concluding remarks.

2. CALCULUS ON TIME SCALES

We give here a short introduction to differential calculus on time scales. This is a generalization of the standard differential calculus, on the one hand, and the calculus of finite differences, on the other hand. Then we describe the inverse operation – integration. This will allow us to solve differential equations on time scales. The proofs and more material on this subject can be found in [6].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, $A = \{t \in \mathbb{R} : t \leq 0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. We assume that $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$. For $t \in \mathbb{T}$ we define

- the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$;
- the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$;
- the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$.

On the basis of defined operators we can classify points on the real line. Namely,

- if $\sigma(t) > t$, then t is called *right-scattered*;
- if $\sigma(t) < t$, then t is called *left-scattered*;
- if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*;
- if $t > \inf \mathbb{T}$ and $\sigma(t) = t$, then t is called *left-dense*.

We define also the set $\mathbb{T}^k$ as follows: $\mathbb{T}^k := \mathbb{T}\setminus(\rho(\sup\mathbb{T}), \sup\mathbb{T}]$ if $\sup\mathbb{T} < \infty$ and $\mathbb{T}^k := \mathbb{T}$ if $\sup\mathbb{T} = \infty$, i.e. $\mathbb{T}^k$ contains those t that are nonmaximal or left-dense. Finally, we will denote $f^\sigma := f \circ \sigma$ for any function $f : \mathbb{T}\to\mathbb{R}$.

Example 2.1.
- If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{R}$, $\sigma(t) = t = \rho(t)$; the graininess function $\mu(t) \equiv 0$.
- If $\mathbb{T} = \mathbb{Z}$, then for every $t \in \mathbb{Z}$, $\sigma(t) = t + 1$, $\rho(t) = t - 1$; the graininess function $\mu(t) \equiv 1$.
- Let $q > 1$. We define the time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}\} \cup \{0\}$. Then $\sigma(t) = qt$, $\rho(t) = \frac{t}{q}$, and $\mu(t) = (q - 1)t$ for all $t \in \mathbb{T}$.

2.1. Delta derivative

Definition 2.2. *Let $f : \mathbb{T}\to\mathbb{R}$ and $t \in \mathbb{T}^k$. The* delta derivative *of f at t, denoted by $f^\triangle(t)$, is the real number (provided it exists) with the property that given any ε, there is a neighbourhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ (for some $\delta > 0$) such that*

$$|(f(\sigma(t)) - f(s)) - f^\triangle(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. Moreover, we say that f is delta differentiable *on $\mathbb{T}^k$ provided $f^\triangle(t)$ exists for all $t \in \mathbb{T}^k$.*

In general, the function σ is not required to be differentiable.

Example 2.3.
- If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R}\to\mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff $f^\triangle(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s} = f'(t)$, i.e. iff f is differentiable in the ordinary sense at t.
- If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z}\to\mathbb{R}$ is always delta differentiable at every $t \in \mathbb{Z}$ with $f^\triangle(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)} = f(t + 1) - f(t) = \triangle f(t)$, where $\triangle$ is the usual forward difference operator defined by the last equation above.
- If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then $f^\triangle(t) = \frac{f(qt)-f(t)}{(q-1)t}$ for all $t \in \mathbb{T} \setminus \{0\}$.

2.2. Properties of delta derivative

Proposition 2.4. *Let $f : \mathbb{T}\to\mathbb{R}$, $g : \mathbb{T}\to\mathbb{R}$ be two delta differentiable functions defined on the time scale $\mathbb{T}$ and let $t \in \mathbb{T}$. The delta derivative satisfies the following properties*:

(i) *if $t \in \mathbb{T}^k$, then f has at most one derivative at t;*

(ii) $(af + bg)^\triangle(t) = af^\triangle(t) + bg^\triangle(t)$ *for any constants a, b;*

(iii) $(fg)^\triangle(t) = f^\triangle(t)g(t) + f(\sigma(t))g^\triangle(t) = f(t)g^\triangle(t) + f^\triangle(t)g(\sigma(t))$;

(iv) *if $g(t)g(\sigma(t)) \neq 0$, then $(\frac{f}{g})^\triangle(t) = \frac{f^\triangle(t)g(t)-f(t)g^\triangle(t)}{g(t)g(\sigma(t))}$.*

Remark 2.5. The delta derivative of t^2 is $t + \sigma(t)$. This means that the second delta derivative of t^2 may not exist.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated*, provided its (finite) right-sided limits exist at all right-dense points in $\mathbb{T}$ and its (finite) left-sided limits exist at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous*, provided it is continuous at right-dense points in $\mathbb{T}$ and its (finite) left-sided limits exist at left-dense points in $\mathbb{T}$. It can be shown [7] that

(i) f is continuous $\Rightarrow$ f is rd-continuous $\Rightarrow$ f is regulated,

(iii) σ is rd-continuous,

(iii) the graininess function μ is rd-continuous.

Proposition 2.6. *If a function $f : \mathbb{T} \to \mathbb{R}$ has delta-derivative, then it is rd-continuous.*

2.3. Integration

A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *pre-differentiable* with (the region of differentiation) D, provided $D \subset \mathbb{T}^k$, $\mathbb{T}^k \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$. It can be proved that if f is regulated, then there exists a function F that is pre-differentiable with the region of differentiation D such that $F^{\triangle}(t) = f(t)$ for all $t \in D$. Any such function is called a pre-antiderivative of f. Then the *indefinite integral* of f is defined by $\int f(t)\triangle t := F(t) + C$, where C is an arbitrary constant. The *Cauchy integral* is

$$\int_r^s f(t)\triangle t = F(s) - F(r) \quad \text{for all} \quad r, s \in \mathbb{T}^k.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$, provided $F^{\triangle}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$.

Remark 2.7. It can be shown that every rd-continuous function has an antiderivative. Moreover, if $f(t) \geq 0$ for all $a \leq t < b$ and $\int_a^b f(\tau)\triangle\tau = 0$, then $f \equiv 0$.

Example 2.8.

- If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(\tau)\triangle\tau = \int_a^b f(\tau)\mathrm{d}\tau$, where the integral on the right-hand side is the usual Riemann integral.
- If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\int_a^b f(\tau)\triangle\tau = \sum_{t=a/h}^{b/h-1} f(th)h$ for $a < b$.

Remark 2.9. An antiderivative of 0 is 1, an antiderivative of 1 is t, but it is not possible to find a closed formula for an antiderivative of t: an antiderivative of $\frac{t^2}{2}$ is $\frac{t+\sigma(t)}{2} = t + \frac{\mu(t)}{2}$.

Under assumptions that $a \in \mathbb{T}$, $\sup\mathbb{T} = \infty$, and f is an rd-continuous function on $[a, \infty)$ we define an *improper integral* by

$$\int\limits_a^\infty f(t)\triangle\tau := \lim_{b\to\infty}\int_a^b f(t)\triangle\tau,$$

provided this limit exists.

3. LAPLACE TRANSFORM

Let $p : \mathbb{T}\to\mathbb{C}$. It has been proved [7] that there exists a unique forward solution of the initial value problem $x^\triangle = p(t)x$, $x(t_0) = x_0$ for $t \geq t_0$.

Definition 3.1. *The exponential function is a unique forward solution of the initial value problem* $x^\triangle = p(t)x$, $x(t_0) = 1$ *for* $t \geq t_0$. *It is denoted by* $e_p(\cdot, t_0)$.

Example 3.2. Let p be any complex constant function.

- If $\mathbb{T} = \mathbb{R}$, then $e_p(t, t_0) = e^{p(t-t_0)}$.
- If $h > 0$ and $\mathbb{T} = h\mathbb{Z}$, then $e_p(t, t_0) = (1 + ph)^{(t-t_0)/h}$ for $t \geq t_0$.
- If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then $e_p(t, t_0) = \prod\limits_{s\in[t_0,t)} [1 + (q-1)ps]$ for $t \geq t_0$.

Let us assume that the time scale $\mathbb{T}_0$ is such that $0 \in \mathbb{T}_0$ and $\sup\mathbb{T}_0 = \infty$. From now on we will assume that z is a constant complex function on $\mathbb{T}$. For any function z we denote

$$(\ominus z)(t) := -\frac{z}{1 + \mu(t)z},$$

where $t \in \mathbb{T}^k$. If it is well defined for all $t \in \mathbb{T}$, then z is called *regressive*. If z is regressive, then $e_{\ominus z}$ is the inverse of e_z (i.e. $e_z e_{\ominus z} = 1$).

Definition 3.3. *Assume that* $x : \mathbb{T}_0\to\mathbb{R}$ *is regulated. The Laplace transform of* x *is defined as*

$$\mathcal{L}\{x\}(z) := \int\limits_0^\infty x(t)e_{\ominus z}(\sigma(t), 0)\triangle t$$

for $z \in D\{x\}$, *where* $D\{x\}$ *is the set of all complex constant functions for which the improper integral exists.*

The Laplace transform is linear, that is the following holds for $z \in D\{x\} \cap D\{y\}$:

$$\mathcal{L}\{\alpha x + \beta y\}(z) = \alpha\mathcal{L}\{x\}(z) + \beta\mathcal{L}\{y\}(z).$$

Moreover, the following holds:

- $\mathcal{L}\{1\}(z) = \frac{1}{z}$, provided $\lim_{t\to\infty} e_{\ominus z}(t, 0) = 0$ holds;

- $\mathcal{L}\{x^{\triangle}\}(z) = z\mathcal{L}\{x\}(z) - x(0)$ and $\mathcal{L}\{x^{\triangle\triangle}\}(z) = z^2\mathcal{L}\{x\}(z) - zx(0) - x^{\triangle}(0)$, provided $\lim_{t\to\infty} x(t)e_{\ominus z}(t,0) = 0$ holds;
- $\mathcal{L}\{e_p(\cdot,0)\}(z) = \frac{1}{z-p}$, provided $\lim_{t\to\infty} e_{p\ominus z}(t,0) = 0$ holds, where $p \in \mathbb{C}$ is regressive;
- Let $x : \mathbb{T}\to\mathbb{C}$ be a regulated function; then $\mathcal{L}\{\int_0^t x(\tau)\triangle\tau\}(z) = \frac{1}{z}\mathcal{L}\{x\}(z)$ for $z \in \mathbb{C}\setminus\{0\}$ satisfying $\lim_{t\to\infty} e_{\ominus z}(t,0)\int_0^t x(\tau)\triangle\tau = 0$.

Remark 3.4. The standard formula for the Laplace transform of the shifted function

$$\mathcal{L}[f(t-t_0)\mathbb{1}(t-t_0)] = e^{-t_0 s}\mathcal{L}(f(t))$$

does not hold, in general.

Example 3.5.
- If $\mathbb{T}_0 = [0,\infty)$, then the Laplace transform defined above coincides with the standard $\mathcal{L}$-transform for the continuous-time case.
- If $\mathbb{T}_0 = \mathbb{N}_0$, then $(z+1)\mathcal{L}\{x\}(z) = \mathcal{Z}\{x\}(z+1)$, where $\mathcal{Z}\{x\}$ is the usual $\mathcal{Z}$-transform of x for the discrete-time case.

Example 3.6. [6] Let us consider the equation

$$x^{\triangle\triangle} + 5x^{\triangle} + 6x = 0, \quad x(0) = 1,\ x^{\triangle}(0) = -5$$

defined on any time scale $\mathbb{T}_0$. Using the Laplace transform defined above, we have

$$0 = z^2\mathcal{L}\{x\}(z) - z + 5 + 5[z\mathcal{L}\{x\}(z) - 1] + 6\mathcal{L}\{x\}(z),$$

$$\mathcal{L}\{x\}(z) = \frac{3}{z+3} - \frac{2}{z+2}$$

and hence

$$x(t) = 3e_{-3}(y,0) - 2e_{-2}(t,0)$$

for all $t \in \mathbb{T}_0$. One can notice that the continuous- and discrete-time cases are included within this example.

4. TRANSFER EQUIVALENCE FOR DISCRETE-TIME SYSTEMS

In this section two descriptions of discrete-time systems are considered. The first description is given in terms of the forward time-shift operator, and the second in terms of the forward time-difference operator.

If $\mathbb{T} = \mathbb{Z}$, then delta derivative $\triangle$ acts on a sequence $\{f(k)\}$ by

$$(\triangle f)(k) = f(k+1) - f(k),$$

so it is now the forward time-difference operator. Let δ denote the forward time-shift operator

$$(\delta f)(k) = f(k+1).$$

If id denotes the identity map, then $\triangle = \delta - \mathrm{id}$ and, conversely, $\delta = \triangle + \mathrm{id}$.

Usually the discrete-time linear systems are defined in terms of the time-shift operator, either in the state-space form

$$\begin{aligned} \delta x &= Ax + Bu, \\ y &= Cx + Du \end{aligned}$$

or in the input-output difference equation form

$$\Gamma : \quad M(\delta)y = N(\delta)u, \tag{1}$$

where M and N are polynomial matrices in the shift operator δ.

Assuming that sufficiently many elements of the sequences $(y(0), y(1), \ldots)$ and $(u(0), u(1), \ldots)$ are zero and applying the standard $\mathcal{Z}$-transform to input-output equation (1), one can compute the standard transfer matrix of system (1)

$$H_\Gamma(z) = M(z)^{-1}N(z).$$

We say that two systems Γ_1 and Γ_2 of the form (1) are *classically transfer equivalent* if their standard transfer matrices coincide.

Now, replacing δ by $\triangle + \mathrm{id}$, we can transform the system Γ into a different representation

$$\Lambda : \quad P(\triangle)y = Q(\triangle)u, \tag{2}$$

where P and Q are certain polynomial matrices in the difference operator Δ.

Observe that transformation of Γ to Λ is invertible. Replacing $\triangle$ by $\delta - \mathrm{id}$ in (2), we can recover the system Γ. Let T denote the map that assigns the system Λ to a system Γ, so we can write $\Lambda = T(\Gamma)$.

For Λ, we can define in analogy with [8] its transfer matrix and study transfer equivalence of two systems of this form. We prove the following.

Proposition 4.1. *Two discrete-time systems Γ_1 and Γ_2 are classically transfer equivalent if and only if the systems $T(\Gamma_1)$ and $T(\Gamma_1)$ are transfer equivalent.*

Proof. Let $\Lambda_i = T(\Gamma_i) : \quad P_i(\triangle)y = Q_i(\triangle)u$ for $i = 1, 2$. Then $P_i(\triangle) = M_i(\triangle + \mathrm{id})$ and $Q_i(\triangle) = N_i(\triangle + \mathrm{id})$ for $i = 1, 2$. Observe that

$$\mathcal{L}\{(\triangle + \mathrm{id})^n\} = (z+1)^n$$

for $n \in \mathbb{N}$ (we treat $(\triangle + \mathrm{id})^n$ as an operator on functions). Thus $\mathcal{L}\{P_i(\triangle)\} = M_i(z+1)$, $\mathcal{L}\{Q_i(\triangle)\} = N_i(z+1)$, $i = 1, 2$, and

$$\begin{aligned} G_{\Lambda_i}(z) &= P_i(z)^{-1}Q_i(z) \\ &= M_i(z+1)^{-1}N_i(z+1) = H_{\Gamma_i}(z+1). \end{aligned}$$

The systems Γ_1 and Γ_2 are classically transfer equivalent if and only if the standard transfer matrices H_{Γ_1} and H_{Γ_2} are equal. This holds if and only if the transfer matrices of Λ_1 and Λ_2 are equal, which means that Λ_1 and Λ_2 are equivalent. $\square$

Note that Proposition 4.1 for SISO systems was proved in [8].

5. TRANSFER MATRIX. INPUT-OUTPUT AND TRANSFER EQUIVALENCE

Let Λ be a system defined by the equation

$$P(\triangle)y = Q(\triangle)u, \tag{3}$$

where $y \in \mathbb{R}^r$, $u \in \mathbb{R}^m$, $\triangle$ is the operator of delta derivative on the time scale $\mathbb{T}$ (i.e. $\triangle f = f^{\triangle}$), $P(\triangle)$ and $Q(\triangle)$ are, respectively, $r \times r$ and $r \times m$ matrices whose entries are polynomials in operator $\triangle$. We assume that $\det P(\triangle) \not\equiv 0$.

Denote the maximal degrees of the entries of P and Q by d_P and d_Q, respectively. Let $y^{(k)}(0) = 0$ for $k = 0, \ldots, d_P$ and $u^{(k)}(0) = 0$ for $k = 0, \ldots, d_Q$. Applying the Laplace transform on both sides of the input-output equation (3), and taking into account the linearity of the transform as well as the fact that

$$\mathcal{L}\{x^{\triangle^n}\}(z) = z^n \mathcal{L}\{x\}(z) - \sum_{k=0}^{n-1} z^k x^{n-k-1}(0),$$

we obtain

$$P(z)Y(z) = Q(z)U(z),$$

where $Y(z)$ and $U(z)$ are the Laplace transforms of y and u, respectively.

As $P(z)$ is invertible for almost all $z \in \mathbb{C}$, we get $Y(z) = P(z)^{-1}Q(z)U(z)$. As usual, $G_\Lambda(z) = P(z)^{-1}Q(z)$ is called *the transfer matrix* of system Λ.

Definition 5.1. *Two systems* Λ_1 *and* Λ_2 *of the form* (3) *are called* transfer equivalent *on the time scale* $\mathbb{T}$ *if their transfer matrices are equal* (*as rational complex matrices*).

Obviously, transfer equivalence is an equivalence relation in the set of all systems of the form (3).

Example 5.2.

- If $\mathbb{T} = \mathbb{R}$, the definition of transfer equivalence is the same as in [9] (though in [9] the Laplace transform is avoided and the transfer matrix is a rational matrix with respect to the differential operator).
- When $\mathbb{T} = \mathbb{Z}$, our delta operator becomes the forward difference operator and not just the forward shift as in [9]. In Section 4 we showed that the two descriptions are equivalent.

Definition 5.3. *Two systems* Λ_1 *and* Λ_2 *of the form* (3) *are called* input-output equivalent *on the time scale* $\mathbb{T}$ *if they are satisfied by the same pairs* (y, u).

Let us recall that a square polynomial matrix $K(z)$ is called *unimodular* if $\det K(z)$ is constant and different from 0. Then the inverse of $K(z)$ is also a polynomial unimodular matrix. It is known [1] that $K(z)$ is unimodular if and only if it can be obtained from the identity matrix by finitely many elementary row operations over the ring of polynomials in z.

Proposition 5.4. *Two input-output systems given by matrices* $[P_1(z), Q_1(z)]$ *and* $[P_2(z), Q_2(z)]$ *are transfer equivalent if and only if there are polynomial matrices* $M_1(z), M_2(z)$ *with* $\det M_i(z) \neq 0$, $i = 1, 2$, *such that*

$$M_1(z)[P_1(z), Q_1(z)] = M_2(z)[P_2(z), Q_2(z)]. \tag{4}$$

Proof. The proof is standard. Let us assume that systems Λ_1 and Λ_2 are transfer equivalent, i.e. $G_{\Lambda_1}(z) = P_1^{-1}(z)Q_1(z) = P_2^{-1}(z)Q_2(z) = G_{\Lambda_2}(z)$. Let $d(z) = \det P_1(z) \det P_2(z)$ and define $M_1(z) := d(z)P_1^{-1}(z)$, $M_2(z) = d(z)P_2^{-1}(z)$. Then $\det M_i(z) \neq 0$, $i = 1, 2$, and

$$\begin{aligned} M_1(z)[P_1(z), Q_1(z)] &= [d(z)I, d(z)P_1^{-1}(z)Q_1(z)], \\ M_2(z)[P_2(z), Q_2(z)] &= [d(z)I, d(z)P_2^{-1}(z)Q_2(z)]. \end{aligned}$$

So we get (4).

Now let us assume that there exist polynomial matrices $M_1(z), M_2(z)$ such that (4) holds. Then

$$\begin{aligned} G_{\Lambda_1}(z) &= P_1^{-1}(z)Q_1(z) = (M_1(z)P_1(z))^{-1}M_1(z)Q_1(z) \\ &= (M_2(z)P_2(z))^{-1}M_2(z)Q_2(z) = P_2^{-1}(z)Q_2(z) = G_{\Lambda_2}(z). \end{aligned} \qquad \square$$

Proposition 5.5. *Two input-output systems given by matrices* $[P_1(z), Q_1(z)]$ *and* $[P_2(z), Q_2(z)]$ *are input-output equivalent if and only if there is a unimodular matrix* $K(z)$ *such that* $[P_1(z), Q_1(z)] = K(z)[P_2(z), Q_2(z)]$.

Proof. See [9] for the continuous-time case. The general case of arbitrary time scale is similar. $\square$

Corollary 5.6. *If two systems* Λ_i, $i = 1, 2$, *are input-output equivalent, then they are transfer equivalent.*

6. CONCLUDING REMARKS

We have generalized the notion of transfer matrix and defined the transfer equivalence for linear control systems described by input-output polynomial equations (in delta derivative operator). Moreover, we have given a characterization of this property and shown what it means for the time scale of integer numbers. A further problem to be studied is the irreducibility of the input-output equations and its reduction if equations are reducible, and the extension of the time-scales approach to the nonlinear case.

ACKNOWLEDGEMENTS

A preliminary version of this paper was presented at the 11th IEEE International Conference on Methods and Models in Automation and Robotics, 2005, Poland. The work was supported by MNiI under the Technical University of Białystok (grant No. W/IMF/1/04), by the Estonian Science Foundation (grant No. 5405), and by the Cooperation Agreement between the Estonian Academy of Sciences and Polish Academy of Sciences.

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Lineaarsete juhtimissüsteemide ekvivalentsus üldistatud ajaskaalal

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On käsitletud lineaarseid juhtimissüsteeme üldistatud ajaskaalal, mis lisaks pidevale ja diskreetsele ajaskaalale hõlmab ka muid võimalusi. Üldistatud ajaskaala on ideaalne tööriist pidevate ja diskreetsete süsteemide teooria ühendamiseks. Üldistatud ajaskaalal on defineeritud ülekandemaatriksi ja juhtimissüsteemi kaht tüüpi ekvivalentsuse – sisend-väljund- ja ülekandeekvivalentsuse – mõisted. Antud kontseptsioonid üldistavad vastavaid mõisteid pideva ja diskreetse ajaga süsteemide jaoks. On esitatud tarvilikud ning piisavad tingimused süsteemide ülekande- ja sisend-väljundekvivalentsuse kontrollimiseks. Põhiliseks tööriistaks on Laplace'i teisendus üldistatud ajaskaalal defineeritud funktsioonide jaoks.