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Equivalence of realizability conditions for nonlinear control systems

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Abstract. The relationship between three state space realizability conditions for nonlinear multi-input multi-output differential equations, formulated in terms of different mathematical tools, is studied. Moreover, explicit formulae are provided for calculating the differentials of the state coordinates which, in case the necessary and sufficient realizability conditions are satisfied, can be integrated to obtain the state coordinates. The main differences in comparison with the single-input single-output case are clarified.

Key words: nonlinear control systems, differential input-output equations, state space realization, algebraic approaches, geometric approaches.

1. INTRODUCTION

The paper compares distinct realizability conditions and realization algorithms in order to systematize the knowledge and to provide the *explicit* formulae for calculating the differentials of the state coordinates which, in case the necessary and sufficient realizability conditions are satisfied, can be integrated to obtain the state coordinates. Our aim is to extend the results of [1] to the multi-input multi-output (MIMO) case. In the above paper three apparently distinct (algebraic, geometric, and Lie brackets based) intrinsic necessary and sufficient realizability conditions [2–5] for input-output differential equations are proved to be equivalent. Moreover, it is shown that the sufficient algorithm-dependent realizability conditions [6,7] are tightly related to the above intrinsic conditions as the algorithm constructs the basis vectors for the algebraic condition. Finally,

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alternative explicit formulae for calculating the differentials of the state coordinates are suggested. Since in [5] only single-input single-output (SISO) systems are studied, we concentrate instead on paper [8] that gives algebraic conditions under which the derivatives of the inputs can be eliminated in the generalized state equations and can thus be viewed as realizability conditions. We also extend the algorithms for calculating the state coordinates from [6,7] and explicit formula from [1] for calculating the differentials of state coordinates to the MIMO case. Note that generalization to the MIMO case, though technically involved, is not difficult *once the extended system* corresponding to the set of input-output equations, *is properly defined* (see Section 4), and the results carry over to the MIMO case.

We stress that an explicit formula for calculating the basis of $\mathcal{H}_{s+2}$ is valid only under the assumption that $\mathcal{H}_1, \ldots, \mathcal{H}_{s+1}$ are completely integrable, though this assumption was not implicitly mentioned in our previous paper [1].

Note that it is not our purpose to compare the results which study realization in *specific* state space form, e.g. bilinear [9], polynomial [10] or state affine [11,12] realizations.

2. THE REALIZATION PROBLEM

Consider a nonlinear system described by p $(i = 1, \ldots, p)$ input-output differential equations where the highest derivatives of y appear linearly

$$y_i^{(n_i)} = \varphi_i(y_k, \ldots, y_k^{(n_{ik})}, u_j, \ldots, u_j^{(s_{ij})}, \;\; k = 1, \ldots, p, j = 1, \ldots, m). \tag{1}$$

Assumption 1. *System* (1) *is strictly proper, i.e.* $s_{ij} < n_i$, *for* $i = 1, \ldots, p$, $j = 1, \ldots, m$.

Assumption 2. *System* (1) *is in a* canonical form, *which means that* $n_i \geq 1$, $n_1 \leq n_2 \leq \ldots \leq n_p$, $n_{ik} < \min(n_i, n_k)$, *and* $n_1 + n_2 + \ldots + n_p = n$ *is the order of the system.*

Assumption 2 implies that whenever (1) admits a Kalmanian realization, the indices n_i, associated with each output y_i, $i = 1, \ldots, p$, are the observability indices of any observable state space realization of order n. The form (1) is an extension of the echelon canonical matrix description, introduced in [13] for linear systems. Note that every strictly proper system can be transformed into the above form [14]. Define $s := \max s_{ij}$ and note that Assumption 2 yields $s < n_p$[1].

A classical state space representation of the form

$$\begin{aligned} \dot{x} &= F(x, u), \\ y &= h(x) \end{aligned} \tag{2}$$

[1] For differences between the geometric conditions [3] and the commutativity conditions [4] for the case $s = n_p$, see [4]

is called a realization of (1) if the external behaviours of the two systems coincide, where the behaviour of (1) or (2) is the set of all pairs (u, y) that satisfy (1) or (2) (for some trajectory x), respectively. We call system (2) observable if almost everywhere

$$\mathrm{rank}\frac{\partial(y, \dot{y}, \ldots, y^{(n-1)})}{\partial x} = n.$$

In dealing with the nonlinear realization problem, we are, like in [15], interested in the *generic* realizability properties, i.e. in the properties that hold almost everywhere, except on a set of measure zero. That is, we look at dimensions (or ranks) over a field of functions, not over $I\!R$. Thus there is no argument either about the points where to evaluate dimensions or about constant dimensionality of distributions and codistributions. Involutivity of distributions and integrability of codistributions are often characterized by conditions which require that specific functions on system variables vanish. Since there are smooth functions that are neither generically zero nor generically different from zero, the notion of generic property does not make sense, in general, for systems defined by smooth functions. The situation is different if we restrict our attention to systems defined by means of analytic (or also meromorphic) functions, and this motivates our choice.

The realization problem studied in this paper is defined as follows. Given Eqs (1), with $\varphi_i(\cdot)$ analytic, find, if possible, the state coordinates $x \in I\!R^n$, $x = \psi(y_i, \ldots, y_i^{(n_i-1)}, u_j, \ldots, u_j^{(s-1)}, \quad i = 1, \ldots, p, \quad j = 1, \ldots, m)$ such that in coordinates x the system takes the form (2), with ψ, F, and h analytic functions.

The solution of the realization problem in [2–4,8] is formulated in terms of the extended state space system,

$$\dot{z} = f(z) + \sum_{j=1}^{m} g_j v_j, \tag{3}$$

associated with (1), with the inputs $v_j = u_j^{(s+1)}$, the state

$$\begin{aligned} z \quad = \quad & [y_1, \ldots, y_1^{(n_1-1)}, \ldots, y_p, \ldots, y_p^{(n_p-1)}, \\ & u_1, \ldots, u_m, \ldots, u_1^{(s)}, \ldots, u_m^{(s)}]^T \in I\!R^{n+m(s+1)} \end{aligned}$$

and the vector fields $f(z)$ and g_j defined respectively as

$$\begin{aligned} f(z) \quad = \quad & [z_2, \ldots, z_{n_1}, \varphi_1(z), \ldots, z_{n_1+\ldots+n_{p-1}+2}, \ldots, z_n, \varphi_p(z), \\ & z_{n+2}, \ldots, z_{n+s+1}, 0, \ldots, z_{n+(m-1)(s+1)+2}, \ldots, z_{n+m(s+1)}, 0]^T \end{aligned} \tag{4}$$

and $g_j = [0 \ldots 0 \; 1 \; 0 \ldots 0]^T$, where the $(n + ms + j)$th element is the only nonzero entry of g_j.

Assumption 3. *In the extended system* (3), *the highest time derivatives of all inputs equal to* $s = \max s_{ij}$ *even if in Eqs* (1) *the highest derivatives of the components are different.*

In many papers on nonlinear control, system (3) is treated as the realization of (1). The disadvantage of the *extended state space realization* is that it uses the $(s+1)$th derivative of control $u^{(s+1)} = v$ as input. For linear systems it is possible to find an extended state coordinate transformation such that the system description in the new coordinates does not involve the explicit differentiation of the input. Unfortunately, this is not always possible for nonlinear systems. Therefore, it is important to characterize the input-output models (1) which do have an observable state space representation (2) of order n and to provide the algorithm to find the state coordinates. In the next section we give a brief exposition of realizability conditions. Note that controllability (accessibility) of the realization is guaranteed by irreducibility of the set of input-output equations (1), see [5].

3. THE REALIZABILITY CONDITIONS

In this paper we work with meromorphic functions. Meromorphic functions are defined as the elements of the quotient field of the ring of analytic functions. The use of meromorphic functions is essential for carrying out division in the algorithms.

All computations in the following algorithms can be performed almost everywhere, i.e. everywhere except on the set of singularities which has measure zero. Similarly, all conditions of Theorems 1–3 hold almost everywhere, or said differently, hold generically. However, the realizations are only locally valid on an open region around some generic point.

3.1. Algebraic realizability conditions

Applying the results of [8] to a realization problem, one has to start not from the input-output differential equations (1), but from the generalized state equations

$$
\begin{array}{rcl}
\dot{z}_1 & = & z_2, \\
 & \vdots & \\
\dot{z}_{n_1-1} & = & z_{n_1}, \\
\dot{z}_{n_1} & = & \varphi_1(z_1, \ldots, z_n, u, \dot{u}, \ldots, u^{(s)}), \\
 & \vdots & \\
\dot{z}_{n_1+\ldots+n_{p-1}+1} & = & z_{n_1+\ldots+n_{p-1}+2}, \\
 & \vdots & \\
\dot{z}_{n-1} & = & z_n, \\
\dot{z}_n & = & \varphi_p(z_1, \ldots, z_n, u, \dot{u}, \ldots, u^{(s)})
\end{array}
\tag{5}
$$

associated with Eqs (1). Equations (5) are, apart from a slight difference in notation, the first n equations of the extended state space description (3).

In [8] the realization problem for MIMO nonlinear systems is studied using the language of differential forms. Necessary and sufficient conditions are formulated in terms of the integrability of certain subspaces of one-forms, classified according to their relative degrees.

Let $\mathcal{K}$ denote the field of meromorphic functions in the variables $\{z, v^{(k)}, k \geq 0\}$, associated with the extended state space system (3). Over the field $\mathcal{K}$ one can define a vector space $\mathcal{E}^* := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$, spanned by the differentials of the elements of $\mathcal{K}$. Consider a one-form $\omega \in \mathcal{E}^*$: $\omega = \sum_i \alpha_i \mathrm{d}\varphi_i, \;\; \alpha_i, \varphi_i \in \mathcal{K}$. Its derivative $\dot{\omega}$ is defined by $\dot{\omega} = \sum_i \dot{\alpha}_i \mathrm{d}\varphi_i + \alpha_i \mathrm{d}\dot{\varphi}_i$ where $\dot{z}$ is defined by (3). The relative degree r of a one-form $\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z\}$ is defined to be the least integer such that the rth derivative of one-form $\omega^{(r)} \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z\}$. If such an integer does not exist, we set $r = \infty$. A decreasing sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}^*$ is defined in [16]:

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}z\}, \\ \mathcal{H}_{k+1} &= \{\omega \in \mathcal{H}_k \mid \dot{\omega} \in \mathcal{H}_k\}, \; k \geq 1. \end{aligned} \tag{6}$$

Note that $\mathcal{H}_k$ is the space of one-forms whose relative degree is greater than or equal to k, and the subspaces $\mathcal{H}_k$ are invariant under (extended) state diffeomorphism [16]. The realizability conditions are formulated in terms of integrability of the subspaces of one-forms.

Theorem 1 [8]. *The input-output differential equations* (1) *are generically realizable in the observable state space form* (2) *iff for* $1 \leq k \leq s+2$ *the subspaces* $\mathcal{H}_k$ *defined by* (6) *for the extended system* (3) *are integrable. The state coordinates can be found by integrating the basis vectors of* $\mathcal{H}_{s+2}$.

3.2. Geometric realizability conditions

The realization problem in [2,3] is studied using the language of vector fields. The increasing sequence of distributions $\{S_k\}$ of $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\partial/\partial y_i, \ldots, \partial/\partial y_i^{(n_i-1)}, \partial/\partial u_j, \ldots, \partial/\partial u_j^{(s+1)}, i = 1, \ldots, p, j = 1, \ldots, m\}$ is defined by[2]

$$\begin{aligned} S_1 &= \mathrm{span}_{\mathcal{K}}\left\{\frac{\partial}{\partial u_j^{(s+1)}}, \;\; j = 1, \ldots, m\right\}, \\ S_{k+1} &= \bar{S}_k + [f, \bar{S}_k \cap \ker \mathrm{d}y \cap \ker \mathrm{d}u], \;\; k \geq 1, \end{aligned} \tag{7}$$

where $\bar{S}$ denotes the involutive closure of the distribution S, and $[f, S]$ denotes the distribution spanned by all Lie brackets $[f, X]$, with X a vector field belonging to S and f defined by (4). The distribution $S^* = S_{s+2}$ is the minimal conditionally invariant distribution for the extended system (3). Using the specific structure of

[2] Note that in [2] $S_1 = \mathrm{span}_{\mathcal{K}}\left\{(\partial/\partial u_j^{(s)})\right\}$.

the extended state space system (3), van der Schaft [3] has proved that if S_k, for $k = 1, \ldots, s+2$, are all involutive, then

$$S_k \subset \ker \mathrm{d}u \cap \ker \mathrm{d}y, \quad k = 1, \ldots, s+1,$$

$$S_{s+2} \cap \ker \mathrm{d}u \cap \ker \mathrm{d}y = S_{s+1}, \tag{8}$$

$$\dim S_k = km, \quad k = 1, \ldots, s+2 \ \text{ globally.}$$

The realizability conditions in [2] are formulated in terms of the involutivity of the distributions.

Theorem 2 [3]. *The input-output differential equations* (1) *are generically realizable in the observable state space form* (2) *iff all the distributions* $S_1, \ldots, S_{s+2}$ *defined by* (7) *for the extended system* (3) *are involutive.*

3.3. Realizability conditions in terms of commutativity of iterative Lie brackets

Delaleau and Respondek [4] also start from Eqs (5). The realizability conditions in [4] are formulated in terms of the iterative Lie brackets of vector fields

$$f = \sum_{i=1}^{p} \left(\dot{y}_i \frac{\partial}{\partial y_i} + \ldots + \varphi_i(\cdot) \frac{\partial}{\partial y_i^{(n_i-1)}} \right) + \sum_{j=1}^{m} \left(\dot{u}_j \frac{\partial}{\partial u_j} + \ldots + u_j^{(s+1)} \frac{\partial}{\partial u_j^{(s)}} \right)$$

and $g_j = \partial / \partial u_j^{(s)}$, $j = 1, \ldots, m$, defined by the extended system (3). Denote for $j = 1, \ldots, m$

$$\begin{aligned} L_f^0 \frac{\partial}{\partial u_j^{(s)}} &= \frac{\partial}{\partial u_j^{(s)}}, \\ L_f^k \frac{\partial}{\partial u_j^{(s)}} &= \left[f, L_f^{k-1} \frac{\partial}{\partial u_j^{(s)}} \right], k \geq 1. \end{aligned}$$

Theorem 3 [4]. *The input-output differential equations* (1) *are generically realizable in the observable state space form* (2) *iff for* $0 \leq q, \mu \leq s, 1 \leq j, l \leq m$

$$\left[L_f^q \frac{\partial}{\partial u_j^{(s)}}, L_f^\mu \frac{\partial}{\partial u_l^{(s)}} \right] \equiv 0. \tag{9}$$

Note that, in order to lower the order of the input derivative in (5) by one, condition (9) has to hold for $0 \leq q, \mu \leq 1, 1 \leq j, l \leq m$. This condition is satisfied only if $\partial^2 \varphi_i(\cdot) / \, (\partial u_j^{(s)})^2 \equiv 0$, or equivalently, if Eqs (1) are linear with respect to the highest derivatives of the inputs. Unlike the SISO case, in the MIMO case linearity with respect to the highest derivatives of controls is not sufficient for lowering the input derivatives by one. The system $\ddot{y} = y\dot{u}_1 + \dot{y}^2\dot{u}_2$ serves as an example. One can easily find that for this system $s = 1$ and

$$f = \dot{y}\frac{\partial}{\partial y} + \left(y\dot{u}_1 + \dot{y}^2\dot{u}_2\right)\frac{\partial}{\partial \dot{y}} + \dot{u}_1\frac{\partial}{\partial u_1} + \dot{u}_2\frac{\partial}{\partial u_2} + \ddot{u}_1\frac{\partial}{\partial \dot{u}_1} + \ddot{u}_2\frac{\partial}{\partial \dot{u}_2}.$$

Therefore,

$$L_f\frac{\partial}{\partial \dot{u}_1} = \left[f, \frac{\partial}{\partial \dot{u}_1}\right] = -\frac{\partial}{\partial u_1} - y\frac{\partial}{\partial \dot{y}},$$

$$L_f\frac{\partial}{\partial \dot{u}_2} = \left[f, \frac{\partial}{\partial \dot{u}_2}\right] = -\frac{\partial}{\partial u_2} - \dot{y}^2\frac{\partial}{\partial \dot{y}},$$

and

$$\left[L_f\frac{\partial}{\partial \dot{u}_1}, L_f\frac{\partial}{\partial \dot{u}_2}\right] = 2y\dot{y}\frac{\partial}{\partial \dot{y}} \not\equiv 0.$$

So, condition (9) is not satisfied for $q = \mu = 1$, though the input-output equation is linear both with respect to $\dot{u}_1$ and $\dot{u}_2$.

Theorems 1 and 2 are valid generically, i.e. they may fail at certain singular points. It may happen that at a certain singular point the conditions of Theorems 1 and 2 are not sufficient for realizability. We will demonstrate this with the following simple example.

Example 1. Consider the nonrealizable system $\ddot{y} = y\dot{u}^2$. For this system we get from (4)

$$f = \dot{y}\frac{\partial}{\partial y} + y\dot{u}^2\frac{\partial}{\partial \dot{y}} + \dot{u}\frac{\partial}{\partial u},$$

and so

$$L_f\frac{\partial}{\partial \dot{u}} = -\frac{\partial}{\partial u} - 2y\dot{u}\frac{\partial}{\partial \dot{y}}.$$

The distribution

$$S_3 = \operatorname{span}_{\mathcal{K}}\left\{\frac{\partial}{\partial u} + 2y\dot{u}\frac{\partial}{\partial \dot{y}}, \frac{\partial}{\partial \dot{u}}, \frac{\partial}{\partial \ddot{u}}\right\}$$

is noninvolutive except at the point $y = 0$ since

$$\left[L_f\frac{\partial}{\partial \dot{u}}, \frac{\partial}{\partial \dot{u}}\right] = 2y\frac{\partial}{\partial \dot{y}}.$$

At the point $y = 0$ the distribution S_3 is involutive, but obviously there exists no realization around the point $y = 0$.

4. MAIN RESULTS: THE EQUIVALENCE OF FOUR REALIZATION METHODS

The main purpose of this section is to prove the equivalence of the three different realizability conditions recalled in the previous section. Moreover, we will provide explicit formulae for calculating the basis vectors of the subspaces of one-forms $\mathcal{H}_k$, for $k = 3, \ldots, s+2$ and extend the algorithm-based solutions [6] to the MIMO case. Finally, we will demonstrate that the MIMO case can be understood as the method to compute the basis vectors for $\mathcal{H}_k$, $k = 3, \ldots, s+2$.

4.1. Relationship of the sequences $\{\mathcal{H}_k\}$ and $\{\mathcal{S}_k\}$

This subsection establishes the relation between the sequences $\{\mathcal{H}_k\}$ and $\{S_k\}$.

Lemma 1. *Assume that the distribution S_k, for $k = 1, \ldots, s+1$, is involutive, and the subspace of one-forms $\mathcal{H}_k$ annihilates the distribution S_k. Then the subspace of one-forms $\mathcal{H}_{k+1}$ annihilates the distribution S_{k+1}, that is $\mathcal{H}_{k+1}(S_{k+1}) \equiv 0$ for $k = 1, 2, \ldots, s+1$.*

This technical Lemma, proved in [1] for the SISO case, can be easily extended to the MIMO case and therefore we omit the proof. Note that the condition of involutivity of S_k is essential to the proof of Lemma 1. If we drop this assumption, $\mathcal{H}_{k+1}$ does not necessarily annihilate S_{k+1}.

Lemma 2. *If for the extended system* (3), *$\mathcal{H}_k$ is completely integrable, then $\dim \mathcal{H}_{k+1} = n + (s+1-k)m$ globally, for $k = 1, \ldots, s+1$.*

Proof. The proof is by induction. According to definition (6),

$$\mathcal{H}_3 = \operatorname{span}_{\mathcal{K}} \left\{ \mathrm{d}y_i, \ldots, \mathrm{d}y_i^{(n_i-2)}, \omega_{i,n_i}^{[2]}, \mathrm{d}u_j, \ldots, \mathrm{d}u_j^{(s-2)}, \ \ i = 1, \ldots, p, \ \ j = 1, \ldots, m \right\},$$

where for $i = 1, \ldots, p$

$$\omega_{i,n_i}^{[2]} = \mathrm{d}y_i^{(n_i-1)} - \sum_{j=1}^{m} \frac{\partial \varphi_i}{\partial u_j^{(s)}} \mathrm{d}u_j^{(s-1)}.$$

Consequently, $\mathcal{H}_3$ is spanned by $n + (s-1)m$ linearly independent one-forms and its dimension is $n + (s-1)m$. Let us assume now that

$$\mathcal{H}_k = \operatorname{span}_{\mathcal{K}} \left\{ \mathrm{d}y_i, \ldots, \mathrm{d}y_i^{(n_i-k+1)}, \omega_{i,n_i-k+3}^{[k-1]}, \ldots, \omega_{i,n_i}^{[k-1]}, \mathrm{d}u_j, \ldots, \mathrm{d}u_j^{(s-k+1)} \right\}$$

is integrable and its dimension is $n + (s-k+2)m$, where

$$\omega_{i,l_i}^{[k-1]} = \mathrm{d}y_i^{(l_i-1)} - \Phi_{i,l_i}^{[k-1]}, \ \ \forall l_i = n_i - k + 3, \ldots, n_i, \tag{10}$$

and $\Phi_{i,l_i}^{[k-1]}$ is a linear combination of differentials $\mathrm{d}u_k^{(s-k+2)},\ldots,\mathrm{d}u_j^{(s-1)}$, chosen so that $\dot{\omega}_{i,l_i}^{[k-1]}\in\mathcal{H}_{k-1}$.

To construct the codistribution $\mathcal{H}_{k+1}$ according to (6), at first we have to remove the differentials $\mathrm{d}u_j^{(s-k+1)}$ from $\mathcal{H}_k$ because their time derivatives do not belong to $\mathcal{H}_k$. So, the dimension of the codistribution reduces by m.

Note that also the time derivatives of one-forms $\mathrm{d}y_i^{(n_i-k+1)}=\omega_{i,n_i-k+2}^{[k-1]}$, $\omega_{i,n_i-k+3}^{[k-1]},\ldots,\omega_{i,n_i}^{[k-1]}$ may contain $\mathrm{d}u_j^{(s-k+2)}$:

$$\dot{\omega}_{i,K_i}^{[k-1]}=\Omega_{i,K_i}^{[k-1]}+\sum_{j=1}^{s}\hat{\Omega}_{i,K_i,j}^{[k-1]}\mathrm{d}u_j^{(s-k+2)},\ \ K_i=n_i-k+2,\ldots,n_i,$$

where the one-forms $\Omega_{i,K_i}^{[k-1]}$ belong to $\mathcal{H}_k$. Next we have to modify the one-forms $\omega_{i,K_i}^{[k-1]}$ so that their time derivatives would also belong to $\mathcal{H}_k$. For that we replace them by the one-forms

$$\omega_{i,K_i}^{[k]}=\omega_{i,K_i}^{[k-1]}-\sum_{j=1}^{s}\Omega_{i,K_i,j}^{[k-1]}\mathrm{d}u_j^{(s-k+1)},$$

which have, according to formula (10), again the form

$$\omega_{i,K_i}^{[k]}=\mathrm{d}y_i^{(K_i-1)}-\Phi_{i,K_i}^{[k]},\quad\forall K_i=n_i-k+2,\ldots,n_i,\tag{11}$$

where $\Phi_{i,K_i}^{[k]}$ as one-forms are the linear combinations of the differentials $\mathrm{d}u_i^{(s-k+1)},\ldots,\mathrm{d}u_j^{(s-1)}$. By their structure, the one-forms (11) are linearly independent and so

$$\begin{aligned}\mathcal{H}_{k+1}=\mathrm{span}_{\mathcal{K}}\Big\{&\mathrm{d}y_i,\ldots,\mathrm{d}y_i^{(n_i-k)},\omega_{i,n_i-k+2}^{[k]},\ldots,\omega_{i,n_i}^{[k]},\mathrm{d}u_j,\ldots,\mathrm{d}u_j^{(s-k)},\\&i=1,\ldots,p,\ \ j=1,\ldots,m\},\end{aligned}$$

is spanned by $n+(s-k+1)m$ linearly independent one-forms. Consequently, $\dim\mathcal{H}_k=n+(s-k+1)m$.

Theorem 4. *The subspaces $\mathcal{H}_k$, $k=3,\ldots,s+2$, defined by* (6) *for the extended system* (3), *are integrable iff the distributions S_k, $k=3,\ldots,s+2$, defined by* (7) *for the extended system* (3), *are involutive.*

Proof. Note that from (8) and from Lemma 2, if S_k, $k=3,\ldots,s+1$, are involutive, or alternatively, $\mathcal{H}_k$, $k=3,\ldots,s+1$, are integrable, then the dimensions of S_k and $\mathcal{H}_k$ $\forall k=1,\ldots,s+2$ are globally defined. From the involutivity of a constant dimensional distribution follows complete integrability of its maximal annihilator and vice versa. Therefore, to prove the theorem, we have to show that $\mathcal{H}_{k+1}$ for $k = 2, \ldots, s+1$ is the maximal annihilator of S_{k+1}, i.e. $\mathcal{H}_{k+1}(S_{k+1}) \equiv 0$ and that $\operatorname{codim} \mathcal{H}_{k+1} = \dim S_{k+1}$, given that either S_k is involutive or $\mathcal{H}_k$ is completely integrable. The codimension of $\mathcal{H}_k$ in $\hat{\mathcal{E}}^* := \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y_i, \ldots, \mathrm{d}y_i^{(n_i-1)}, \mathrm{d}u_j, \ldots, \mathrm{d}u_j^{(s+1)},\ i = 1, \ldots, p,$ $j = 1, \ldots, m\} \subset \mathcal{E}^*$ is defined to be the dimension of $\hat{\mathcal{E}}^*/\mathcal{H}_k$. Consider the subspace $\mathcal{H}_2 = \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y_i, \ldots, \mathrm{d}y_i^{(n_i-1)}, i = 1, \ldots, p, \mathrm{d}u_j, \ldots, \mathrm{d}u_j^{(s-1)},$ $j = 1, \ldots, m\}$, which is obviously a maximal annihilator of $S_2 = \operatorname{span}_{\mathcal{K}}\{\partial/\partial u_j^{(s)}, \partial/\partial u_j^{(s+1)}\}$, i.e. $\mathcal{H}_2(S_2) \equiv 0$, and moreover, $\operatorname{codim}\mathcal{H}_2 = \dim S_2$. The proof is now by induction on k. We will show that if S_k is involutive, then $\mathcal{H}_{k+1}$ is a *maximal* annihilator of S_{k+1}. From Lemma 1, $\mathcal{H}_{k+1}(S_{k+1}) \equiv 0$. Next, since by Lemma 2 $\dim \mathcal{H}_{k+1} = \dim \mathcal{H}_k - m$, or equivalently, $\operatorname{codim} \mathcal{H}_{k+1} = \operatorname{codim} \mathcal{H}_k + m$, the proof is completed by the fact that from (8) $\dim S_{k+1} = \dim S_k + m$.

Note that $\mathcal{H}_1$ is integrable by the definition and integrability of $\mathcal{H}_2$ follows from the special structure of the extended system (3). In a similar manner S_1 is involutive by the definition and involutivity of S_2 comes from the specific structure of (3). □

Note that Theorem 4 is valid only under Assumption 3. Otherwise, (8) is not valid since $S_k \not\subset \ker \mathrm{d}u \cap \ker \mathrm{d}y$ for some k values, $1 \leq k \leq s+1$, and therefore, starting from $k+1$, $\mathcal{H}_{k+1}$ is not necessarily anymore a maximal annihilator of S_{k+1}, even if $\mathcal{H}_k$ is a maximal annihilator of S_k. Really, $\mathcal{H}_k(S_k) \equiv 0$. Since $\mathcal{H}_{k+1} \subset \mathcal{H}_k$,

$$\mathcal{H}_{k+1}(S_k) \equiv 0 \tag{12}$$

as well. Next, taking the time derivative of (12) (or equivalently, Lie derivative L_f with respect to vector field f) yields

$$\dot{\mathcal{H}}_{k+1}(S_k) + \mathcal{H}_{k+1}([f, S_k]) \equiv 0. \tag{13}$$

Since $\dot{\mathcal{H}}_{k+1} \subset \mathcal{H}_k$,

$$\dot{\mathcal{H}}_{k+1}(S_k) \equiv 0. \tag{14}$$

From (13) and (14), $\mathcal{H}_{k+1}([f, S_k]) \equiv 0$ and since $[S_k \cap \ker \mathrm{d}u \cap \ker \mathrm{d}y] \subset S_k$,

$$\mathcal{H}_{k+1}(S_k + [f, S_k]) \equiv \mathcal{H}_{k+1}(S_{k+1}) \equiv 0,$$

which means that $\mathcal{H}_{k+1}$ is the annihilator of $S_{k+1} = \bar{S}_k + [f, \bar{S}_k \cap \ker \mathrm{d}u \cap \ker \mathrm{d}y]$, but not necessarily the maximal annihilator.

We demonstrate this in the example below.

Example 2. Consider a third-order input-output equation with two inputs

$$y^{(3)} = y\dot{u}_1 + \ddot{u}_2. \tag{15}$$

The total time derivative operator of (15) has the form

$$f = \dot{y}\frac{\partial}{\partial y} + \ddot{y}\frac{\partial}{\partial \dot{y}} + (y\dot{u}_1 + \ddot{u}_2)\frac{\partial}{\partial \ddot{y}} + \dot{u}_1\frac{\partial}{\partial u_1} + \ddot{u}_1\frac{\partial}{\partial \dot{u}_1} + \dot{u}_2\frac{\partial}{\partial u_2} + \ddot{u}_2\frac{\partial}{\partial \dot{u}_2} + \dddot{u}_2\frac{\partial}{\partial \ddot{u}_2}.$$

If we do not take in the extended state equations the highest derivatives of all inputs equal to $s = \max s_{ij} = 2$ and work with the coordinates $\{y, \dot{y}, \ddot{y}, u_1, u_2, \dot{u}_1, \dot{u}_2, \ddot{u}_2\}$, then the first two distributions are

$$S_1 = \operatorname{span}_{\mathcal{K}} \left\{ \frac{\partial}{\partial \ddot{u}_1}, \frac{\partial}{\partial u_2^{(3)}} \right\}, \; S_2 = \operatorname{span}_{\mathcal{K}} \left\{ \frac{\partial}{\partial \dot{u}_1}, \frac{\partial}{\partial \ddot{u}_2}, \frac{\partial}{\partial \ddot{u}_1}, \frac{\partial}{\partial u_2^{(3)}} \right\}.$$

To compute the distribution S_3, we find the following Lie brackets

$$\left[f, \frac{\partial}{\partial \dot{u}_1} \right] = -y\frac{\partial}{\partial \ddot{y}} - \frac{\partial}{\partial u_1}, \quad \left[f, \frac{\partial}{\partial \ddot{u}_2} \right] = -\frac{\partial}{\partial \ddot{y}} - \frac{\partial}{\partial \dot{u}_2},$$

and according to (7), the distribution

$$S_3 = \operatorname{span}_{\mathcal{K}} \left\{ y\frac{\partial}{\partial \ddot{y}} + \frac{\partial}{\partial u_1}, \frac{\partial}{\partial \dot{u}_1}, \frac{\partial}{\partial \ddot{y}} + \frac{\partial}{\partial \dot{u}_2}, \frac{\partial}{\partial \ddot{u}_1}, \frac{\partial}{\partial \ddot{u}_2}, \frac{\partial}{\partial u_2^{(3)}} \right\}.$$

Using again formula (7) to compute S_4, we must take into account that

$$y\frac{\partial}{\partial \ddot{y}} + \frac{\partial}{\partial u_1} \notin \ker \mathrm{d}u,$$

and therefore its Lie derivative with respect to f does not, according to (7), belong to S_4. Next, calculating the Lie brackets

$$\left[f, -\frac{\partial}{\partial \ddot{y}} - \frac{\partial}{\partial \dot{u}_2} \right] = \frac{\partial}{\partial \dot{y}} + \frac{\partial}{\partial u_2} \subset \ker \mathrm{d}y \cap \ker \mathrm{d}u,$$

we obtain

$$S_4 = \operatorname{span}_{\mathcal{K}} \left\{ \frac{\partial}{\partial u_1} + y\frac{\partial}{\partial \ddot{y}}, \frac{\partial}{\partial \dot{u}_1}, \frac{\partial}{\partial u_2} + \frac{\partial}{\partial \dot{y}}, \frac{\partial}{\partial \dot{u}_2} + \frac{\partial}{\partial \ddot{y}}, \frac{\partial}{\partial \ddot{u}_2}, \frac{\partial}{\partial u_2^{(3)}} \right\}.$$

This distribution is obviously integrable and, according to Theorem 2, Eq. (15) is realizable.

The maximal annihilators $S_k^{\perp}$ of distributions S_k are

$$\begin{aligned} S_2^{\perp} &= \operatorname{span}_{\mathcal{K}} \{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}\ddot{y}, \mathrm{d}u_1, \mathrm{d}u_2, \mathrm{d}\dot{u}_2\}, \\ S_3^{\perp} &= \operatorname{span}_{\mathcal{K}} \{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}\ddot{y} - y\mathrm{d}u_1 - \mathrm{d}\dot{u}_2, \mathrm{d}u_2\}, \\ S_4^{\perp} &= \operatorname{span}_{\mathcal{K}} \{\mathrm{d}y, \mathrm{d}\dot{y} - \mathrm{d}u_2, \mathrm{d}\ddot{y} - y\mathrm{d}u_1 - \mathrm{d}\dot{u}_2\}. \end{aligned} \tag{16}$$

According to (6),

$$\begin{aligned} \mathcal{H}_2 &= \operatorname{span}_{\mathcal{K}} \{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}\ddot{y}, \mathrm{d}u_1, \mathrm{d}u_2, \mathrm{d}\dot{u}_2\}, \\ \mathcal{H}_3 &= \operatorname{span}_{\mathcal{K}} \{\mathrm{d}y, \mathrm{d}\dot{y}, \mathrm{d}\ddot{y} - y\mathrm{d}u_1 - \mathrm{d}\dot{u}_2, \mathrm{d}u_2\}, \\ \mathcal{H}_4 &= \operatorname{span}_{\mathcal{K}} \left\{\mathrm{d}y, \dot{y}\mathrm{d}\dot{y} + y\mathrm{d}\ddot{y} - y^2\mathrm{d}u_1 - \dot{y}\mathrm{d}u_2 - y\mathrm{d}\dot{u}_2\right\}. \end{aligned} \tag{17}$$

We see that as far as the corresponding distributions S_k belong to $\ker \mathrm{d}u \cap \ker \mathrm{d}y$, i.e. for $k = 2, 3$, $S_k^{\perp} = \mathcal{H}_k$ is valid and the codistributions $\mathcal{H}_k$ are the maximal annihilators of S_k. It is obvious from (16) and (17) that $\mathcal{H}_4$ is not a maximal annihilator of S_4 since dimensions of the subspaces $S_4^{\perp}$ and $\mathcal{H}_4$ are not equal. Note that in the calculation of S_4 we had to drop the vector field $\partial/\partial\ddot{y} + \partial/\partial\dot{u}_2$ in S_3 belonging to $\ker \mathrm{d}u$, and therefore

$$\begin{aligned} S_4 \neq S_3 + [f, S_3] \quad = \quad & \mathrm{span}_{\mathcal{K}} \left\{ \dot{y}\frac{\partial}{\partial \ddot{y}} - \frac{\partial}{\partial \dot{y}}, \frac{\partial}{\partial \dot{y}} + \frac{\partial}{\partial u_2}, y\frac{\partial}{\partial \ddot{y}} + \frac{\partial}{\partial u_1}, \frac{\partial}{\partial \ddot{y}} \right. \\ & \left. - \frac{\partial}{\partial \dot{u}_2}, \frac{\partial}{\partial \dot{u}_1}, \frac{\partial}{\partial \ddot{u}_2}, \frac{\partial}{\partial \ddot{u}_1}, \frac{\partial}{\partial u_2^{(3)}} \right\}. \end{aligned}$$

Note that $\mathcal{H}_4$ is the maximal annihilator of $S_3 + [f, S_3]$. Simple calculation shows that $\mathcal{H}_4$ is even not an integrable codistribution.

This example also demonstrates that without Assumption 3 the distributions $\mathcal{H}_k$ are not necessarily integrable even if the system of input-output equations is realizable. This means that *without Assumption* the geometric realizability conditions and the conditions in terms of the Lie brackets are valid in the MIMO case, but the algebraic conditions are not.

4.2. The relationship between the geometric conditions and conditions in terms of iterative Lie brackets

Theorem 5. *Involutivity of the distributions* $S_1, \ldots, S_k$, *for* $k = 3, \ldots, s+2$ *is equivalent to condition* (9) *for* $0 \leq q,\ r \leq k-2,\ 1 \leq j,\ l \leq m.$

This Theorem, proved in [1] for the SISO case, can again be easily extended to the MIMO case under Assumption 3. We omit the proof.

4.3. Algorithms for calculating the basis vectors of $\mathcal{H}_{s+2}$

In principle, $\mathcal{H}_{s+2}$ can be found using either definition (6) or the algorithm given in [16]. However, neither of them take into account the specific simple structure of the extended system (3). If we take this structure into account, and assume integrability of $\mathcal{H}_k$, $k = 1, \ldots, s+1$, the following recursive explicit algorithm can be obtained to compute the basis of $\mathcal{H}_{k+2} = \mathrm{span}_{\mathcal{K}}\{\omega_{1,1}^{[k+1]}, \ldots, \omega_{1,n_1}^{[k+1]}, \ldots, \omega_{p,1}^{[k+1]}, \ldots, \omega_{p,n_p}^{[k+1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{(s-k-1)}\}$ from definition (6) and Theorem 4:

$$\begin{aligned} \omega_{i,l_i}^{[0]} &= \omega_{i,l_i}^{[1]} := \mathrm{d}y_{l_i}^{(l_i-1)}, \\ \omega_{i,l_i}^{[k+1]} &= \omega_{i,l_i}^{[k]} - (-1)^k \sum_{j=1}^{m} \left\langle \omega_{i,l_i}^{[k]}, L_f^k \frac{\partial}{\partial u_j^{(s)}} \right\rangle \mathrm{d}u_j^{(s-k)}, \\ i &= 1, \ldots, p,\ \ l_i = 1, \ldots, n_i,\ \ k = 1, \ldots, s. \end{aligned} \tag{18}$$

At the kth step of the algorithm the one-form $\omega_{i,l_i}^{[k]}$, obtained at the previous step, is orthogonalized with respect to the vector fields $L_f^k(\partial/\partial u_j^{(s)})$, $j = 1, \ldots, m$. From direct computation we get that $\omega_{i,l_i}^{[k+1]}$ annihilate, or equivalently, the subspace of one-forms $\mathcal{H}_{k+2}$ annihilates all the vector fields $L_f^l\left(\partial/\partial u_j^{(s)}\right)$, $l = 0, \ldots, k$, $j = 1, \ldots, m$.

Alternatively, instead of (18), another formula can be derived to compute $\omega_{i,l_i}^{[k+1]}$, $k = 1, \ldots, s$, in terms of Lie derivatives of one-forms, and not in terms of Lie derivatives of vector fields as in (18):

$$\omega_{i,l_i}^{[k+1]} = \omega_{i,l_i}^{[k]} - \sum_{j=1}^{m} \left\langle L_f^k \omega_{i,l_i}^{[k]}, \frac{\partial}{\partial u_j^{(s)}} \right\rangle \mathrm{d}u_j^{(s-k)}. \tag{19}$$

The advantage of using algorithms (18) or (19) lies in the fact that they can be directly and easily implemented in the computer algebra program Mathematica. However, integration of the subspace $\mathcal{H}_{s+2}$ to obtain the state coordinates can be difficult.

Formulae (18) and (19) are equivalent. The proof is a straightforward extension of the proof in the SISO case ([1]) and is therefore omitted.

4.4. Algorithmic realizability conditions

We extend to the MIMO case the constructive algorithm (up to the solution of a set of partial differential equations) for finding, if possible, the state coordinates from the input-output differential equations given in [6,7] for the SISO case.

Lemma 3. *The M-dimensional distribution* $\Delta = \mathrm{span}_{\mathcal{K}}\{X_1, \ldots, X_M\}$ *on an N-dimensional manifold* $(M < N)$ *has* $N - \dim\bar{\Delta}$ *functionally independent*[3] *solutions* (*invariants*), *where* $\bar{\Delta}$ *denotes the involutive closure of* Δ.

Proof. Denote the solutions of the distribution Δ by I_α. Then for all i, α

$$L_{X_i} I_\alpha \equiv 0. \tag{20}$$

Moreover, since

$$L_{[X_i,X_j]} I_\alpha = L_{X_i}(L_{X_j} I_\alpha) - L_{X_j}(L_{X_i} I_\alpha)$$

and because of (20), we have for all i, α

$$L_{[X_i,X_j]} I_\alpha \equiv 0.$$

[3] Functional independence of solutions means that none of them can be expressed in terms of the others.

So, I_α is also a solution of the distribution $\Delta \cup [\Delta, \Delta]$. Proceeding analogously, one can demonstrate that I_α are the solutions of the involutive closure of Δ. Therefore, the codistribution $\Delta^* = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}I_\alpha\}$ is a maximal annihilator of the involutive closure of Δ, and so

$$\dim \Delta^* = N - \dim \bar{\Delta}. \qquad \square$$

Define $\bar{y} = (y_1, \ldots, y_1^{(n_1-1)}, \ldots, y_p, \ldots, y_p^{(n_p-1)})$ and $\bar{u} = (u_1, \ldots, u_1^{(s-1)}, \ldots, u_m, \ldots, u_m^{(s-1)})$. The starting point for the algorithm is not Eqs (1), but the equations where the highest derivatives of controls, $u_j^{(s)}$, appear already linearly:

$$y_i^{(n_i)} = \sum_{j=1}^{m} \alpha_{ij}(\bar{y}, \bar{u}) u_j^{(s)} + \beta_i(\bar{y}, \bar{u}), \quad i = 1, \ldots, p. \tag{21}$$

The goal of the first step of the algorithm is to find the new generalized state variables $\tilde{z}_1, \ldots, \tilde{z}_n$ such that $\dot{\tilde{z}}$ does not depend on $u^{(s)}$. Note that only the n_1th, the $(n_1 + n_2)$th, ... and the nth equations of (5) depend on $u^{(s)}$. So, one can define $\tilde{z}_i = z_i$, for $i = 1, \ldots, n_1 - 1, n_1 + 1, \ldots, n_1 + n_2 - 1, \ldots, n_1 + \ldots + n_{p-1} + 1, \ldots, n - 1$ and find for $k = n_1, n_1 + n_2, \ldots, n$

$$\tilde{z}_k = r_k(\bar{y}, \bar{u}) \tag{22}$$

such that

$$\dot{\tilde{z}}_k = \sum_{i=1}^{p} \left(\dot{y}_i \frac{\partial r_k}{\partial y_i} + \ldots + y_i^{(n_i-1)} \frac{\partial r_k}{\partial y_i^{(n_i-2)}} + \left[\sum_{j=1}^{m} \alpha_{ij}(\cdot) u_j^{(s)} + \beta_i(\cdot) \right] \frac{\partial r_k}{\partial y_i^{(n_i-1)}} \right) + \sum_{j=1}^{m} \left(\dot{u}_j \frac{\partial r_k}{\partial u_j} + \ldots + u_j^{(s)} \frac{\partial r_k}{\partial u_j^{(s-1)}} \right)$$

does not depend on $u^{(s)}$, which means that $r_k(\cdot)$ has to be a solution of the set of m partial differential equations in variables $\bar{y}$ and $\bar{u}$:

$$\left\langle \mathrm{d}r, -L_f \frac{\partial}{\partial u_j^{(s)}} \right\rangle = \sum_{i=1}^{p} \alpha_{ij}(\cdot) \frac{\partial r}{\partial y_i^{(n_i-1)}} + \frac{\partial r}{\partial u_j^{(s-1)}} = 0. \tag{23}$$

Equation (23) is solvable if (9) is satisfied for $0 \le q,\ r \le 1$. Then there exist, because of Lemma 3 at least locally, $n + (s-1)m$ independent solutions $r_1 = y_1, \ldots, r_{n_1-1} = y_1^{(n_1-2)}, \ldots, r_{n_1+\ldots+n_{p-1}+1} = y_p, \ldots, r_{n-1} = y_p^{(n_p-1)}$, $r_{n+j} = u_j, \ldots, r_{n+m(s-1)+j} = u_j^{(s-2)}$ and p solutions $r_{n_1}, r_{n_1+n_2}, \ldots, r_n$ of the form (22), whose Jacobian with respect to $y_i, \ldots, y_i^{(n_i-1)}, u_j, \ldots, u_j^{(s-1)}$ is nonsingular and that satisfy (23). The generalized state equations in the new coordinates become

$$\begin{array}{rcl}
\dot{\tilde{z}}_1 &=& \tilde{z}_2, \\
&\vdots& \\
\dot{\tilde{z}}_{n_1-2} &=& \dot{\tilde{z}}_{n_1-1}, \\
\dot{\tilde{z}}_{n_1-1} &=& \tilde{\varphi}_{11}(\tilde{z}_1,\ldots,\tilde{z}_n,u,\dot{u}\ldots,u^{(s-1)}), \\
\dot{\tilde{z}}_{n_1} &=& \tilde{\varphi}_{12}(\tilde{z}_1,\ldots,\tilde{z}_n,u,\dot{u},\ldots,u^{(s-1)}), \\
&\vdots& \\
\dot{\tilde{z}}_{n_1+\ldots+n_{p-1}+1} &=& \tilde{z}_{n_1+\ldots+n_{p-1}+2}, \\
&\vdots& \\
\dot{\tilde{z}}_{n-2} &=& \tilde{z}_{n-1}, \\
\dot{\tilde{z}}_{n-1} &=& \tilde{\varphi}_{p1}(\tilde{z}_1,\ldots,\tilde{z}_n,u,\dot{u},\ldots,u^{(s-1)}), \\
\dot{\tilde{z}}_n &=& \tilde{\varphi}_{p2}(\tilde{z}_1,\ldots,\tilde{z}_n,u,\dot{u},\ldots,u^{(s-1)}).
\end{array} \tag{24}$$

At the next step, if $\tilde{\varphi}_{ij}(z,u,\ldots,u^{(s-1)}),$, $i=1,\ldots,p$ $j=1,2$, are linear in the highest time derivative of controls $u_j^{(s-1)}$, then the same procedure can be repeated for them to produce a new generalized state space representation with $u^{(s-2)}$ as the highest time derivative of the input.

Next, we will demonstrate that the algorithm described above constructs exact basis vectors for the subspaces of one-forms $\mathcal{H}_3$, whenever possible. Note that the basis vectors $\mathcal{H}_1$ are always exact by definition and the basis vectors for $\mathcal{H}_2$ are exact by the specific structure of (3). For input-output differential equations of the form (21), Eq. (23) is solvable iff $\mathcal{H}_3$ is integrable, and the solutions $r_k(\cdot)$, $k=n_1,n_1+n_2,\ldots,n$, of the form (22) define the new state coordinates $\tilde{z}_k = r_k(\bar{y},\bar{u})$. We will demonstrate that $\mathrm{d}r_k = \mathrm{d}\tilde{z}_k \in \mathcal{H}_3$. According to (18), $\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_i,\ldots,\mathrm{d}y_i^{(n_i-2)},\omega_{i,n_i}^{[2]}, i=1,\ldots,p,\mathrm{d}u,\ldots,\mathrm{d}u^{(s-2)}\}$, where for (21)

$$\begin{aligned}
\omega_{i,n_i}^{[2]} &= \mathrm{d}y_i^{(n_i-1)} + \sum_{j=1}^{m}\left\langle \mathrm{d}y_i^{(n_i-1)}, L_f\frac{\partial}{\partial u_j^{(s)}}\right\rangle \mathrm{d}u_j^{(s-1)} \\
&= \mathrm{d}y_i^{(n_i-1)} - \alpha_{ij}(\cdot)\mathrm{d}u_j^{(s-1)}.
\end{aligned} \tag{25}$$

Note that the one-form $\omega_{i,n_i}^{[2]}$ annihilates $L_f\partial/\partial u_j^{(s)}$. So, if the one-form $\omega_{i,n_i}^{[2]}$ is exact, the solution of (23) can be obtained by integrating $\omega_{i,n_i}^{[2]}$. Though the one-form (25) is not necessarily exact, from the integrability of $\mathcal{H}_3$ it is possible to find the integrating factors that make the solution exact and equal to $\mathrm{d}r_i$ with r_i being the solution of (23).

In a similar manner it can be shown that the subsequent steps of the algorithm construct exact basis vectors for $\mathcal{H}_4,\ldots,\mathcal{H}_{s+2}$, whenever possible.

5. EXAMPLES

Example 3. Consider the special class of the SISO differential equation [17]

$$y^{(n)} + b_1 y^{(n-1)} + \ldots + b_{n-1} y^{(1)} + b_n y = a_0 u + \ldots + a_s u^{(s)} + N(u, y, y^{(1)}, \ldots, y^{(n-s)}), \tag{26}$$

linear with respect to all input time derivatives. The realization of Eq. (26) in [17] was completed via a nonrecursive method which defines the state variables as follows:

$$\begin{aligned} z_i &= y^{(i-1)} + b_1 y^{(i-2)} + \ldots + b_{i-2} y^{(1)} - a_{n-i+2} u^{(1)} \\ &\quad - a_s u^{(s-n+i-1)} + [b_{i-1} y - a_{n-i+1} u], \;\; i = n-s+1, \ldots, n, \\ z_i &= y^{(i-1)} + b_1 y^{(i-2)} + \ldots + b_{i-1} y, \;\; i = 2, \ldots, n-s, \\ x_1 &= y. \end{aligned} \tag{27}$$

Comparing (27) with the recursive algorithmic method described in Subsection 4.3, one can see that, although the state variables (27) differ from those defined in the present paper (18), they also satisfy Eq. (23) for a SISO case,

$$a_s \frac{\partial r}{\partial y^{(n-1)}} + \frac{\partial r}{\partial u^{(s-1)}} = 0$$

and also the equations defined in the next steps of the algorithm. This shows that the method of [17] can be understood as a special case of the algorithmic method of Section 4, applied to a restricted special input-output differential equation.

Example 4. We will demonstrate, using the example below, the equivalence of the considered methods. Consider the system

$$\ddot{y}_1 = y_2 u_1 + \dot{u}_2, \;\; \ddot{y}_2 = y_1 \dot{u}_1, \tag{28}$$

where

$$\begin{aligned} f = \dot{y}_1 \frac{\partial}{\partial y_1} + (y_2 u_1 + \dot{u}_2) \frac{\partial}{\partial \dot{y}_1} + \dot{y}_2 \frac{\partial}{\partial y_2} + y_1 \dot{u}_1 \frac{\partial}{\partial \dot{y}_2} \\ + \dot{u}_1 \frac{\partial}{\partial u_1} + \dot{u}_2 \frac{\partial}{\partial u_2} + \ddot{u}_1 \frac{\partial}{\partial \dot{u}_1} + \ddot{u}_2 \frac{\partial}{\partial \dot{u}_2}. \end{aligned}$$

In order to calculate the sequence of subspaces $\{\mathcal{H}_k\}$ by (18), we first find

$$L_f \frac{\partial}{\partial \dot{u}_1} = -\frac{\partial}{\partial u_1} - y_1 \frac{\partial}{\partial \dot{y}_2}, \quad L_f \frac{\partial}{\partial \dot{u}_2} = -\frac{\partial}{\partial u_2} - \frac{\partial}{\partial \dot{y}_1},$$

which also yields that Lie brackets based conditions (9) are satisfied for $q = \mu = 1$. So,

$$\omega_{11}^{[2]} = \mathrm{d}y_1,\ \ \omega_{21}^{[2]} = \mathrm{d}y_2,\ \ \omega_{12}^{[2]} = \mathrm{d}\dot{y}_1 - \mathrm{d}u_2,\ \ \omega_{22}^{[2]} = \mathrm{d}\dot{y}_2 - y_1\mathrm{d}u_1$$

and

$$\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y_1, \mathrm{d}\dot{y}_1 - \mathrm{d}u_2, \mathrm{d}y_2, \mathrm{d}\dot{y}_2 - y_1\mathrm{d}u_1\},$$

which is obviously completely integrable.

Next calculate according to (7),

$$S_3 = \mathrm{span}_{\mathcal{K}}\left\{\frac{\partial}{\partial u_1} + y_1\frac{\partial}{\partial \dot{y}_2}, \frac{\partial}{\partial \dot{u}_1}, \frac{\partial}{\partial u_1^{(2)}}, \frac{\partial}{\partial u_2} + \frac{\partial}{\partial \dot{y}_1}, \frac{\partial}{\partial \dot{u}_2}, \frac{\partial}{\partial u_2^{(2)}}\right\},$$

which is involutive, and the maximal annihilator of $\mathcal{H}_3$. We can find the coordinates

$$x_1 = y_1,\ \ x_2 = \dot{y}_1 - u_2,\ \ x_3 = y_2,\ \ x_4 = \dot{y}_2 - u_1y_1 \tag{29}$$

of the integrable classical state space realization by integrating the integrable basis vectors of $\mathcal{H}_3$, that is after changing $\omega_{22}^{[2]}$ by $\omega_{22}^{[2]} - u_1\omega_{11}^{[2]}$. So, the state equations are

$$\begin{aligned}\dot{x}_1 &= x_2 + u_2,\\ \dot{x}_2 &= x_3u_1,\\ \dot{x}_3 &= x_4 + u_1x_1,\\ \dot{x}_4 &= -u_1(x_2 + u_2).\end{aligned}$$

The state coordinates r_2 and r_4 can be found from (23) as the solutions of the set of two partial differential equations

$$\frac{\partial r}{\partial u_1} + y_1\frac{\partial r}{\partial \dot{y}_2} = 0,\ \ \frac{\partial r}{\partial u_2} + \frac{\partial r}{\partial \dot{y}_1} = 0,$$

whereas $r_1 = y_1$ and $r_3 = y_2$. It is easy to see that x_2 and x_4 in (29) provide the solution.

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Mittelineaarsete juhtimissüsteemide realiseeritavustingimuste ekvivalentsus

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On võrreldud erinevaid tarvilikke ja piisavaid tingimusi mitme sisendi ning väljundiga kõrgemat järku diferentsiaalvõrrandite mittelineaarse süsteemi realiseeritavuseks esimest järku diferentsiaalvõrrandite süsteemina ja tõestatud reali-

seeritavustingimuste ekvivalentsus. Vaatluse all on esiteks geomeetrilised tingimused, milleks on võrrandisüsteemile vastavas laiendatud olekuruumis nende võrrandite põhjal defineeritud teatud jaotuste involutiivsus. Teiseks on käsitletud algebralisi tingimusi, mis seisnevad nimetatud võrranditele vastavate kaasjaotuste integreeritavuses. Kolmas tingimus on esitatud laiendatud olekuruumis defineeritud vektorväljade Lie sulgude kommutatiivsuse kaudu. Uued muutujad laiendatud olekuruumis, millele üleminek võimaldab kõrgemat järku diferentsiaalvõrrandite süsteemi realisatsiooni, on ühtlasi esimeses tingimuses sisalduvate jaotuste lahendid, teises tingimuses sisalduvate kaasjaotuste esimesed integraalid ja kolmandas tingimuses sisalduvate vektorväljade invariandid. Ühtlasi on antud valemid selliste diferentsiaalvormide arvutamiseks, mille integreerimine annab realisatsiooni võimaldavad olekumuutujad.