Proc. Estonian Acad. Sci. Phys. Math., 2006, **55**, 1, 3–15

Generalization of superconnection in noncommutative geometry

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Received 21 November 2005, in revised form 22 December 2005

Abstract. We propose the notion of a $\mathbb{Z}_N$-connection, where $N \geq 2$, which can be viewed as a generalization of the notion of a $\mathbb{Z}_2$-connection or superconnection. We use the algebraic approach to the theory of connections to give the definition of a $\mathbb{Z}_N$-connection and to explore its structure. It is well known that one of the basic structures of the algebraic approach to the theory of connections is a graded differential algebra with differential d satisfying $d^2 = 0$. In order to construct a $\mathbb{Z}_N$-generalization of a superconnection for any $N > 2$, we make use of a $\mathbb{Z}_N$-graded q-differential algebra, where q is a primitive Nth root of unity, with N-differential d satisfying $d^N = 0$. The concept of a graded q-differential algebra arises naturally within the framework of noncommutative geometry and the use of this algebra in our construction involves the appearance of q-deformed structures such as graded q-commutator, graded q-Leibniz rule, and q-binomial coefficients. Particularly, if $N = 2, q = -1$, then the notion of a $\mathbb{Z}_N$-connection coincides with the notion of a superconnection. We define the curvature of a $\mathbb{Z}_N$-connection and prove that it satisfies the Bianchi identity.

Key words: superconnection, covariant derivative, graded differential algebra, graded q-differential algebra.

1. INTRODUCTION

The concept of a superconnection was proposed by Mathai and Quillen [1] (see also [2]) in the 1980s to represent the Thom class of a vector bundle by a differential form having a Gaussian shape. Later, Atiyah and Jeffrey [3] proposed the geometric approach to a topological quantum field theory on a four-dimensional manifold [4] based on the superconnection formalism. Assuming that a vector bundle $\pi : E \to M$ has a $\mathbb{Z}_2$-graded structure, i.e. it is a superbundle, the total grading of an E-valued differential form can be defined as the sum of two gradings, one of which comes from the $\mathbb{Z}_2$-graded structure of the algebra of differential forms on a base manifold M and the other from a $\mathbb{Z}_2$-graded structure of a superbundle E. A superconnection is a linear mapping of odd degree with respect to this total grading, behaving like a graded differentiation with respect to the multiplication by differential forms. Consequently, if we wish to generalize the notion of a superconnection to any integer $N > 2$, we must have a $\mathbb{Z}_N$-graded analogue of an algebra of differential forms, and assuming that a vector bundle has also a $\mathbb{Z}_N$-graded structure, we can elaborate a generalization of a superconnection following the scheme proposed by Mathai and Quillen. In the present paper we introduce the notion of a $\mathbb{Z}_N$-connection, where N is any integer satisfying $N \geq 2$, within the framework of an algebraic approach to the theory of connections. The first component of our construction is a $\mathbb{Z}_N$-graded q-differential algebra [5−8], where q is a primitive Nth root of unity, denoted by $\mathcal{B}$. This algebra plays the role of an analogue of an algebra of differential forms. It should be mentioned that a differential d of $\mathcal{B}$ satisfies $d^N = 0$. The second component is a $\mathbb{Z}_N$-graded left module $\mathcal{E}$ over the subalgebra $\mathcal{A} \subset \mathcal{B}$ of the elements of grading zero of $\mathcal{B}$. From a geometric point of view, a module $\mathcal{E}$ can be considered as an analogue of the space of sections of a $\mathbb{Z}_N$-graded vector bundle. Taking the tensor product $\mathcal{E}_{\mathcal{B}} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$, which can be viewed as an analogue of a space of $\mathbb{Z}_N$-graded vector bundle valued differential forms, and defining the $\mathbb{Z}_N$-graded structure on this product, we give the definition of a $\mathbb{Z}_N$-connection D in the spirit of Mathai and Quillen. We show that the Nth power of a $\mathbb{Z}_N$-connection is the grading zero endomorphism of the left $\mathcal{B}$-module $\mathcal{E}_{\mathcal{B}}$, and we define the curvature F_D of a $\mathbb{Z}_N$-connection by $F_D = D^N$. It is proved that the curvature of a $\mathbb{Z}_N$-connection satisfies the Bianchi identity.

2. GRADED q-DIFFERENTIAL ALGEBRAS

In this section we describe a generalization of a graded differential algebra, which naturally arises in the framework of q-deformed structures. This generalization is called a graded q-differential algebra, where q is a primitive Nth root of unity. We show that given a graded unital associative algebra over $\mathbb{C}$ with element v satisfying $v^N = e$, where e is the identity element of this algebra, one can construct the graded q-differential algebra by means of a q-commutator.

Let $\mathcal{B} = \oplus_{k\in\mathbb{Z}}\mathcal{B}^k$ be an associative unital $\mathbb{Z}$-graded algebra over $\mathbb{C}$. We shall denote the identity element of $\mathcal{B}$ by e and the grading of a homogeneous element $\omega \in \mathcal{B}$ by $|\omega|$, i.e. if $\omega \in \mathcal{B}^k$, then $|\omega| = k$. An algebra $\mathcal{B}$ is said to be a graded q-differential algebra ([5,6]), where q is a primitive Nth root of unity ($N \geq 2$), if it is endowed with a linear mapping $d : \mathcal{B}^k \to \mathcal{B}^{k+1}$ of degree 1 satisfying the graded q-Leibniz rule $d(\omega\,\omega') = d(\omega)\,\omega' + q^{|\omega|}\omega\, d(\omega')$, where $\omega, \omega' \in \mathcal{B}$, and $d^N(\omega) = 0$ for any $\omega \in \mathcal{B}$. A mapping d is called an N-differential of a graded q-differential algebra. It is easy to see that a graded q-differential algebra is a generalization of the notion of a graded differential algebra, since a graded differential algebra is a particular case of a graded q-differential algebra if $N = 2$ and $q = -1$.

From the graded structure of an algebra $\mathcal{B}$ it follows that the subspace $\mathcal{B}^0 \subset \mathcal{B}$ of elements of grading zero is the subalgebra of an algebra $\mathcal{B}$. The pair $(\mathcal{B}, d)$ is said to be an N-differential calculus on a unital associative algebra $\mathcal{A}$ if $\mathcal{B}$ is a graded q-differential algebra with N-differential d and $\mathcal{A} = \mathcal{B}^0$. For any $k \in \mathbb{Z}$ the subspace $\mathcal{B}^k$ of elements of grading k has the structure of a bimodule over the subalgebra $\mathcal{B}^0$ and a graded q-differential algebra can be viewed as an N-differential complex ([6])

$$\ldots \xrightarrow{d} \mathcal{B}^{k-1} \xrightarrow{d} \mathcal{B}^k \xrightarrow{d} \mathcal{B}^{k+1} \xrightarrow{d} \ldots,$$

with differential d satisfying the graded q-Leibniz rule. If $\mathcal{B}$ is a $\mathbb{Z}$-graded q-differential algebra, then we can define the $\mathbb{Z}_N$-graded structure on an algebra $\mathcal{B}$ by putting $\mathcal{B}^{\bar{p}} = \oplus_{i\in\mathbb{Z}} \mathcal{B}^{Ni+p}$, where $p = 0, 1, 2, \ldots, N-1$, and $\bar{p}$ is the residue class of an integer p modulo N. Then $\mathcal{B} = \oplus_{p\in\mathbb{Z}_N} \mathcal{B}^p$. In what follows, if a graded structure of an algebra $\mathcal{B}$ is concerned, we shall always mean the above-described $\mathbb{Z}_N$-graded structure of $\mathcal{B}$. Since all graded structures considered in this paper are $\mathbb{Z}_N$-graded structures, we always assume that the values of each index related to a graded structure are elements of $\mathbb{Z}_N$. If there is no confusion, we shall denote the values of indices by $0, 1, 2, \ldots, N-1$ meaning the residue classes modulo N.

Let us now show that if a graded unital associative algebra contains an element v satisfying $v^N = e$, where e is the identity element of this algebra, then one equips this algebra with the N-differential satisfying the graded q-Leibniz rule, turning this algebra into a graded q-differential algebra. Let $\mathcal{A}$ be an associative unital $\mathbb{Z}_N$-graded algebra over the complex numbers $\mathbb{C}$ and $\mathcal{A}^k \subset \mathcal{A}$ be the subspace of homogeneous elements of a grading k. Given a complex number $q \neq 1$, one defines a q-commutator of two homogeneous elements $w, w' \in \mathcal{A}$ by the formula

$$[w, w']_q = ww' - q^{|w||w'|} w'w.$$

Using the associativity of an algebra $\mathcal{A}$ and the property $|ww'| = |w| + |w'|$ of its graded structure, it is easy to show that for any homogeneous elements $w, w', w'' \in \mathcal{A}$ it holds that

$$[w, w'w'']_q = [w, w']_q w'' + q^{|w||w'|} w'[w, w'']_q. \tag{1}$$

Given an element v of grading 1, i.e. $v \in \mathcal{A}^1$, one can define the mapping $d_v : \mathcal{A}^k \to \mathcal{A}^{k+1}$ by the formula $d_v w = [v, w]_q$, $w \in \mathcal{A}^k$. It follows from the property of q-commutator (1) that d_v is the linear mapping of degree 1 satisfying the graded q-Leibniz rule $d_v(ww') = d_v(w)w' + q^{|w|} w d_v(w')$, where w, w' are homogeneous elements of $\mathcal{A}$. Let $[k]_q = 1 + q + q^2 + \ldots + q^{k-1}$ and $[k]_q! = [1]_q [2]_q \ldots [k]_q$.

Lemma 1. *For any integer $k \geq 2$ the kth power of the mapping d_v can be written as follows*:

$$d_v^k w = \sum_{i=0}^{k} p_i^{(k)} v^{k-i} w v^i,$$

where w is a homogeneous element of $\mathcal{A}$ and

$$p_i^{(k)} = (-1)^i q^{|w|_i} \frac{[k]_q!}{[i]_q![k-i]_q!} = (-1)^i q^{|w|_i} \left[\begin{array}{c} k \\ i \end{array} \right]_q,$$

$$|w|_i = i|w| + \frac{i(i-1)}{2}.$$

The proof of this lemma is based on the following identities:

$$p_0^{(k)} = p_0^{(k+1)} = 1, \quad p_{k+1}^{(k+1)} = -q^{|w|+k} p_k^{(k)},$$

$$p_i^{(k+1)} = p_i^{(k)} - q^{|w|+k} p_{i-1}^{(k)}, \quad 1 \leq i \leq k.$$

Theorem 1. *If N is an integer such that $N \geq 2$, q is a primitive Nth root of unity, $\mathcal{A}$ is a $\mathbb{Z}_N$-graded algebra containing an element v satisfying $v^N = e$, where e is the identity element of an algebra $\mathcal{A}$, then $\mathcal{A}$ equipped with the linear mapping $d_v = [v,\ ]_q$ is a graded q-differential algebra with N-differential d_v, i.e. d_v satisfies the graded q-Leibniz rule and $d_v^N w = 0$ for any $w \in \mathcal{A}$.*

Proof. It follows from Lemma 1 that if q is a primitive Nth root of unity, then for any integer $l = 1, 2, \ldots, N-1$ the coefficient $p_l^{(N)}$ contains the factor $[N]_q$ which vanishes in the case of q being a primitive Nth root of unity. This implies $p_l^{(N)} = 0$. Thus $d_v^N(w) = v^N w + (-1)^N q^{|w|_N} w v^N$. Taking into account that $v^N = e$, we obtain $d_v^N(w) = (1 + (-1)^N q^{|w|_N}) w = \lambda\, w$. The coefficient $\lambda = 1 + (-1)^N q^{|w|_N}$ vanishes if q is a primitive Nth root of unity. Indeed, if N is an odd number, then $1 - (q^N)^{(N-1)/2} = 0$. In the case of an even integer N we have $1 + (q^{N/2})^{N-1} = 1 + (-1)^{N-1} = 0$, and this ends the proof of the theorem.

For applications in differential geometry it is important to have a realization of a graded q-differential algebra as an algebra of analogues of differential forms on a geometric space. The proved theorem allows us to construct a graded q-differential algebra taking as a starting point a generalized Clifford algebra. The structure of a generalized Clifford algebra suggests that we shall get an analogue of an algebra of differential forms with an N-differential on a noncommutative space. Indeed, let us remind that a generalized Clifford algebra $\mathcal{C}_{p,N}$ is a unital associative algebra over $\mathbb{C}$ generated by $\gamma_1, \gamma_2, \ldots, \gamma_p$ which are subjected to the relations

$$\gamma_i \gamma_j = q^{\mathrm{sg}(j-i)} \gamma_j \gamma_i, \quad \gamma_i^N = 1, \quad i, j = 1, 2, \ldots, p, \tag{2}$$

where q is a primitive Nth root of unity and $\mathrm{sg}(x)$ is the usual sign function. The structure of a graded q-differential algebra in the case of the generalized Clifford algebra with two generators is studied in [9]. In this case the corresponding generalized Clifford algebra $\mathcal{C}_{2,N}$ can be interpreted as an algebra of polynomial functions on a reduced quantum plane. Let us denote by x, y the generators of the algebra in this case. The relations (2) take on the form $xy = q\,yx, x^N = y^N = 1$. The algebra $\mathcal{C}_{2,N}$ becomes a $\mathbb{Z}_N$-graded algebra if we assign the grading zero to the generator x, the grading 1 to the generator y and define the grading of any monomial made up of generators x, y as the sum of gradings of its factors. The differential d is defined by $dw = [y, w]_q, w \in \mathcal{C}_{2,N}$. Since $y^N = 1$, it follows from Theorem 1 that the algebra $\mathcal{C}_{2,N}$ is a graded q-differential algebra and d is its N-differential. We give this graded q-differential algebra and its N-differential d the following geometric interpretation: the subalgebra of polynomials of grading zero is the algebra of functions on a one-dimensional space with "coordinate" x, and the elements of higher gradings expressed in terms of "coordinate" x and its "differential" dx are the analogues of differential forms with exterior differential d. We have $dx = y\Delta_q x = y(x - qx)$. Since $d^k \neq 0$ for $k < N$, a differential k-form w may be expressed either by means of $(dx)^k$ or by means of $d^k x$, where

$$d^k x = \frac{[k]_q}{q^{k(k-1)/2}}(dx)^k\, x^{1-k}.$$

If $w = (dx)^k f(x)$, where $f(x)$ is a polynomial of grading zero, and $dw = (dx)^{k+1}\delta_x^{(k)}(f)$, then

$$\delta_x^{(k)}(f) = (\Delta_q x)^{-1}(q^{-k} f - q^k A(f)),$$

where A is the homomorphism of the algebra of polynomials of grading zero determined by $A(x) = qx$. The higher-order derivatives $\delta_x^{(k)}$ have the property

$$\delta_x^{(k)}(fg) = \delta_x^{(k)}(f)\, g + q^k\, A(f)\, \delta_x^{(0)}(g), \quad k = 0, 1, 2, \ldots, N-1,$$

where $\delta_x^{(0)}(g) = \frac{\partial g}{\partial x} = (\Delta_q x)^{-1}(g - A(g))$ is the A-twisted derivative. A higher-order derivative $\delta_x^{(k)}$ can be expressed in terms of the derivative $\frac{\partial}{\partial x}$ as follows:

$$\delta_x^{(k)} = q^k\, \frac{\partial}{\partial x} + \frac{q^{-k} - q^k}{1 - q}\, x^{-1}.$$

The realization of a graded q-differential algebra as an algebra of analogues of differential forms on an ordinary (commutative) space is constructed in [10]. Let $x_1, x_2, \ldots, x_n$ be the coordinates of an n-dimensional space $\mathbb{R}^n$, $C^\infty(\mathbb{R}^n)$ be the algebra of smooth $\mathbb{C}$-valued functions, and $dx_1, dx_2, \ldots, dx_n$ be the differentials of the coordinates. Let $\mathcal{N} = \{1, 2, \ldots, n\}$ be the set of integers, I be a subset of $\mathcal{N}$, and $|I|$ be the number of elements in I. Given any subset I of $\mathcal{N}$, i.e.

$I = \{i_1, i_2, \ldots, i_k\} \subset \mathcal{N}, 1 \le i_1 < i_2 < \ldots i_k \le n$, we associate to I the formal monomial dx_I, where $dx_I = dx_{i_1} dx_{i_2} \ldots dx_{i_n}$ and $dx_\emptyset = 1$. Let $\Omega(\mathbb{R}^n)$ be the free left $C^\infty(\mathbb{R}^n)$-module generated by all formal monomials dx_I. It is evident that $\Omega(\mathbb{R}^n)$ has a natural $\mathbb{Z}$-graded structure $\Omega(\mathbb{R}^n) = \oplus_k \Omega^k(\mathbb{R}^n)$, where $\Omega^k(\mathbb{R}^n)$ is the left $C^\infty(\mathbb{R}^n)$-module freely generated by all dx_I, where I contains k elements. An element of the module $\Omega^k(\mathbb{R}^n)$ has the form

$$\omega = \sum_{I,|I|=k} f_I \, dx_I = \sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} f_{i_1 i_2 \ldots i_k} dx_{i_1} dx_{i_2} \ldots dx_{i_k}, \tag{3}$$

where $f_I = f_{i_1 i_2 \ldots i_k} \in C^\infty(\mathbb{R}^n)$. Let us define the degree 1 linear operator $d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ by the formula

$$d\omega = \sum_{J,|J|=k+1} g_J dx_J, \; g_{j_1 j_2 \ldots j_{k+1}} = \sum_{m=1}^{k+1} q^{m-1} \frac{\partial f_{j_1 j_2 \ldots \hat{j}_m \ldots j_{k+1}}}{\partial x_{j_m}}, \tag{4}$$

where ω has the form (3) and $J = \{j_1, j_2, \ldots, j_{k+1}\}$. It can be shown that $d^N \omega = 0$ for any $\omega \in \Omega(\mathbb{R}^n)$. Thus we have the N-differential complex

$$\ldots \xrightarrow{d} \Omega^{k-1}(\mathbb{R}^n) \xrightarrow{d} \Omega^k(\mathbb{R}^n) \xrightarrow{d} \Omega^{k+1}(\mathbb{R}^n) \xrightarrow{d} \ldots \tag{5}$$

We shall call (5) an N-differential de Rham complex. In order to define the structure of an algebra on the N-differential de Rham complex (5), we introduce the following notations: if I, J are two subsets of $\mathcal{N}$ satisfying $I \cap J = \emptyset$, then we denote by $b(I,J)\,(a(I,J))$ the number of pairs $(i,j) \in I \times J$ such that $i > j\,(i < j)$, and $c(I,J) = b(I,J) - a(I,J)$. It is easy to check that for any subsets I, J we have $b(I,J) = a(J,I)$, $a(I,J) + b(I,J) = |I||J|$ and $c(I,J) = -c(J,I)$. Let us define the multiplication on the left $C^\infty(\mathbb{R}^n)$-module $\Omega(\mathbb{R}^n)$ by the following rules:

$$f\, dx_I = dx_I\, f, \;\; dx_I\, dx_J = \begin{cases} 0 & \text{if } I \cap J \neq \emptyset, \\ q^{b(I,J)}\, dx_{I \cup J} & \text{if } I \cap J = \emptyset. \end{cases} \tag{6}$$

The left module $\Omega(\mathbb{R}^n)$ with the product defined by the rules (6) is a graded associative algebra, and it can be shown that the N-differential d defined by (4) satisfies the graded q-Leibniz rule with respect to this product, which implies that $\Omega(\mathbb{R}^n)$ is a graded q-differential algebra with N-differential d. We shall call an element of this algebra a differential form and d the N-exterior differential. It is evident that taking $q = -1$ in (4), (6), we get the classical algebra of differential forms with exterior differential d satisfying $d^2 = 0$. It follows from (6) that $dx_I dx_J = q^{c(I,J)} dx_J dx_I = q^{|I||J| - 2a(I,J)} dx_J dx_I$, and in the special case of $q = -1$ this commutation relation depends only on the gradings $|I|, |J|$. This leads to the supercommutativity of the algebra of differential forms in the classical case.

3. $\mathbb{Z}_N$-CONNECTION AND ITS CURVATURE

In this section we give the definition of a $\mathbb{Z}_N$-connection, the curvature of a $\mathbb{Z}_N$-connection, and prove the Bianchi identity. We also study the structure of a $\mathbb{Z}_N$-connection and show that a superconnection is a particular case of a $\mathbb{Z}_N$-connection for $N = 2$.

Let $\mathcal{A}$ be a unital associative $\mathbb{C}$-algebra, $(\mathcal{B}, d)$ be an N-differential calculus over $\mathcal{A}$, and $\mathcal{E} = \oplus_{k\in\mathbb{Z}_N}\mathcal{E}^k$ be a $\mathbb{Z}_N$-graded left $\mathcal{A}$-module. It is evident that $\mathcal{E}$ has the structure of $\mathbb{Z}_N$-graded $\mathbb{C}$-vector space induced by a left $\mathcal{A}$-module structure if one defines $\alpha\xi = (\alpha\, e)\,\xi$, where $\alpha \in \mathbb{C}, \xi \in \mathcal{E}$, e is the identity element of $\mathcal{A}$. Let $\mathcal{E}_{\mathcal{B}} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$ be the tensor product $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$ of the right $\mathcal{A}$-module $\mathcal{B}$ and the left $\mathcal{A}$-module $\mathcal{E}$. A graded q-differential algebra $\mathcal{B}$ can be viewed as a $(\mathcal{B}, \mathcal{B})$-bimodule, which implies that the tensor product $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$ has the structure of the left $\mathcal{B}$-module. Since an algebra $\mathcal{B}$ can also be viewed as an $(\mathcal{A}, \mathcal{A})$-bimodule, the tensor product $\mathcal{E}_{\mathcal{B}} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$ has also the left $\mathcal{A}$-module structure. It should be mentioned that $\mathcal{E}_{\mathcal{B}}$ has also the structure of $\mathbb{C}$-vector space which is the tensor product of $\mathbb{C}$-vector space structures of $\mathcal{B}$ and $\mathcal{E}$.

Each factor in the tensor product $\mathcal{E}_{\mathcal{B}} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$ has the $\mathbb{Z}_N$-graded structure. Using these $\mathbb{Z}_N$-graded structures, one can construct a $\mathbb{Z}_N$-graded structure on the tensor product $\mathcal{E}_{\mathcal{B}}$ as follows: given two homogeneous elements $\omega \in \mathcal{B}, \xi \in \mathcal{E}$, one defines the total grading of the element $\omega \otimes_{\mathcal{A}} \xi \in \mathcal{E}_{\mathcal{B}}$ by $|\omega \otimes_{\mathcal{A}} \xi| = |\omega| + |\xi|$. Then

$$\mathcal{E}_{\mathcal{B}} = \oplus_{k\in\mathbb{Z}_N}\mathcal{E}^k_{\mathcal{B}}, \quad \mathcal{E}^k_{\mathcal{B}} = \oplus_{m+l=k}\mathcal{E}^{m,l}_{\mathcal{B}} = \oplus_{m+l=k}\, \mathcal{B}^m \otimes_{\mathcal{A}} \mathcal{E}^l,$$

where $k, l, m \in \mathbb{Z}_N$. If we consider the tensor product $\mathcal{E}_{\mathcal{B}}$ as the left $\mathcal{B}$-module, then multiplication by a homogeneous element $\omega \in \mathcal{B}$ of grading k maps an element $\xi \in \mathcal{E}^{m,l}_{\mathcal{B}}$ into the element $\omega\xi \in \mathcal{E}^{m+k,l}_{\mathcal{B}}$, i.e. $\mathcal{E}^n_{\mathcal{B}} \xrightarrow{\omega} \mathcal{E}^{n+k}_{\mathcal{B}}$. If we consider the tensor product $\mathcal{E}_{\mathcal{B}}$ as the left $\mathcal{A}$-module, then multiplication by any element $u \in \mathcal{A}$ preserves the $\mathbb{Z}_N$-graded structure of $\mathcal{E}_{\mathcal{B}}$. Consequently, if $m + l = k$, then $\mathcal{E}^{m,l}_{\mathcal{B}}$ is the left $\mathcal{A}$-submodule of a left $\mathcal{A}$-module $\mathcal{E}^k_{\mathcal{B}}$. Let us denote

$$\Gamma_{\mathcal{B}}(\mathcal{E}) = \oplus_l\mathcal{E}^{0,l}_{\mathcal{B}}, \;\; \Omega^k_{\mathcal{B}}(\mathcal{E}) = \oplus_l\mathcal{E}^{k,l}_{\mathcal{B}}, \;\; k \geq 1.$$

The $\mathbb{Z}_N$-graded left $\mathcal{A}$-module $\Gamma_{\mathcal{B}}(\mathcal{E})$ is isomorphic to a left $\mathcal{A}$-module $\mathcal{E}$. The corresponding isomorphism $\varrho : \mathcal{E} \to \Gamma_{\mathcal{B}}(\mathcal{E})$ is defined for any $\xi \in \mathcal{E}$ by $\varrho(\xi) = e \otimes_{\mathcal{A}} \xi \in \Gamma_{\mathcal{B}}(\mathcal{E})$, where e is the identity element of $\mathcal{A}$. It is worth mentioning that the isomorphism ϱ preserves the graded structures of the $\mathcal{A}$-modules $\mathcal{E}$ and $\Gamma_{\mathcal{B}}(\mathcal{E})$, i.e. $\varrho : \mathcal{E}^k \to \mathcal{E}^{0,k}_{\mathcal{B}}$.

Let $\mathrm{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ be the space of endomorphisms of the vector space $\mathcal{E}_{\mathcal{B}}$, $\mathrm{End}_{\mathcal{A}}(\mathcal{E}_{\mathcal{B}})$ be the space of endomorphisms of the left $\mathcal{A}$-module $\mathcal{E}_{\mathcal{B}}$, and $\mathrm{End}_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$ be the space of endomorphisms of the left $\mathcal{B}$-module $\mathcal{E}_{\mathcal{B}}$. Obviously, $\mathrm{End}_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}}) \subset \mathrm{End}_{\mathcal{A}}(\mathcal{E}_{\mathcal{B}}) \subset \mathrm{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$. The space $\mathrm{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ is a $\mathbb{Z}_N$-graded unital associative algebra if one takes the product $A \circ B$ of two endomorphisms of the space $\mathcal{E}_{\mathcal{B}}$

as an algebra multiplication. The $\mathbb{Z}_N$-graded structure of this algebra as well as the $\mathbb{Z}_N$-graded structures of the spaces $\mathrm{End}_{\mathcal{A}}(\mathcal{E}_{\mathcal{B}}), \mathrm{End}_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$ are induced by the total $\mathbb{Z}_N$-graded structure of $\mathcal{E}_{\mathcal{B}}$. Thus we have the decomposition $\mathrm{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}}) = \oplus_k \mathrm{End}^k_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$, where $\mathrm{End}^k_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ is the space of homogeneous endomorphisms of grading k of the space $\mathcal{E}_{\mathcal{B}}$, and the similar decompositions for the spaces $\mathrm{End}_{\mathcal{A}}(\mathcal{E}_{\mathcal{B}})$, $\mathrm{End}_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$. The structure of a graded associative algebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ allows us to use the q-commutator $[A,B]_q = A\circ B - q^{|A||B|} B\circ A$, where $A, B \in \mathrm{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$, and $|A|, |B|$ are the corresponding gradings.

Definition 1. *A $\mathbb{Z}_N$-graded $\mathcal{B}$-connection on the $\mathbb{Z}_N$-graded left $\mathcal{B}$-module $\mathcal{E}_{\mathcal{B}}$ is an endomorphism D of degree 1 of the vector space $\mathcal{E}_{\mathcal{B}}$ satisfying the condition*

$$D(\omega\,\xi) = d(\omega)\,\xi + q^{|\omega|}\omega\, D(\xi), \tag{7}$$

where $\omega \in \mathcal{B},\ \xi \in \mathcal{E}_{\mathcal{B}}$, and d is the N-differential of a $\mathbb{Z}_N$-graded q-differential algebra $\mathcal{B}$.

Since we use the same graded q-differential algebra $\mathcal{B}$ in many of our constructions, and in order to simplify the terminology, we shall call D a $\mathbb{Z}_N$-connection on the module $\mathcal{E}_{\mathcal{B}}$. Hence, a $\mathbb{Z}_N$-connection D can be viewed as an element of grading 1 of the $\mathbb{Z}_N$-graded algebra $\mathrm{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$, i.e. $D \in \mathrm{End}^1_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$, and the behaviour of this element with respect to the structure of the left $\mathcal{B}$-module of $\mathcal{E}_{\mathcal{B}}$ is fixed by the condition (7).

We can extend a $\mathbb{Z}_N$-connection D to act on the $\mathbb{Z}_N$-graded algebra $\mathrm{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ in a way consistent with the graded q-Leibniz rule if we put

$$D(A) = [D, A]_q = D\circ A - q^{|A|} A\circ D, \quad A \in \mathrm{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}}).$$

It is evident that $D : \mathrm{End}^k_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}}) \to \mathrm{End}^{k+1}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ and

$$D(A\,B) = D(A)\circ B + q^{|A|} A\circ D(B).$$

Proposition 1. *For any homogeneous endomorphism A of the left $\mathcal{B}$-module $\mathcal{E}_{\mathcal{B}}$, homogeneous element ω of the algebra $\mathcal{B}$, and $\xi \in \mathcal{E}_{\mathcal{B}}$ it holds that*

$$D(A)(\omega\xi) = (1 - q^{|A|})\, d\omega\, A(\xi) + q^{|\omega|}\, \omega\, D(A)(\xi).$$

It follows from Proposition 1 that if A is a grading zero endomorphism of the left $\mathcal{A}$-module $\mathcal{E}_{\mathcal{B}}$, i.e. $A \in \mathrm{End}^0_{\mathcal{A}}(\mathcal{E}_{\mathcal{B}})$, then $D(A)(u\xi) = u\, D(A)(\xi)$ for any $u \in \mathcal{A}$ and $\xi \in \mathcal{E}_{\mathcal{B}}$. Consequently, $D(A) \in \mathrm{End}^1_{\mathcal{A}}(\mathcal{E}_{\mathcal{B}})$. Particularly, if $A \in \mathrm{End}^0_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$, then $D(A) \in \mathrm{End}^1_{\mathcal{A}}(\mathcal{E}_{\mathcal{B}})$.

Proposition 2. *For any $\mathbb{Z}_N$-connection D the Nth power of an endomorphism $D \in \mathrm{End}^1_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ is the grading zero endomorphism of the left $\mathcal{B}$-module $\mathcal{E}_{\mathcal{B}}$, i.e. $D^N \in \mathrm{End}^0_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$.*

Proof. It suffices to show that for any homogeneous $\omega \in \mathcal{B}$ an endomorphism $D \in \mathrm{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ satisfies $D^N(\omega\,\xi) = \omega\, D^N(\xi)$, where $\xi \in \mathcal{E}_{\mathcal{B}}$. We can expand the kth power of an endomorphism D as follows:

$$D^k(\omega\,\xi) = \sum_{m=0}^{k} q^{m|\omega|} \left[\begin{array}{c} k \\ m \end{array}\right]_q d^{k-m}(\omega)\; D^m(\xi). \tag{8}$$

Since d is the N-differential of an algebra $\mathcal{B}$ which implies $d^N\omega = 0$, and $\left[\begin{array}{c} N \\ m \end{array}\right]_q = 0$ for q being a primitive Nth root of unity, where $1 \leq m \leq N-1$, the expansion (8) takes on the form

$$D^N(\omega\xi) = q^{N|\omega|}\;\omega\; D^N(\xi) = \omega\; D^N(\xi).$$

Definition 2. *The curvature F_D of a $\mathbb{Z}_N$-connection D is the endomorphism D^N of grading zero of the left $\mathbb{Z}_N$-graded $\mathcal{B}$-module $\mathcal{E}_{\mathcal{B}}$, i.e. $F_D = D^N \in \mathrm{End}^0_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$.*

Proposition 3. *For any $\mathbb{Z}_N$-connection D the curvature F_D of this connection satisfies the Bianchi identity $D(F_D) = 0$.*

Proof. We have $D(F_D) = [D, F_D]_q = D \circ F_D - F_D \circ D = D^{N+1} - D^{N+1} = 0$.

In order to understand better the structure of a $\mathbb{Z}_N$-connection, we shall need a notion of a covariant derivative. Let $(\mathfrak{M}, \mathfrak{d})$ be a differential calculus over an algebra $\mathfrak{A}$, i.e. $\mathfrak{M}$ is a $\mathfrak{A}$-bimodule and $\mathfrak{d} : \mathfrak{A} \to \mathfrak{M}$ is a linear mapping satisfying the Leibniz rule $\mathfrak{d}(\mathfrak{a}\mathfrak{b}) = \mathfrak{d}(\mathfrak{a})\,\mathfrak{b} + \mathfrak{a}\,\mathfrak{d}(\mathfrak{b})$ for any $\mathfrak{a}, \mathfrak{b} \in \mathfrak{A}$. Let $\mathfrak{F}$ be a left $\mathfrak{A}$-module. A linear mapping $\nabla : \mathfrak{F} \to \mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{F}$ is said to be a covariant derivative on $\mathfrak{F}$ with respect to a differential calculus $(\mathfrak{M}, \mathfrak{d})$ if it satisfies

$$\nabla(\mathfrak{a}\mathfrak{f}) = \mathfrak{d}(\mathfrak{a}) \otimes_{\mathfrak{A}} \mathfrak{f} + \mathfrak{a}\nabla(\mathfrak{f}), \tag{9}$$

for any $\mathfrak{a} \in \mathfrak{A}, \mathfrak{f} \in \mathfrak{F}$.

Let us show that a $\mathbb{Z}_N$-connection D induces the covariant derivative. According to the definition of a $\mathbb{Z}_N$-connection, we have $D : \mathcal{E}^k_{\mathcal{B}} \to \mathcal{E}^{k+1}_{\mathcal{B}}$. The left $\mathcal{A}$-modules $\mathcal{E}^k_{\mathcal{B}}, \mathcal{E}^{k+1}_{\mathcal{B}}$ split into the direct sums

$$\begin{aligned} \mathcal{E}^k_{\mathcal{B}} &= \oplus_{m+l=k}\, \mathcal{E}^{m,l}_{\mathcal{B}} = \mathcal{E}^{0,k}_{\mathcal{B}} \oplus \mathcal{E}^{1,k-1}_{\mathcal{B}} \oplus \mathcal{E}^{2,k-2}_{\mathcal{B}} \oplus \ldots \oplus \mathcal{E}^{N-1,k+1}_{\mathcal{B}}, \\ \mathcal{E}^{k+1}_{\mathcal{B}} &= \oplus_{m+l=k+1}\, \mathcal{E}^{m,l}_{\mathcal{B}} = \mathcal{E}^{0,k+1}_{\mathcal{B}} \oplus \mathcal{E}^{1,k}_{\mathcal{B}} \oplus \mathcal{E}^{2,k-1} \oplus \ldots \oplus \mathcal{E}^{N-1,k+2}_{\mathcal{B}}. \end{aligned}$$

Let $p_{i,j} : \mathcal{E}_{\mathcal{B}} \to \mathcal{E}^{i,j}_{\mathcal{B}}, p_i : \mathcal{E}_{\mathcal{B}} \to \Omega^i_{\mathcal{B}}(\mathcal{E}), \pi_k : \mathcal{B} \to \mathcal{B}^k, \rho_l : \mathcal{E} \to \mathcal{E}^l$ be the projections of the left $\mathcal{A}$-modules onto their $\mathcal{A}$-submodules. It is evident that each projection is the homomorphism of the corresponding left $\mathcal{A}$-modules, $p_{k,l} = \pi_k \otimes_{\mathcal{A}} \rho_l$ and

$$p_k = \sum_l p_{k,l}, \quad p_{k,l}(\omega \otimes_{\mathcal{A}} \xi) = \pi_k(\omega) \otimes_{\mathcal{A}} \rho_l(\xi), \quad \forall \omega \in \mathcal{B}, \xi \in \mathcal{E}.$$

The pair $(\mathcal{B}^1, d)$ is the differential calculus over an algebra $\mathcal{A}$ and $\mathcal{E}$ is a left $\mathcal{A}$-module. Let us consider the linear mapping $\nabla_D : \mathcal{E} \to \Omega^1_{\mathcal{B}}(\mathcal{E})$ defined by the formula $\nabla_D = p_1 \circ D \circ \varrho$.

Proposition 4. *The linear mapping ∇_D is the covariant derivative on a left $\mathcal{A}$-module $\mathcal{E}$ with respect to the differential calculus $(\mathcal{B}^1, d)$. The covariant derivative ∇_D preserves the $\mathbb{Z}_N$-graded structures of the left $\mathcal{A}$-modules $\mathcal{E}$ and $\Omega^1_{\mathcal{B}}(\mathcal{E})$, i.e. $\nabla_D : \mathcal{E}^k \to \mathcal{E}^{1,k}_{\mathcal{B}}$.*

Proof. In order to show that ∇_D is the covariant derivative on a left $\mathcal{A}$-module $\mathcal{E}$, we check the covariant derivative condition (9). For any $u \in \mathcal{A}, \xi \in \mathcal{E}$ we have

$$\begin{aligned}
\nabla_D(u\xi) &= p_1(D(\varrho(u\xi))) = p_1(D(u\varrho(\xi))) = p_1(du\,\varrho(\xi) + u\,D\varrho(\xi)) \\
&= p_1(du\,(e \otimes_{\mathcal{A}} \xi)) + u\,p_1\,(D\varrho(\xi)) = p_1(du \otimes_{\mathcal{A}} \xi) + u\nabla_D(\xi) \\
&= \sum_l p_{1,l}(du \otimes_{\mathcal{A}} \xi) + u\nabla_D(\xi) = \pi_1(du) \otimes_{\mathcal{A}} \sum_l \rho_l(\xi) + u\nabla_D(\xi) \\
&= du \otimes_{\mathcal{A}} \xi + u\nabla_D(\xi).
\end{aligned}$$

The algebraic approach to a $\mathbb{Z}_N$-connection described in this section has a geometric realization on a vector bundle in the case of $N = 2, q = -1$ leading to the known notion of a superconnection. Let us consider a superbundle $E = E^+ \oplus E^-$ over a smooth n-dimensional manifold M^n. In this case let $\mathcal{B}$ be the algebra of differential forms on a manifold M^n and d be the exterior differential of this algebra. It is evident that $\mathcal{B}$ is a $\mathbb{Z}_2$-graded algebra, where the grading of a homogeneous differential form is equal to its degree modulo 2. The exterior differential satisfies $d^2 = 0$ and we see that the algebra of differential forms on a finite-dimensional manifold is a special case of a graded q-differential algebra for $N = 2$ and $q = -1$. Let $\mathcal{B}^0 = \mathcal{A}$ be the algebra of smooth functions on a manifold M^n, and $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be the left $\mathbb{Z}_2$-graded $\mathcal{A}$-module of smooth sections of a superbundle E. Then the tensor product $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{E}$ is the space of smooth differential forms on a manifold M^n with values in a superbundle E. The total grading of a homogeneous differential form with values in E is the sum of two gradings, where the first is determined by the graded structure of the algebra of differential forms and the second is determined by the graded structure of a superbundle E. The space $\mathrm{End}_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$ is the space of differential forms on a manifold M with the values in the superbundle $\mathrm{Hom}(E, E)$. The q-commutator becomes the supercommutator if we take $q = -1$. Finally, the definition of a $\mathbb{Z}_N$-connection coincides in this special case with the definition of a superconnection as it is given in [2].

Let us construct an example of a $\mathbb{Z}_N$-connection. One can extend the N-differential of an algebra $\mathcal{B}$ to act on the $\mathbb{Z}_N$-graded left $\mathcal{B}$-module $\mathcal{E}_{\mathcal{B}}$ in a way consistent with the graded q-Leibniz rule by putting $d(\omega \otimes_{\mathcal{A}} \xi) = d(\omega) \otimes_{\mathcal{A}} \xi$, where $\omega \in \mathcal{B}, \xi \in \mathcal{E}$. It is evident that $d \in \mathrm{End}^1_{\mathbb{C}}(\mathcal{E}_{\mathcal{B}})$. Let L be an endomorphism of grading 1 of a left $\mathcal{A}$-module $\mathcal{E}$, i.e. $L \in \mathrm{End}^1_{\mathcal{A}}(\mathcal{E})$. This endomorphism can be extended to the $\mathcal{B}$-endomorphism of the left $\mathcal{B}$-module $\mathcal{E}_{\mathcal{B}}$ in a way consistent with the $\mathbb{Z}_N$-graded structure of $\mathcal{E}_\mathcal{B}$ by means of $L(\omega \otimes_\mathcal{A} \xi) = q^{|\omega|}\, \omega \otimes_\mathcal{A} L(\xi)$. Indeed, if $\zeta = \omega \otimes_\mathcal{A} \xi \in \mathcal{E}_\mathcal{B}$, then

$$\begin{aligned} L(\theta\zeta) &= L(\theta(\omega \otimes_\mathcal{A} \xi)) = L((\theta\omega) \otimes_\mathcal{A} \xi) = q^{|\theta|+|\omega|}(\theta\omega) \otimes_\mathcal{A} L(\xi) \\ &= q^{|\theta|}\, \theta\, (q^{|\omega|}\omega \otimes_\mathcal{A} L(\xi)) = q^{|\theta|}\, \theta\, L(\zeta). \end{aligned}$$

Obviously, $L \in \mathrm{End}^1_\mathcal{B}(\mathcal{E}_\mathcal{B}) \subset \mathrm{End}^1_\mathbb{C}(\mathcal{E}_\mathcal{B})$. The endomorphism $D = d+L$ of grading 1 of the vector space $\mathcal{E}_\mathcal{B}$ is a $\mathbb{Z}_N$-connection. Indeed, for any $\omega \in \mathcal{B}, \zeta \in \mathcal{E}_\mathcal{B}$ we have

$$\begin{aligned} D(\omega\zeta) &= (d+L)(\omega\zeta) = d(\omega\zeta) + L(\omega\zeta) \\ &= d(\omega)\,\zeta + q^{|\omega|}\omega\, d(\zeta) + q^{|\omega|}\omega L(\zeta) \\ &= d(\omega)\zeta + q^{|\omega|}\omega\, D(\zeta). \end{aligned}$$

If A is an endomorphism of a left $\mathcal{A}$-module $\mathcal{E}$, then we can decompose it into the homogeneous parts $A_{ij}, i, j \in \mathbb{Z}_N$ with respect to the $\mathbb{Z}_N$-grading of $\mathcal{E}$, where $A_{ij} : \mathcal{E}^j \to \mathcal{E}^i$. Then, for any element $\xi = \xi_0 + \xi_1 + \ldots + \xi_{N-1} \in \mathcal{E}$ we have $A(\xi)_i = \sum_{j\in\mathbb{Z}_N} A_{ij}(\xi_j)$, where $A(\xi)_i$ is the component of grading $i \in \mathbb{Z}_N$ of the element $A(\xi)$. We can extend the action of homogeneous parts A_{ij} to the $\mathbb{Z}_N$-graded left $\mathcal{B}$-module $\mathcal{E}_\mathcal{B}$. If we associate the $N \times N$-matrix $(A_{ij}), i, j \in \mathbb{Z}_N$ to an endomorphism A, where the entries of the matrix are the homogeneous parts of A, then in the case of L we get

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & L_{0,N-1} \\ L_{1,0} & 0 & 0 & \ldots & 0 & 0 \\ 0 & L_{2,1} & 0 & \ldots & 0 & 0 \\ 0 & 0 & L_{3,2} & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & L_{N-1,N-2} & 0 \end{pmatrix}. \tag{10}$$

Let us denote by $\{d^m, L_1\, L_2 \ldots L_k\}$, where m, k are non-negative integers, the sum of all possible products made up of the mappings $d, L_1, L_2, \ldots, L_k$, where each product contains m-times the differential d and k mappings $L_1, L_2, \ldots, L_k$ succeeding in the same order. For instance, for $k = 0$ we have $\{d^m, L_1\, L_2 \ldots L_k\} = d^m$, and for $m = 2, k = 1$ we have $\{d^2, L\} = d^2\, L + d\, L\, d + L\, d^2$. The curvature of the $\mathbb{Z}_N$-connection $D = d + L$ can be written as follows:

$$F_D = D^N = \sum_{m+k=N} \{d^m, L^k\}.$$

Using the matrix associated to L, we obtain the $N \times N$-matrix corresponding to the curvature F_D, where the entry $F_{D,ij}$ of this matrix can be written as follows:

$$F_{D,ij} = \sum_{m+k=N, i,j\in\mathbb{Z}_N} \{d^m, L_{i,i-1}\, L_{i-1,i-2} \ldots L_{j+1,j}\}, \tag{11}$$

where m, k are non-negative integers running $m, k = 0, 1, \ldots, N$, and each product in $\{d^m, L_{i,i-1}\, L_{i-1,i-2} \ldots L_{j+1,j}\}$ contains k entries of the matrix associated to L, which means that $i - j = k$. For instance, if $N = 2$, then from (10) and (11) we obtain the matrix of a superconnection $D = d + L$ and the matrix of its curvature, which can be written in the standard notations of supergeometry $\mathcal{E}_{\bar{0}} = \mathcal{E}^+, \mathcal{E}_{\bar{1}} = \mathcal{E}^-, L^+ = L_{\bar{0}\bar{1}}, L^- = L_{\bar{1}\bar{0}}$ as follows:

$$L \to \begin{pmatrix} 0 & L^- \\ L^+ & 0 \end{pmatrix}, \quad F_D \to \begin{pmatrix} L^-L^+ & dL^- \\ dL^+ & L^+L^- \end{pmatrix}.$$

The appearance of the terms generated by an endomorphism L is a peculiar property of the curvature of a $\mathbb{Z}_N$-connection. Just this property of the curvature of a superconnection makes it possible to construct the representative of the Thom class of a fibre bundle rapidly decreasing in a fibre direction. This property plays also an essential part in the Atiyah–Jeffrey geometric approach [3] to the Lagrangian of a topological field theory on a four-dimensional manifold [4]. If, following the algebraic scheme described in this paper, we construct a $\mathbb{Z}_N$-connection on a noncommutative space with the help of analogues of differential forms with exterior differential d satisfying $d^N = 0$, then the extra terms (11) of the curvature will enable us to construct an analogue of the representative of a Thom class rapidly decreasing along a fibre in the case of a noncommutative space. In turn, this representative could be taken as a starting point for an analogue of a topological field theory on a noncommutative space with BRST-like symmetry δ_{BRST} satisfying $\delta^N_{\text{BRST}} = 0$ up to gauge transformation. Since a topological field theory on a four-dimensional manifold is related to a supersymmetric Yang-Mills theory [11], we may expect that a noncommutative analogue of a topological field theory could be related to a field theory with a fractional supersymmetry [12].

ACKNOWLEDGEMENTS

The author is grateful to the referees for helpful suggestions and remarks. This work was supported by the Estonian Science Foundation (grant No. 6206).

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Superseostuse üldistus mittekommutatiivses geomeetrias

Viktor Abramov

On sisse toodud $\mathbb{Z}_N$-seostuse mõiste, mida võib vaadelda superseostuse ehk $\mathbb{Z}_2$-seostuse üldistusena. Superseostuse mõiste on antud V. Mathai ja D. Quilleni artiklis [1], kus autorid on superseostust kasutanud vektorkonna topoloogiliste invariantide konstrueerimiseks. Hiljem on M. F. Atiyah ja L. Jeffrey [3] kasutanud superseostuste formalismi E. Witteni [4] topoloogilise kvantväljateooria geomeetrilise interpretatsiooni kirjeldamiseks. Superseostus tugineb oluliselt diferentsiaalvormide algebra ja supervektorkonna $\mathbb{Z}_2$-gradueeringutele. $\mathbb{Z}_N$-seostus defineeritakse seostuste teooria algebralise formalismi raames. Konstruktsioon baseerub $\mathbb{Z}_N$-gradueeringuga q-diferentsiaalalgebral $\mathcal{B}$ N-diferentsiaaliga d, mis rahuldab tingimust $d^N = 0$ ([5,6]), ja $\mathbb{Z}_N$-gradueeringuga vasakpoolsel moodulil üle algebra $\mathcal{B}$ gradueeringuga 0 elementide alamalgebra. On defineeritud $\mathbb{Z}_N$-seostuse kõverus ja tõestatud, et kõverus rahuldab Bianchi samasust.