Proc. Estonian Acad. Sci. Phys. Math., 2001, **50**, 2, 86–97

COERCIVITY INEQUALITY FOR THE QUASILINEAR FINITE DIFFERENCE OPERATOR

Malle FISCHER

Institute of Applied Mathematics, University of Tartu, Liivi 2, 50409 Tartu, Estonia; malle_f@ut.ee

Received 22 December 2000, in revised form 12 February 2001

Abstract. The coercivity inequality is established for the finite difference operator which approximates in two- or three-dimensional unit cube the quasilinear elliptic operator for the second boundary value problem. The obtained results are based on special discrete analogues of the Sobolev imbedding theorems.

Key words: finite difference method.

1. INTRODUCTION: NOTATIONS AND AUXILIARY RESULTS

The coercivity inequalities have been used in the study of convergence of the finite difference method in strong Sobolev norms (see [1,2]). In the present paper we establish a coercivity inequality for the finite difference operator that approximates the m-dimensional ($m = 2, 3$) Neumann boundary value problem for the quasilinear elliptic operator of the 2nd order on the unit cube. In case $m = 3$ we assume that the differential operator contains no mixed derivatives. Note that in the case of the Dirichlet boundary value problem ($m = 2, 3$) the coercivity inequality holds without this restriction (see [1]).

Let

$$\Omega = \{0 < x_i < 1, i = 1, \dots, m\}$$

be the m-dimensional unit cube with the boundary $\partial\,\Omega$ and closure $\overline{\Omega}$. Introduce the grid

$$\overline{\Omega}_h = \{\xi = (k_1 h, \dots, k_m h), \quad k_i = 0, 1, \dots, n;\ i = 1, \dots, m\}, \quad h = \frac{1}{n},$$

and denote

$$\Omega_h = \overline{\Omega}_h \cap \Omega\,, \quad \partial\,\Omega_h = \overline{\Omega}_h \cap \partial\Omega\,.$$

Given a grid functions $y\colon \Omega_h \to R$, we prolong it to the boundary $\partial\,\Omega_h$ using the values of y at the nearest grid points of Ω_h, e.g.

$$y(k_1 h, \dots, k_{l-1}h, 0, k_{l+1}h, \dots, k_m h) = \\ y(k_1 h, \dots, k_{l-1}h, h, k_{l+1}h, \dots, k_m h),$$

$$y(k_1 h, \dots, k_{l-1}h, 1, k_{l+1}h, \dots, k_m h) = \\ y\big(k_1 h, \dots, k_{l-1}h, (n-1)h, k_{l+1}h, \dots, k_m h\big);$$

$$k_i = 1, \dots, n-1, \quad i = 1, \dots, m.$$

Into the remaining grid points of $\partial\Omega_h$ we prolong the grid function y also with the value at the nearest point of the set Ω_h. For the prolonged grid functions we define the discrete Laplace operator by the formula

$$-\Delta_h y(x) = -\sum_{i=1}^{m} \partial_i \overline{\partial}_i y(x)\,, \quad x \in \Omega_h\,,$$

where

$$\partial_i y = \frac{1}{h}(y^{+1_i} - y), \quad \overline{\partial}_i y = \frac{1}{h}(y - y^{-1_i})\,,$$

$$y^{+1_i} = y(x + he_i)\,, \quad y^{-1_i} = y(x - he_i)\,, \quad e_i = (\delta_{i1}, \dots, \delta_{im})\,.$$

For the prolonged $y, v\colon \Omega_h \to R$, we use the following notations:

$$\|y\|_{L_p(\Omega_h)} = \left(h^m \sum_{\xi\in\Omega_h} |y(\xi)|^p\right)^{1/p}, \quad 1 \le p < \infty$$

(in the case $p = 2$ we also use the notation $|y|_0 = \|y\|_0 = \|y\|_{L_2(\Omega_h)}$),

$$\|y\|_{C(\Omega_h)} = \|y\|_{L_\infty(\Omega_h)} = \max_{\xi\in\Omega_h} |y(\xi)|\,,$$

$$(y, v) = h^m \sum_{\xi\in\Omega_h} y(\xi)v(\xi)\,,$$

$$|y|_1 = \left(\sum_{i=1}^{m} \|\partial_i y\|_0^2\right)^{1/2},$$

$$|y|_2 = \left(\sum_{i,j=1}^{m} \|\partial_i \overline{\partial}_j y\|_0^2\right)^{1/2},$$

$$\|y\|_k = \left(\sum_{s=0}^{k} |y|_s^2\right)^{1/2}, \quad k = 1, 2\,.$$

Further, by $H^k(\Omega_h)$ ($k = 1, 2$) we denote the discrete Sobolev space supplied with the norm $\|y\|_k$.

Using the formulas of summation by parts, we get

$$\|y\|_1^2 = \|y\|_0^2 + (-\Delta_h y, y)\,,$$
$$\|y\|_2^2 = \|y\|_1^2 + (-\Delta_h y, -\Delta_h y)\,.$$

Consequently,

$$\|y\|_1^2 = \big((-\Delta_h + I_h)y, y\big)$$

and

$$\frac{1}{2}\big((-\Delta_h + I_h)y, (-\Delta_h + I_h)y\big) \le \|y\|_2^2 \le \big((-\Delta_h + I_h)y, (-\Delta_h + I_h)y\big)\,, \quad (1)$$

where I_h is the identity operator. This means that the Sobolev norms $\|y\|_1$ and $\|y\|_2$ are equivalent to the norms $\|(-\Delta_h + I_h)^{1/2} y\|_0$ and $\|(-\Delta_h + I_h)y\|_0$, respectively.

Now with the help of the operator

$$\Phi_h = -\Delta_h + I_h$$

we define the interpolation space $H^{2\beta}(\Omega_h)$ ($\frac{1}{2} < \beta < 1$) with the norm

$$\|y\|_{2\beta} = \|\Phi_h^\beta y\|_0\,.$$

For the interpolation spaces $H^{2\beta}(\Omega_h)$ the following theorem [3] holds.

Theorem 1. *For the grid functions* $y\colon \Omega_h \to R$, *prolonged to the boundary with the value from the nearest inner grid point, the following inequalities hold*:

$$\|y\|_{C(\Omega_h)} \le c\|y\|_{2\beta}\,, \quad \beta > \frac{m}{4}\,, \quad m = 2, 3\,,$$

$$\|\hat{\partial}_i y\|_{L_q(\Omega_h)} \le c\|y\|_{2\beta}\,, \quad \beta \ge \frac{(q-2)m + 2q}{4q}, \quad m < \frac{2q}{q-2}\,,$$

where $\hat{\partial}_i$ *denotes either* ∂_i, $\overline{\partial}_i$, *or* $\tilde{\partial}_i = \frac{1}{2}(\partial_i + \overline{\partial}_i)$.

Also the following result (see [1]) is valid.

Corollary 1. *For any* $\varepsilon > 0$, *we have*

$$\|y\|_{2\beta} \le 2^{\beta/2}(\varepsilon^{1-\beta}\|y\|_2 + \varepsilon^{-\beta}\|y\|_0)\,. \quad (2)$$

2. COERCIVITY INEQUALITY

Let $\partial u/\partial\nu$ denote the outer normal derivative of the boundary $\partial\Omega$. Consider the quasilinear elliptic operator of the Neumann boundary value problem defined by

$$Au = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right) + b(x,u)u, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = 0, \quad m = 2,3,$$

$$a_{ij}(x,u) = a_{ji}(x,u)\,,$$

and its discrete analogue in the following form:

$$A_h y = -\frac{1}{2}\sum_{i,j=1}^{m}\left[\partial_i\left(a_{ij}(x,y)\overline{\partial}_j y\right) + \overline{\partial}_i\big(a_{ij}(x,y)\partial_j y\big)\right] + b(x,y)y\,,$$
$$y \in H^2(\Omega_h)\,.$$

In what follows we assume that the following conditions hold:

(I) For any $a > 0$ there exists a number $\kappa_a > 0$ such that, for all $\xi_i \in R$, $x \in \overline{\Omega}$, $u \in [-a,a]$, there hold

$$\sum_{i,j=1}^{m} a_{ij}(x,u)\xi_i\xi_j \ge \kappa_a \sum_{i=1}^{m}\xi_i^2, \quad b(x,u) \ge \kappa_a\,.$$

(II) The functions $a_{ij}(x,u)$, $b(x,u)$ are continuously differentiable, and for $x \in \overline{\Omega}$, $u \in [-a,a]$, there hold

$$\left|\frac{\partial^{\mu_1+\mu_2}a_{ij}(x,u)}{\partial x_l^{\mu_1}\partial u^{\mu_2}}\right| \le d_a\,, \quad \left|\frac{\partial^{\mu_3+\mu_4}b(x,u)}{\partial x_l^{\mu_3}\partial u^{\mu_4}}\right| \le d_a\,,$$

where $0 \le \mu_1,\mu_2 \le 3$, $\mu_1+\mu_2 \le 3$, $0 \le \mu_3,\mu_4 \le 2$, $\mu_3+\mu_4 \le 2$, $l = 1,\dots,m$. Here d_a is a positive constant dependent on a.

Theorem 2. *Let conditions* (I), (II) *be fulfilled. In case* $m = 3$ *assume also that* $a_{ij} = 0$ *for* $j \ne i$*. Then, for all* $y, v \in H^2(\Omega_h)$ *such that* $\max(\|y\|_{C(\Omega_h)}, \|v\|_{C(\Omega_h)}, \|y\|_2, \|v\|_2) \le a$*, there holds*

$$\big(A_h y - A_h v, \Phi_h(y-v)\big) \ge \frac{\kappa_a}{2}\|y-v\|_2^2 - c_a\|y-v\|_1\|y-v\|_2\,, \tag{3}$$

and together with it the following coercivity inequality is valid:

$$\|A_h y - A_h v\|_0 \ge \frac{\kappa_a}{2\sqrt{2}}\|y-v\|_2 - \frac{c_a}{\sqrt{2}}\|y-v\|_1\,. \tag{4}$$

Here

$$c_a = d_a\big(5m^2\sqrt{2} + c(a+a^2)\big) + \Big(\frac{\kappa_a}{2}\Big)^{\beta/(\beta-1)} \big[d_a(a+a^2)\big]^{1/(1-\beta)} ,$$

$\beta = \frac{3}{4}$ *for* $m=2$*,* $\beta = \frac{7}{8}$ *for* $m=3$*,* c *is a positive constant.*

Proof. It follows from condition (II) that the operator $A_h \colon H^2(\Omega_h) \to L_2(\Omega_h)$ possesses the Frechet derivative:

$$\begin{aligned} A_h'(y)w = &\left\{ -\frac{1}{2}\sum_{i,j=1}^{m} \big[\partial_i\big(a_{ij}(x,y)\overline{\partial}_j w\big) + \overline{\partial}_i\big(a_{ij}(x,y)\partial_j w\big)\big] + b(x,y)w \right\} \\ &+ \left\{ -\frac{1}{2}\sum_{i,j=1}^{m} \left[\partial_i\left(\frac{\partial a_{ij}(x,y)}{\partial u}\overline{\partial}_j y \cdot w\right) + \overline{\partial}_i\left(\frac{\partial a_{ij}(x,y)}{\partial u}\partial_j y \cdot w\right)\right] \right. \\ &\left. + \frac{\partial b(x,y)}{\partial u} y w \right\} \\ =& A_{h1}(y)w + A_{h2}(y)w\,, \quad y,w \in H^2(\Omega_h)\,. \end{aligned}$$

Since

$$A_h y - A_h v = \int_0^1 A_h'\big(\theta y + (1-\theta)v\big)(y-v)d\theta, \tag{5}$$

we have

$$\begin{aligned} (A_h y - A_h v, \Phi_h z) &= \int_0^1 \big(A_h'\big(\theta y + (1-\theta)v\big)(y-v), \Phi_h z\big)d\theta \\ &= \int_0^1 \big(A_{h1}(\eta)z, \Phi_h z\big)d\theta + \int_0^1 \big(A_{h2}(\eta)z, \Phi_h z\big)d\theta\,, \\ z &= y - v, \quad \eta = \theta y + (1-\theta)v\,. \end{aligned}$$

First we consider the scalar product

$$\big(A_{h1}(\eta)z, \Phi_h z\big) = \big(A_{h1}(\eta)z, -\Delta_h z\big) + \big(A_{h1}(\eta)z, z\big),$$

where

$$A_{h1}(\eta)z = -\frac{1}{2}\sum_{i,j=1}^{m} \big[\partial_i\big(a_{ij}(x,\eta)\overline{\partial}_j z\big) + \overline{\partial}_i\big(a_{ij}(x,\eta)\partial_j z\big)\big] + b(x,\eta)z\,. \tag{6}$$

Using the Leibniz formula for the difference of the product, we get

$$\begin{aligned}
A_{h1}(\eta)z = &-\frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(x,\eta)(\partial_i\overline{\partial}_j z + \overline{\partial}_i\partial_j z)\\
&-\frac{1}{2}\sum_{i,j=1}^{m}\left[\partial_i\big(a_{ij}(x,\eta)\big)(\overline{\partial}_j z)^{+1_i} + \overline{\partial}_i\big(a_{ij}(x,\eta)\big)(\partial_j z)^{-1_i}\right]\\
&+ b(x,\eta)z\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
&(A_{h1}(\eta)z, -\Delta_h z)\\
&\quad= \frac{1}{2}\left(\sum_{i,j=1}^{m} a_{ij}(x,\eta)(\partial_i\overline{\partial}_j z + \overline{\partial}_i\partial_j z), \sum_{k=1}^{m}\partial_k\overline{\partial}_k z\right)\\
&\qquad+\frac{1}{2}\left(\sum_{i,j=1}^{m}\left[\partial_i\big(a_{ij}(x,\eta)\big)(\overline{\partial}_j z)^{+1_i} + \overline{\partial}_i\big(a_{ij}(x,\eta)\big)(\partial_j z)^{-1_i}\right], \sum_{k=1}^{m}\partial_k\overline{\partial}_k z\right)\\
&\qquad+\left(b(x,\eta)z, -\sum_{k=1}^{m}\partial_k\overline{\partial}_k z\right)\\
&\quad= S^{(1)} + S^{(2)} + S^{(3)}\,.
\end{aligned}$$

The addend $S^{(1)}$ occupies the central place in the estimation of the scalar product $(A_{h1}(\eta)z, -\Delta_h z)$:

$$\begin{aligned}
S^{(1)} &= \frac{1}{2}\left(\sum_{i,j=1}^{m} a_{ij}(x,\eta)(\partial_i\overline{\partial}_j z + \overline{\partial}_i\partial_j z), \sum_{k=1}^{m}\partial_k\overline{\partial}_k z\right)\\
&= \frac{1}{2}\sum_{\substack{i,j,k=1\\ k=i \text{ or } k=j}}^{m}\big(a_{ij}(x,\eta)(\partial_i\overline{\partial}_j z + \overline{\partial}_i\partial_j z), \partial_k\overline{\partial}_k z\big)\\
&\quad+\frac{1}{2}\sum_{\substack{i,k=1\\ k\neq i}}^{m}\big(a_{ii}(x,\eta)(\partial_i\overline{\partial}_i z + \overline{\partial}_i\partial_i z), \partial_k\overline{\partial}_k z\big) = S_1^{(1)} + S_2^{(1)}\,,
\end{aligned}$$

because for $m = 3$, $a_{ij} = 0$ if $j \neq i$. Using the formulas of summation by parts (cf. [4]) and the way of prolongations to the boundary $\partial\,\Omega_h$, we get

$$S_2^{(1)} = \frac{1}{2}\sum_{\substack{i,k=1\\ k\neq i}}^{m}\left[\big(a_{ii}(x,\eta)\partial_i\overline{\partial}_k z, \partial_i\overline{\partial}_k z\big) + \big(a_{ii}(x,\eta)\overline{\partial}_i\partial_k z, \overline{\partial}_i\partial_k z\big)\right] + S_{22}^{(1)}, \quad (7)$$

where

$$S_{22}^{(1)} = \frac{1}{2}\sum_{\substack{i,k=1\\k\neq i}}^{m} \Big[-\big(\overline{\partial}_k\big(a_{ii}(x,\eta)\big)(\partial_i\overline{\partial}_i z)^{-1_k}, \overline{\partial}_k z\big) + \big(\partial_i\big(a_{ii}(x,\eta)\big)(\overline{\partial}_k z)^{+1_i}, \partial_i\overline{\partial}_k z\big)$$

$$- \big(\partial_k\big(a_{ii}(x,\eta)\big)(\overline{\partial}_i\partial_i z)^{+1_k}, \partial_k z\big) + \big(\overline{\partial}_i\big(a_{ii}(x,\eta)\big)(\partial_k z)^{-1_i}, \overline{\partial}_i\partial_k z\big)\Big] .$$

Thus,

$$S^{(1)} = S_1^{(1)} + S_2^{(1)} = S_0^{(1)} + S_{22}^{(1)} ,$$

where

$$S_0^{(1)} = \frac{1}{2}\sum_{k=1}^{m}\sum_{i,j=1}^{m} \Big[\big(a_{ij}(x,\eta)\partial_i\overline{\partial}_k z, \partial_j\overline{\partial}_k z\big) + \big(a_{ij}(x,\eta)\overline{\partial}_i\partial_k z, \overline{\partial}_j\partial_k z\big)\Big]$$

($a_{ij} = 0$, $j \neq i$ for $m = 3$). Summation by parts yields

$$\begin{aligned} S^{(3)} &= \Big(b(x,\eta)z, -\sum_{k=1}^{m}\partial_k\overline{\partial}_k z\Big) \\ &= \sum_{k=1}^{m}\big(b(x,\eta)\partial_k z, \partial_k z\big) + \sum_{k=1}^{m}\big(\partial_k\big(b(x,\eta)\big)z^{+1_k}, \partial_k z\big) = S_1^{(3)} + S_2^{(3)} . \end{aligned}$$

From (I) it follows that

$$S_1^{(3)} \geq \kappa_a |z|_1^2$$

and

$$S_0^{(1)} \geq \kappa_a \sum_{i,k=1}^{m}\big(\partial_i\overline{\partial}_k z, \partial_i\overline{\partial}_k z\big) = \kappa_a |z|_2^2 .$$

Using condition (II) and the mean-value theorem, we get

$$\big|S_2^{(3)}\big| \leq d_a\big(\sqrt{2m}|z|_0|z|_1 + \|z\|_{C(\Omega_h)}|\eta|_1|z|_1\big) .$$

Analogously

$$\big|S_{22}^{(1)}\big| \leq 2d_a\sqrt{2m}\big(|z|_1|z|_2 + |\eta|_{1,4}|z|_{1,4}|z|_2\big) ,$$

$$\big|S^{(2)}\big| \leq d_a m^2\sqrt{2}\big(|z|_1|z|_2 + |\eta|_{1,4}|z|_{1,4}|z|_2\big) ,$$

where

$$|y|_{1,4} = \left(\sum_{i=1}^{m}\|\partial_i y\|_{L_4(\Omega_h)}^4\right)^{1/4} .$$

Summing up, we obtain

$$\begin{aligned}
&\big(A_{h1}(\eta)z, -\Delta_h z\big) \\
&\qquad \geq \kappa_a|z|_2^2 + \kappa_a|z|_1^2 - \sqrt{2}(2\sqrt{m}+m^2)d_a\big(|z|_1 + |\eta|_{1,4}|z|_{1,4}\big)|z|_2 \\
&\qquad\quad - d_a\big(\sqrt{2m}|z|_0 + \|z\|_{C(\Omega_h)}|\eta|_1\big)|z|_1 \,.
\end{aligned}$$

Similarly, it follows from conditions (I), (II) that

$$\big(A_{h1}(\eta)z, z\big) \geq \kappa_a|z|_0^2 - d_a m\big(|z|_2|z|_0 + \sqrt{2m}|z|_1|z|_0 + \sqrt{2}\|z\|_{C(\Omega_h)}|\eta|_1|z|_1\big)\,.$$

Thus,

$$\begin{aligned}
\big(A_{h1}(\eta)z, \Phi_h z\big) \geq \kappa_a\|z\|_2^2 - d_a m^2\sqrt{2}\big(&|z|_0|z|_2 + 2|z|_0|z|_1 + 2|z|_1|z|_2 \\
&+ 2|\eta|_{1,4}|z|_{1,4}|z|_2 + |\eta|_1\|z\|_{C(\Omega_h)}|z|_1\big)\,. \qquad (8)
\end{aligned}$$

Now we estimate the scalar product $(A_{h2}(\eta)z, \Phi_h z)$. We have

$$\begin{aligned}
A_{h2}(\eta)z = &-\frac{1}{2}\sum_{i,j=1}^{m}\left[\partial_i\left(\frac{\partial a_{ij}(x,\eta)}{\partial u}\overline{\partial}_j\eta\cdot z\right) + \overline{\partial}_i\left(\frac{\partial a_{ij}(x,\eta)}{\partial u}\partial_j\eta\cdot z\right)\right] \\
&+ \frac{\partial b(x,\eta)}{\partial u}\eta\cdot z\,.
\end{aligned}$$

Using the Leibniz formula, we get

$$\begin{aligned}
A_{h2}(\eta)z = &-\frac{1}{2}\sum_{i,j=1}^{m}\Bigg\{\frac{\partial a_{ij}(x,\eta)}{\partial u}\left[\partial_i(\overline{\partial}_j\eta\cdot z) + \overline{\partial}_i(\partial_j\eta\cdot z)\right] \\
&+ \partial_i\left(\frac{\partial a_{ij}(x,\eta)}{\partial u}\right)(\overline{\partial}_j\eta\cdot z)^{+1_i} + \overline{\partial}_i\left(\frac{\partial a_{ij}(x,\eta)}{\partial u}\right)(\partial_j\eta\cdot z)^{-1_i}\Bigg\} \\
&+ \frac{\partial b(x,\eta)}{\partial u}\eta\cdot z\,.
\end{aligned}$$

With the help of the mean-value theorem we obtain

$$\begin{aligned}
&\big|\big(A_{h2}(\eta)z,\Phi_h z\big)\big| \\
&\qquad \leq d_a m^2\sqrt{2}|z|_2\big[|z|_{1,4}|\eta|_{1,4} + \|z\|_{C(\Omega_h)}(|\eta|_0 + |\eta|_1 + |\eta|_2 + |\eta|_{1,4}^2)\big] \\
&\qquad\quad + d_a m^2\sqrt{2}\|z\|_{C(\Omega_h)}\big[|z|_1|\eta|_1 + |z|_0(|\eta|_0 + |\eta|_1 + |\eta|_2 + |\eta|_{1,4}^2)\big]\,.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\big(A_h'(\eta)z, \Phi_h z\big)\\
&\quad = \big(A_{h1}(\eta)z + A_{h2}(\eta)z, \Phi_h z\big)\\
&\quad \geq \kappa_a\|z\|_2^2 - d_a m^2\sqrt{2}|z|_2\big[3|z|_{1,4}|\eta|_{1,4} + \|z\|_{C(\Omega_h)}\big(|\eta|_0 + |\eta|_1 + |\eta|_2 + |\eta|_{1,4}^2\big)\big]\\
&\qquad - d_a m^2\sqrt{2}\big[|z|_0|z|_2 + 2|z|_0|z|_1 + 2|z|_1|z|_2 + 2\|z\|_{C(\Omega_h)}|z|_1|\eta|_1\\
&\qquad + \|z\|_{C(\Omega_h)}|z|_0\big(|\eta|_0 + |\eta|_1 + |\eta|_2 + |\eta|_{1,4}^2\big)\big]\,.
\end{aligned}
$$

Further, using Theorem 1 and Corollary 1, we get

$$
\begin{aligned}
&\big(A_h'(\eta)z,\Phi_h z\big)\\
&\qquad \geq \kappa_a\|z\|_2^2 - cd_a\|z\|_2\|z\|_{2\beta}\big(\|\eta\|_2 + \|\eta\|_2^2\big)\\
&\qquad\quad - d_a m^2\sqrt{2}\big[\|z\|_0\|z\|_2 + 2\|z\|_0\|z\|_1 + 2\|z\|_1\|z\|_2 + c\|z\|_2\|z\|_1\|\eta\|_2\\
&\qquad\quad + c\|z\|_2\|z\|_0\big(\|\eta\|_2 + \|\eta\|_2^2\big)\big]\,,
\end{aligned}
$$

where $c =$ const > 0, $\beta = \frac{3}{4}$ for $m = 2$, $\beta = \frac{7}{8}$ for $m = 3$. Here we used the inequalities

$$
\begin{aligned}
&\|\hat{\partial}_i y\|_{L_4(\Omega_h)} \leq c\|y\|_2\,, && i = 1,\ldots,m;\ m = 2,3\,;\\
&\|y\|_{C(\Omega_h)} \leq c\|y\|_2\,, && m = 2,3\,,
\end{aligned}
$$

which follow from Theorem 1 and Corollary 1.

Using inequality (2) with $\|\eta\|_2 \leq a$, we get

$$
\begin{aligned}
\big(A_h'(\eta)z, \Phi_h z\big) \geq{}& \big[\kappa_a - cd_a\varepsilon^{1-\beta}(a + a^2)\big]\|z\|_2^2 - d_a 5m^2\sqrt{2}\|z\|_1\|z\|_2\\
&- cd_a(a + a^2)(1 + \varepsilon^{-\beta})\|z\|_1\|z\|_2\,.
\end{aligned}
$$

Choosing $\varepsilon > 0$ so that

$$cd_a\varepsilon^{1-\beta}(a + a^2) = \frac{\kappa_a}{2},$$

we obtain

$$\big(A_h'(\eta)z, \Phi_h z\big) \geq \frac{\kappa_a}{2}\|z\|_2^2 - c_a\|z\|_1\|z\|_2\,,$$

where

$$c_a = d_a\big(5m^2\sqrt{2} + c(a + a^2)\big) + \Big(\frac{\kappa_a}{2}\Big)^{\beta/(\beta-1)}\big[d_a(a + a^2)\big]^{1/(1-\beta)}\,.$$

From (5) it follows that

$$\big(A_h y - A_h v, \Phi_h(y-v)\big) \geq \frac{\kappa_a}{2}\|y-v\|_2^2 - c_a\|y-v\|_2\|y-v\|_1\,,$$

i.e. (3) holds. Together with (1) this implies (4).

The next theorem shows that if A contains no mixed derivatives, then the corresponding operator A_{h1} (see (6)) is invertible.

Theorem 3. *In case of the difference operator*

$$\overline{A}_h y = -\frac{1}{2}\sum_{i=1}^{m}\big[\partial_i\big(a_{ii}(x,y)\overline{\partial}_i y\big)+\overline{\partial}_i\big(a_{ii}(x,y)\partial_i y\big)\big] + b(x,y)y\,,$$
$$y \in H^2(\Omega_h)\,,$$

under conditions (I), (II), *the following inequality holds*:

$$\|\overline{A}_{h1}(\eta)z\|_0 \geq \frac{\kappa_a^2}{2(\sqrt{2}\kappa_a+\tau_a)}\|z\|_2\,, \tag{9}$$
$$z,\eta \in H^2(\Omega_h)\,, \quad \max\big(\|\eta\|_{C(\Omega_h)}, \|\eta\|_2\big) \leq a,$$

where (*cf.* (6))

$$\overline{A}_{h1}(\eta)z = -\frac{1}{2}\sum_{i=1}^{m}\big[\partial_i\big(a_{ii}(x,\eta)\overline{\partial}_i z\big)+\overline{\partial}_i\big(a_{ii}(x,\eta)\partial_i z\big)\big] + b(x,\eta)z\,,$$

$$\tau_a = d_a(5m^2\sqrt{2}+c\,a) + \Big(\frac{\kappa_a}{2}\Big)^{\beta/(\beta-1)}\big(c\,d_a a\big)^{1/(1-\beta)},$$

$\beta=\frac{3}{4}$ *for* $m=2$, $\beta=\frac{7}{8}$ *for* $m=3$, c *is a positive constant.*

Proof. From inequality (8) and Corollary 1 with $\|\eta\|_2 \leq a$ we get

$$\begin{aligned}&\big(\overline{A}_{h1}(\eta)z, \Phi_h z\big)\\ &\quad\geq \big(\kappa_a - c\,a\,d_a\varepsilon^{1-\beta}\big)\|z\|_2^2 - d_a\|z\|_2\|z\|_1\big(5m^2\sqrt{2}+c\,a+c\,a\varepsilon^{-\beta}\big)\,.\end{aligned}$$

Let us denote

$$\tau_a = d_a\big(5m^2\sqrt{2}+c\,a+c\,a\varepsilon^{-\beta}\big)\,,$$

where we choose $\varepsilon>0$ so that

$$c\,a\,d_a\varepsilon^{1-\beta} = \frac{\kappa_a}{2}\,.$$

Thus

$$\big(\overline{A}_{h1}(\eta)z, \Phi_h z\big) \geq \frac{\kappa_a}{2}\|z\|_2^2 - \tau_a\|z\|_2\|z\|_1\,,$$

which together with (1) implies

$$\|\overline{A}_{h1}(\eta)z\|_0 \geq \frac{\kappa_a}{2\sqrt{2}}\|z\|_2 - \frac{\tau_a}{\sqrt{2}}\|z\|_1 \,. \tag{10}$$

Using the formula of summation by parts and (I), we get

$$\big(\overline{A}_{h1}(\eta)z, z\big) \geq \kappa_a \|z\|_1^2 \,.$$

Consequently,

$$\|\overline{A}_{h1}(\eta)z\|_0 \geq \kappa_a \|z\|_1 \,,$$

or

$$\|z\|_1 \leq \frac{1}{\kappa_a}\|\overline{A}_{h1}(\eta)z\|_0 \,.$$

Now it follows from (10) that

$$\begin{aligned}\frac{\kappa_a}{2\sqrt{2}}\|z\|_2 &\leq \|\overline{A}_{h1}(\eta)z\|_0 + \frac{\tau_a}{\sqrt{2}}\frac{1}{\kappa_a}\|\overline{A}_{h1}(\eta)z\|_0 \\ &= \frac{\sqrt{2}\kappa_a + \tau_a}{\sqrt{2}\kappa_a}\|\overline{A}_{h1}(\eta)x\|_0 \,.\end{aligned}$$

Theorem 3 is proven.

It should by noted that in case of the Dirichlet boundary value problem the result like (9) is valid also for the finite difference operator (6) (see [1]) with mixed differences. In [5,6] one can find the coercivity inequalities for linear finite difference operators. The coercivity inequalities for the finite difference operator, which approximates a nonlinear monotone elliptic operator, can be found in [2,7].

ACKNOWLEDGEMENT

This investigation was supported by the Estonian Science Foundation (grant No. 4410).

REFERENCES

1. Fischer, M. Convergence of the difference method in the strong norm for a nonlinear problem of elliptic type. *Acta Comment. Univ. Tartuensis*, 1983, 633, 55–66 (in Russian).
2. Lapin, A. V. A study of convergence of the difference schemes in the norm $W_2^{(2)}$ for quasilinear elliptic equations. *Z. Vychisl. Mat. i Mat. Fiz.*, 1974, 14, 1516–1525 (in Russian).

3. Fischer, M. Certain discrete analogoue of the Sobolev imbedding theorems. *Proc. Estonian Acad. Sci. Phys. Math.*, 1996, **45**, 23–29.
4. Tikhonov, A. N. and Samarskij, A. A. *Equations of Mathematical Physics.* Dover Publ., New York, 1990.
5. D′yakonov E. G. *Difference Methods of the Solution of Boundary Value Problems*, 1st edn. Izd-vo MGU, 1971 (in Russian).
6. Sobolevskij, P. E. and Tyunchik, M. F. The difference method of approximate solution for elliptic equations. *Voronezh. Gos. Univ. Tr. Mat. Fak.*, 1970, 4, 117–127 (in Russian).
7. D′yakonov, E. G. The convergence of certain iterative method. *Uspekhi Mat. Nauk*, 1966, **21**, 179–182 (in Russian).

KOERTSITIIVSUSE VÕRRATUS KVAASILINEAARSE DIFERENTSOPERAATORI JAOKS

Malle FISCHER

On vaadeldud teist järku kvaasilineaarse diferentsiaaloperaatori Neumanni rajaülesande diskreetset analoogi kahe- ja kolmemõõtmelises ühikkuubis. Nimetatud operaatori jaoks on tõestatud koertsitiivsuse võrratus, kusjuures kolmedimensioonilisel juhul kehtib tulemus vaid siis, kui diferentsiaaloperaator ei sisalda segaosatuletisi.