

TWO-BASED DUPLICATE-CLONES

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Abstract. The notion “two-based clone” on a pair of sets (universes) is defined. Some properties of two-based duplicate-clones are proved. The lattice of all double-dually closed duplicate-clones on a pair of 2-element sets is described.

Key words: two-based clone, lattice, two-based duplicate-clone, double-dually closed clone.

1. INTRODUCTION

The notion “clone on A ” was introduced for classifying algebras on a fixed universe A . Two algebras on A are term equivalent if and only if the clones generated by all fundamental operations of them coincide. A review of the results on clones is given by Sichler and Trnková ([¹]). Under inclusion the set of all clones on A forms the lattice \mathcal{L}_A . The structure of the lattice \mathcal{L}_A has been studied in general (see, e.g., [²]) and for some $k = |A|$. The lattice \mathcal{L}_A is completely known for Boolean functions, i.e. for $|A| = 2$ (see [³]). As for $|A| \geq 3$ the lattice \mathcal{L}_A is uncountable, it seems hopeless to find a satisfactory description of \mathcal{L}_A in general. Special parts of \mathcal{L}_A (with $|A| = k > 2$) are described, for example, by Burle [⁴] and Hoa [⁵].

Let S_A be the full symmetric group on A . The notion “ S_A -clone” was introduced in [⁶]. From [^{6,7}] we know that the lattice of all S_A -clones is finite if $|A| = 2, 3$. In the present paper we define the notion “two-based clone”. This notion is justified by the fact that many algebraic structures (acts, modules, linear spaces, etc.) are two-based (called also “two-sorted”). The set of all two-based clones on a fixed pair \mathbf{A} forms (with respect to the set inclusion) the lattice $\mathcal{L}_{\mathbf{A}}$. As expected, the lattice $\mathcal{L}_{\mathbf{A}}$ has a very complicated structure.

In Section 3 we define the notions “two-based duplicate-clone”, “1-component” and “2-component” of a two-based clone. We prove some properties of two-based duplicate-clones and describe the 1-component and 2-component of a two-based duplicate-clone.

In Section 4 we apply these results to the 2-Boolean clones, i.e. to two-based clones on a pair of 2-element sets. The first results in this direction were obtained by Kudrjavcev and Burosch [8] who studied generating sets of closed classes of the two-based full iterative algebra on a pair of 2-element sets. Here, for any doubly closed 2-Boolean clone the subset of all unary functions is described (see Proposition 3.3).

The main result of the present paper is a full description of the sublattice consisting of all duplicate- dd' -clones in $\mathcal{L}_{2 \times 2}$ (see Theorem 4.1).

2. NOTATIONS AND PRELIMINARIES

Let $\mathbf{A} := (A_1, A_2)$ be a pair of (finite) disjoint sets containing at least two elements each. The sets A_1, A_2 will be called *the first* and *the second universe*, respectively. Let us denote

$$O_{\mathbf{A}} := \{f : A_{i_1} \times \dots \times A_{i_n} \rightarrow A_{i_{n+1}} \mid i_1, \dots, i_{n+1} \in \{1, 2\}, \quad n \in \mathbf{N}^+\}$$

and let $\tau = (i_1, \dots, i_n; i_{n+1})$ be called the *signature* of the mapping f . We denote by $J_{\mathbf{A}}$ the set of all projections

$$e_k^{i_1 \dots i_n} : A_{i_1} \times \dots \times A_{i_n} \rightarrow A_{i_k} : (x_1, \dots, x_n) \mapsto x_k$$

with $k \in \{1, \dots, n\}$, $i_1, \dots, i_n \in \{1, 2\}$. Let all five Mal'tsev's operations (see [9]) be acting on $O_{\mathbf{A}}$. Then superposition, composition, and linearized composition of mappings are defined on $O_{\mathbf{A}}$ too.

Definition 2.1. *If a subset $F \subseteq O_{\mathbf{A}}$ contains $J_{\mathbf{A}}$ and is closed under composition, then we write $F \leq O_{\mathbf{A}}$ and call F a two-based clone on \mathbf{A} . We denote by $\langle F \rangle_{O_{\mathbf{A}}}$ (or simply by $\langle F \rangle$) the two-based clone generated by $F \subseteq O_{\mathbf{A}}$.*

For any subset $F \subseteq O_{\mathbf{A}}$ and any signature $\tau \in \{1, 2\}^{n+1}$ we introduce the set

$$F^{\tau} = \{f^{\tau} \in F \mid f \text{ is of signature } \tau\}.$$

Functions with values in A_1 (or in A_2) are called 1-functions (or 2-functions). We denote by F_1 and F_2 the subsets of all 1-functions and 2-functions of a set $F \subseteq O_{\mathbf{A}}$. Let O_{A_1} and O_{A_2} be the sets of all functions on the first universe A_1 and on the second universe A_2 , respectively. Then

$$F_{A_1} := F \cap O_{A_1}, \quad F_{A_2} := F \cap O_{A_2}$$

will be called the 1-component and the 2-component of $F \subseteq O_{\mathbf{A}}$, respectively.

Example 2.1. Let both universes be 2-element sets:

$$A_1 = E_2 := \{0, 1\}, \quad A_2 = E'_2 := \{0', 1'\}.$$

The set of all functions on this pair will be denoted by $O_{2 \times 2}$. A two-based clone $F \leq O_{2 \times 2}$ will be called a *2-Boolean clone*. The 1-component (2-component) of a 2-Boolean clone is the clone of Boolean functions over $\{0, 1\}$ (over $\{0', 1'\}$, respectively).

Let \neg be the negation on E_2 , i.e. $\neg(0) = 1, \neg(1) = 0$, and \neg' be the negation on E'_2 . Besides the identity functions and negations there are only four other unary nonconstant functions

$$d_1(0') = 0, \quad d_1(1') = 1; \quad d_2(0) = 0', \quad d_2(1) = 1'$$

and their negations $\neg d_1, \neg' d_2$. An n -ary function ($n \geq 1$) is called *essentially unary* if it depends only on one of the variables.

Kudrjavcev and Burosch [⁸] investigated closed under composition classes of functions over a pair of 2-element sets. They found the subset of all unary nonconstant functions for all closed classes. Let us remark that all closed classes containing $J_{\mathbf{A}}$, and only such classes, are 2-Boolean clones. The results about 2-Boolean clones contained in [⁸] can be systematized and represented as in the next Proposition 2.1.

Proposition 2.1. *There are 19 2-Boolean clones generated by a subset of unary nonconstant functions in $O_{2 \times 2}$:*

$J_{\mathbf{A}} = \langle G_4 \rangle$	(projections),
$\langle \neg \rangle = \langle G_2 \rangle$	(1-negations of projections),
$\langle \neg' \rangle = \langle G_3 \rangle$	(2-negations of projections),
$\langle \neg, \neg' \rangle = \langle G_1 \rangle$	(negations of projections),
$\langle d_1 \rangle = \langle F_{14} \rangle$	(1-duplicates of 2-projections),
$\langle d_2 \rangle = \langle F_{12} \rangle$	(2-duplicates of 1-projections),
$\langle \neg d_1 \rangle = \langle F_{15} \rangle$	(neg-1-duplicates of 2-projections),
$\langle \neg' d_2 \rangle = \langle F_{13} \rangle$	(neg-2-duplicates of 1-projections),
$\langle d_1, \neg d_1 \rangle = \langle F_{11} \rangle$	(1-duplicates and neg-1-duplicates of 2-projections),
$\langle d_2, \neg' d_2 \rangle = \langle F_{10} \rangle$	(2-duplicates and neg-2-duplicates of 1-projections),
$\langle d_1, d_2 \rangle = \langle F_8 \rangle$	(duplicates of projections),
$\langle \neg d_1, \neg' d_2 \rangle = \langle F_9 \rangle$	(neg-duplicates of projections),
$\langle \neg, d_1, (\neg d_1) \rangle = \langle F_5 \rangle$	(all (essentially) unary 1-functions),
$\langle \neg, d_2, (\neg' d_2) \rangle = \langle F_4 \rangle$	(negations, 2-duplicates and neg-2-duplicates of 1-projections),
$\langle \neg', d_2, (\neg' d_2) \rangle = \langle F_7 \rangle$	(all (essentially) unary 2-functions),
$\langle \neg', d_1, (\neg d_1) \rangle = \langle F_6 \rangle$	(negations, 1-duplicates and neg-1-duplicates of 2-projections),

$$\begin{aligned}
\langle \neg, \neg', d_1, (\neg d_1) \rangle &= \langle F_3 \rangle && \text{(negations; all (essentially) unary 1-functions),} \\
\langle \neg, \neg', d_2, (\neg' d_2) \rangle &= \langle F_2 \rangle && \text{(negations; all (essentially) unary 2-functions),} \\
\langle \neg, \neg', d_1, (\neg d_1), d_2, (\neg' d_2) \rangle &= \langle F_1 \rangle && \text{(all (essentially) unary functions).}
\end{aligned}$$

Remark 2.1. Here the functions in round brackets may be omitted (for example, the parts $(\neg d_1)$ and $(\neg' d_2)$ in the last line).

3. DUPLICATE-CLONES AND dd' -CLONES

We define the notion of duplication over a pair $\mathbf{A} = (A_1, A_2)$ as follows.

Definition 3.1. *Let both universes have the same power, i.e. $|A_1| = |A_2|$ and assume that a two-based clone F contains bijections $d_1 : A_2 \rightarrow A_1$, $d_2 : A_1 \rightarrow A_2$ which are inverses of each other. Then we say that F is a two-based duplicate-clone (for short, $d_1 d_2$ -clone). The functions d_1 and d_2 will be called 1-duplication and 2-duplication, respectively.*

Proposition 3.1. *The 1-component F_{A_1} and the 2-component F_{A_2} of a $d_1 d_2$ -clone $F \leq O_{\mathbf{A}}$ are clones on A_1 and A_2 , respectively, and they are isomorphic.*

Proof. The 1-component F_{A_1} and the 2-component F_{A_2} are both closed under composition. So they are clones on A_1 and on A_2 , respectively. An isomorphism from F_{A_1} to F_{A_2} can be given by the correspondence

$$f \mapsto f^{d_2}, \quad \text{where} \quad f^{d_2}(y_1, \dots, y_n) = d_2(f(d_1(y_1), \dots, d_1(y_n))). \quad (1)$$

Proposition 3.2. *For any two signatures*

$$\tau_1 = (i_1, \dots, i_n; i_{n+1}), \quad \tau_2 = (j_1, \dots, j_n; j_{n+1})$$

of the same length, and for any $d_1 d_2$ -clone F we have

$$|F^{\tau_1}| = |F^{\tau_2}|,$$

where both sets determine each other uniquely.

Proof. Let F be a $d_1 d_2$ -clone and let

$$\tau_1 = (i_1, \dots, i_n; i_{n+1}), \quad \tau_2 = (j_1, \dots, j_n; j_{n+1})$$

be signatures of the same length. Let

$$u = \begin{cases} \text{id}_{A_l} & \text{if } i_{k+1} = j_{k+1} = l, \\ d_{j_{k+1}} & \text{if } i_{k+1} \neq j_{k+1}; \end{cases} \quad v = \begin{cases} \text{id}_{A_l} & \text{if } i_{k+1} = j_{k+1} = l, \\ d_{i_{k+1}} & \text{if } i_{k+1} \neq j_{k+1} \end{cases}$$

and for all $k = 1, \dots, n$ let us have the mappings

$$u_k = \begin{cases} \text{id}_{A_l} & \text{if } i_k = j_k = l, \\ d_{i_k} & \text{if } i_k \neq j_k; \end{cases} \quad v_k = \begin{cases} \text{id}_{A_l} & \text{if } i_k = j_k = l, \\ d_{j_k} & \text{if } i_k \neq j_k. \end{cases}$$

For any $f \in F^{\tau_1}$ and any $g \in F^{\tau_2}$ we define functions $f' \in F^{\tau_2}$, $g' \in F^{\tau_1}$ as follows:

$$f'(y_1, \dots, y_n) = u(f(u_1(y_1), \dots, u_n(y_n))), \quad (2)$$

$$g'(x_1, \dots, x_n) = v(g(v_1(x_1), \dots, v_n(x_n))) \quad (3)$$

for all $y_1 \in A_{j_1}, \dots, y_n \in A_{j_n}$, $x_1 \in A_{i_1}, \dots, x_n \in A_{i_n}$.

The correspondences $f \mapsto f'$ and $g \mapsto g'$, defined by formulas (2) and (3), respectively, are bijections between the sets F^{τ_1} and F^{τ_2} . \square

Let F be again a 2-Boolean clone and let \mathbf{s} denote the pair of negations, i.e. $\mathbf{s} := (\neg, \neg')$. For the functions

$$f : E_2^m \times E_2'^k \rightarrow E_2 \text{ and } g : E_2^m \times E_2'^k \rightarrow E_2'$$

the \mathbf{s} -dual functions are defined by the formulas

$$f^{\mathbf{s}}(x_1, \dots, x_n, y_1, \dots, y_m) := \neg f(\neg x_1, \dots, \neg x_n, \neg' y_1, \dots, \neg' y_m)$$

and

$$g^{\mathbf{s}}(x_1, \dots, x_n, y_1, \dots, y_m) := \neg' g(\neg x_1, \dots, \neg x_n, \neg' y_1, \dots, \neg' y_m).$$

For functions f and g with a different order of variables the functions $f^{\mathbf{s}}$ and $g^{\mathbf{s}}$ are defined similarly. For a set $F \subseteq O_{2 \times 2}$, let $F^{\mathbf{s}} := \{f^{\mathbf{s}} \mid f \in F\}$.

Definition 3.2. A two-based clone $F \leq O_{2 \times 2}$ is called a double-dually closed 2-Boolean clone (in short, dd' -clone) if $F^{\mathbf{s}} = F$.

Proposition 3.3. The subset of all unary nonconstant functions of a dd' -clone has one of the 19 forms $(Q_1, \dots, Q_4, F_1, \dots, F_{15})$ listed in Proposition 2.1. The subset of all unary nonconstant functions of a duplicate- dd' -clone is F_1, F_8 , or F_9 .

Proof. Any dd' -clone contains a minimal two-based clone listed in Proposition 2.1, because any unary nonconstant function is \mathbf{s} -dual to itself. It is easy to verify that just the subset F_8 is closed under both duplications (d_1 and d_2), the subset F_9 is closed under negations of both duplications ($\neg d_1$ and $\neg' d_2$), and F_1 is closed under all four of these functions. \square

4. LATTICE OF DUPLICATE- dd' -CLONES

Now we will focus on the most interesting part of the lattice $\mathcal{L}_{2 \times 2}$ consisting of all duplicate- dd' -clones F . In such a dd' -clone F the set F_1 depends on F_2 and vice versa. Gorlov and Pöschel described in [6] the lattice \mathcal{L}_{2, S_2} of all dually closed clones (i.e. S_2 -clones) of Boolean functions (on one universe). This lattice consists of 14 elements and has the structure pictured in Fig. 1.

The list of clones shown in Fig. 1 and sets generating them (in notations of [10] and [6]) is as follows:

$\mathbf{O}_1 = J_{\mathbf{A}}$	(projections),
$\mathbf{O}_4 = \langle \neg \rangle$	(projections and their negations),
$\mathbf{O}_8 = \langle c_0, c_1 \rangle$	(constants),
$\mathbf{O}_9 = \langle c_0, c_1, \neg \rangle$	(essentially unary functions),
$\mathbf{L}_1 = \langle c_1, + \rangle$	(all linear functions),
$\mathbf{L}_4 = \langle g \rangle$	(linear idempotent functions (where $g(x, y, z) := x + y + z$)),
$\mathbf{L}_3 = \langle g, \neg \rangle$	(linear self-dual functions),
$\mathbf{D}_2 = \langle h \rangle$	(self-dual monotone functions (where $h(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$)),
$\mathbf{D}_1 = \langle g, h \rangle$	(self-dual idempotent functions),
$\mathbf{D}_3 = \langle h, \neg \rangle$	(self-dual functions),
$\mathbf{A}_4 = \langle \wedge, \vee \rangle$	(monotone idempotent functions),
$\mathbf{A}_1 = \langle c_0, c_1, \wedge, \vee \rangle$	(monotone functions),
$\mathbf{C}_4 = \langle g, \wedge, \vee \rangle$	(idempotent functions),
$\mathbf{C}_1 = O_{\mathbf{A}}$	(all functions).

Theorem 4.1. *There are exactly 22 duplicate- dd' -clones in $\mathcal{L}_{2 \times 2}$. Together with the minimal 2-Boolean clone \mathbf{O}_1 they form a lattice pictured in Fig. 2.*

Proof. Let F be a duplicate- dd' -clone. There are three possibilities for the duplication functions: 1) d_1 and d_2 , 2) $\neg d_1$ and $\neg' d_2$, 3) d_1 , d_2 , $\neg d_1$, and $\neg' d_2$. In case of 1, 2, or 3 we will say that F has type 1, 2, or 3, respectively. By Proposition 3.1 the 1-component F_{E_2} and the 2-component $F_{E'_2}$ of the duplicate- dd' -clone F are clones of Boolean functions on E_2 and E'_2 , respectively, and these clones are isomorphic. It follows immediately from the definitions of dd' -clones and S_2 -clones that F_{E_2} and $F_{E'_2}$ are S_2 -clones. The set of 1-components F_{E_2} (2-components $F_{E'_2}$) of all duplicate- dd' -clones F of type 1 (or 2 or 3) under inclusion forms a lattice which is isomorphic to a sublattice of the lattice \mathcal{L}_{2, S_2} (given in Fig. 1).

An immediate calculation shows that any of these 14 clones on E_2 is the 1-component for some duplicate- dd' -clone of type 1. Namely, we get from a fixed clone \mathbf{C} on E_2 a duplicate- dd' -clone F of type 1 if we construct all subsets F^{τ_2} for all signatures $\tau_2 \in \{1, 2\}^{n+1}$ by the formulas (2), (3) with the condition $F_{A_1} = \mathbf{C}$. The duplicate- dd' -clone of type 1, just constructed, will be denoted by \mathbf{C}_d . It follows from Proposition 3.2 that \mathbf{C}_d is uniquely determined by \mathbf{C} . It is easy to

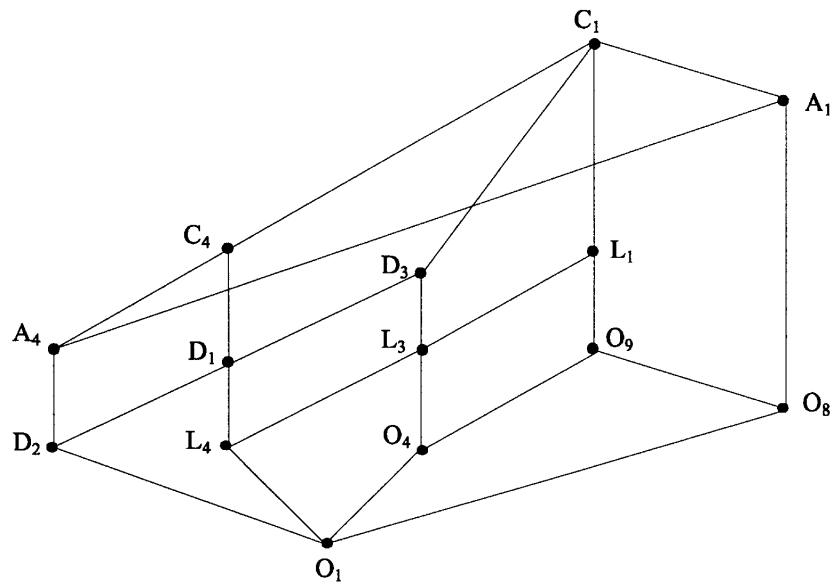


Fig. 1. The lattice \mathcal{L}_{2,S_2} of S_2 -clones.

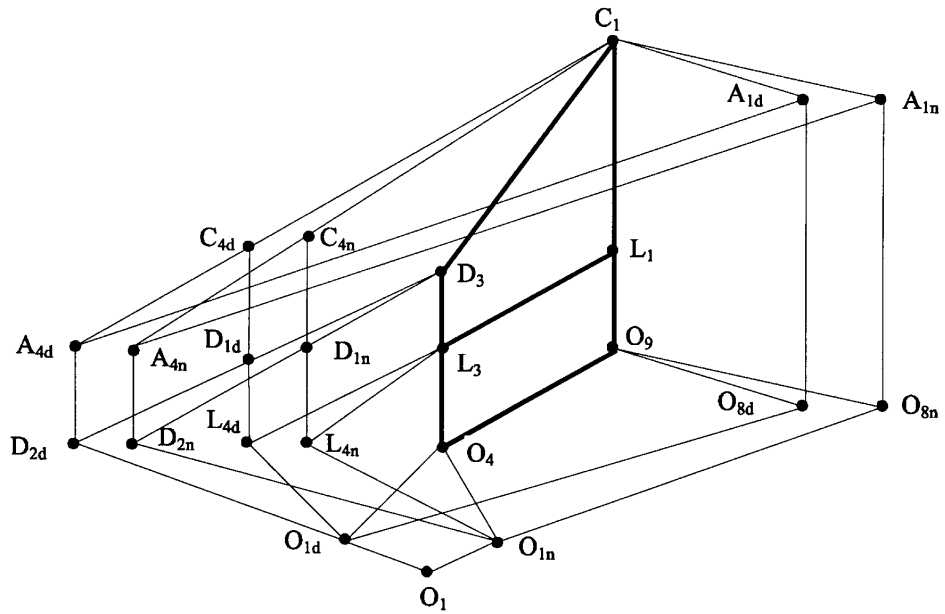


Fig. 2. The lattice of duplicate- dd' -clones.

verify that \mathbf{C}_d is a dd' -clone. Hence the lattice of all duplicate- dd' -clones of type 1 is isomorphic to the lattice \mathcal{L}_{2,S_2} .

Similarly, in order to describe all duplicate- dd' -clones of type 2, we have to use the functions $\neg d_1$ and $\neg' d_2$ instead of the functions d_1 and d_2 , respectively, in the formulas (2), (3). By \mathbf{C}_n we denote the duplicate- dd' -clone of type 2, constructed from a fixed clone \mathbf{C} on E_2 in the same way as \mathbf{C}_d (but using $\neg d_1$ and $\neg' d_2$). We see that the lattice of all duplicate- dd' -clones of type 2 is also isomorphic to the lattice \mathcal{L}_{2,S_2} .

Now we consider duplicate- dd' -clones F of type 3. First we notice that the set of unary nonconstant functions of F consists of all such functions. In particular it contains the negation \neg . Thus the 1-component of a duplicate- dd' -clone of type 3 can be one of the following: \mathbf{O}_4 , \mathbf{O}_9 , \mathbf{L}_1 , \mathbf{L}_3 , \mathbf{D}_3 , and \mathbf{C}_1 . If we take all duplicates (or all neg-duplicates) of all functions of these clones, then we get a uniquely determined duplicate- dd' -clone of type 3. The duplicate- dd' -clone of type 3, just constructed, we denote also by \mathbf{O}_4 , \mathbf{O}_9 , \mathbf{L}_1 , \mathbf{L}_3 , \mathbf{D}_3 , and \mathbf{C}_1 , respectively. Hence they form the lattice of all duplicate- dd' -clones of type 3, which is shown in Fig. 2 by bold lines.

By an easy checking we see that the equations $\mathbf{O}_{4d} = \mathbf{O}_{4n} = \mathbf{O}_4$, $\mathbf{O}_{9d} = \mathbf{O}_{9n} = \mathbf{O}_9$, $\mathbf{L}_{1d} = \mathbf{L}_{1n} = \mathbf{L}_1$, $\mathbf{L}_{3d} = \mathbf{L}_{3n} = \mathbf{L}_3$, $\mathbf{D}_{3d} = \mathbf{D}_{3n} = \mathbf{D}_3$, and $\mathbf{C}_{1d} = \mathbf{C}_{1n} = \mathbf{C}_1$ hold. Altogether we got 22 different duplicate- dd' -clones. We have to add \mathbf{O}_1 to the set of all duplicate- dd' -clones to get a lattice because $\mathbf{O}_{1d} \cap \mathbf{O}_{1n} = \mathbf{O}_1$. But the minimal 2-Boolean clone $\mathbf{O}_1 = J_A$ is not a duplicate-clone. Hence we got the lattice graphed in Fig. 2. This completes our proof. \square

Two-based clones that are not duplicate-clones will be considered in a forthcoming paper.

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KAHEALUSELISED DUBLIKAATKLOONID

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On defineeritud kahealuselise klooni mõiste ühisosata hulkade paaril, tehtud kindlaks kahealuseliste dublikaatkloonide omadusi ja esitatud topeltduaalsete dublikaatkloonide võre täielik kirjeldus.