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NUMERICAL STABILITY IN A MODEL OF THE LAYERED STRUCTURE OF THERMOHALINE FIELDS

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Abstract. The paper presents a set-up and the stability criteria for numerical realization of the model of the vertical structure of thermohaline fields caused by double diffusion effects. The model considers the turbulence accompanying the environment layering process as rotationally isotropic in the sense suggested by the theory of rotationally anisotropic turbulence. It is formulated through the set of three nonlinear differential equations for the environment's temperature, salinity, and turbulence kinetic energy. The numerical realization of the model is based on an implicit difference method and the uniform rectangular time-space grid. The numerical stability criterion of the algorithm for the model's approximated solution is estimated by the maximum values of the solutions of system components.

Key words: turbulence, double diffusion, numerical stability.

1. INTRODUCTION

The layering of initially stable vertical profiles of hydrophysical fields into stair-like profiles is well known from oceanographic observations [1–4] and laboratory investigations [5–7]. The physical reason for this process proceeds from the difference of coefficients of molecular diffusion for different environmental constituents (in oceans – heat and salt, characterized by temperature and salinity fields). Depending on the character of initial stratification, two different scenarios of layering processes are possible, called salt-fingering and diffusive layering regimes [7,8].

The success of the mathematical modelling of the layering processes depends on the description of accompanying turbulent processes and on the skill of their

modelling. Our present model is a development of the model discussed in [9]. The latter model is based on the traditional approach in solving the "turbulent problems" [10,11] using the classical semiempirical theory of turbulence [12] and postulating the absence of a prevailing orientation of eddy rotation in turbulent media. In contrast, the model proposed by us considers this postulate as the assumption superposed on a more general situation characterized by prevailing orientation of eddy rotation (called the property of rotational anisotropy of the turbulent flow field [13–15]). The proposed approach not only involves a more general point of view, but also allows formulation of a new interpretation for mixing-length and the closing assumptions for a dissipation function and turbulent viscosity.

Within the mathematical problems arising from the model of [9] (as well as from its earlier version [16]), the central problem is the stability of the numerical realization of the model. The difficulties arise from the variation of effective diffusion coefficients from molecular to turbulent, which differ in space and time by several orders of magnitude.

The choice in favour of an implicit method is based on the fact that the advantage of an explicit method, improved by trivial matrix inversion and minimal number of arithmetic operations per time step, is counterbalanced by the stability and convergence conditions which impose severe restrictions on the admissible time step. We have chosen the implicit iterative method, as for nonlinear problems all approaches will necessarily be iterative [17]. It is shown that the variation of effective diffusion coefficients in space and time by several orders of magnitude cannot introduce bifurcation in stability criteria.

2. THE THEORY

2.1. The structure of the turbulent flow field

In the theory of rotationally anisotropic turbulence [13–15] the rotational isotropy of turbulence (postulated in classical semiempirical theory [12]) is determined by the conditions

$$\boldsymbol{M} = \left\langle \boldsymbol{M}^* \right\rangle = 0, \quad \Omega = \left\langle \Omega^* \right\rangle = 0. \tag{1}$$

In (1), $\boldsymbol{M}^* = \boldsymbol{v}' \times \boldsymbol{R}$ and $\Omega^* = \boldsymbol{M}^* / R^2$, where $R = |\boldsymbol{R}|$, are the instantaneous moment of momentum and the internal rotation velocity of the turbulent flow field. The brackets $\langle \cdot \rangle$ denote an arbitrary averaging operator, satisfying Reynolds averaging rules (see, e.g., [12]). Here $\boldsymbol{v}'$ is the fluctuating component of velocity and $\boldsymbol{R}$ is the curvature radius of the $\boldsymbol{v}'$ field, whereas

$$\langle v' \rangle = 0. \tag{2}$$

In terms of Ω^* and $\boldsymbol{R}$, $\boldsymbol{v}'$ is represented as

$$\boldsymbol{v}' = \Omega^* \times \boldsymbol{R}. \tag{3}$$

For turbulent energy (per unit mass) K, we have

$$K = \frac{1}{2}\left\langle v'^2 \right\rangle = \frac{1}{2}\left\langle R^2 \Omega^{*2} \right\rangle, \tag{4}$$

where $\Omega^* = \left|\boldsymbol{\Omega}^*\right|$. From (2) and (3) it follows that Ω is not correlated with R and the expression (4) for turbulent energy K is represented as

$$K = \frac{1}{2}\left\langle R^2 \right\rangle \left\langle \Omega^{*2} \right\rangle. \tag{5}$$

Defining further the characteristic length and time scales, l and t_0, as

$$l = \sqrt{R^2}, \quad t_0 = \frac{1}{\left|\Omega^*\right|},$$

we get from (5)

$$l = t_0 \sqrt{2K}. \tag{6}$$

2.2. The balance equations

Let the state of the environment be determined by a set of scalar state parameters q_i ($q_1 \equiv T$, $q_2 \equiv S$, $q_{i+2} \equiv C_i$, where T and S are the environment's temperature and salinity; C_i are concentrations of the environment's constituents; $i = 1, ..., m-2$; m is the total number of state parameters) and turbulent kinetic energy K, depending on time t and vertical coordinate z, i.e.

$$q_i = q_i(z, t) \text{ and } K = K(z, t).$$

The balance equations for q_i and K in this case are [9,11]:

$$\frac{\partial q_i}{\partial t} = \frac{\partial}{\partial z}\left[\left(k_i + k^{\text{turb}}\right)\frac{\partial q_i}{\partial z}\right], \tag{7}$$

$$\frac{\partial K}{\partial t} = \frac{\partial}{\partial z}\left[k^{\text{turb}}\frac{\partial K}{\partial z}\right] - \psi - \frac{g}{\rho_0}k^{\text{turb}}\frac{\partial \rho}{\partial z}. \tag{8}$$

In (7) and (8), k_i are the coefficients of molecular diffusion of q_i; k^{turb} is the coefficient of turbulent diffusion; $g = |\boldsymbol{g}|$, where $\boldsymbol{g}$ is the gravity acceleration; ψ

is the dissipation function for turbulent energy; ρ_0 and $\rho = \rho(q_i)$ are characteristic and actual density of the environment; the *z*-axis is directed downwards.

Classical semiempirical formulae for k^{turb} and ψ are [12]:

$$k^{\text{turb}} = c_1 \ell \sqrt{K} \text{ and } \psi = c_2 \frac{K\sqrt{K}}{\ell}, \tag{9}$$

where c_1 and c_2 are nondimensional positive coefficients and ℓ defines the mixing length. After postulating the identity

$$\ell \equiv l, \tag{10}$$

according to (6), we get

$$k^{\text{turb}} = ct_K K, \ \psi = \frac{1}{t_K} K, \tag{11}$$

where $c = \sqrt{2} c_1$ and $t_K = \frac{1}{\sqrt{2} c_2} t_0$.

Substituting k^{turb} and ψ from (11) into Eqs. (7) and (8), we have for q_i and K:

$$\frac{\partial q_i}{\partial t} = \frac{\partial}{\partial z}\left[\left(k_i + ct_K K\right) \frac{\partial q_i}{\partial z} \right], \quad i = 1, 2, ..., m, \tag{12}$$

$$\frac{\partial K}{\partial t} = ct_K \frac{\partial}{\partial z}\left[K \frac{\partial K}{\partial z} \right] - \left(\frac{1}{t_K} + \frac{gct_K}{\rho_0} \frac{\partial \rho}{\partial z} \right) K. \tag{13}$$

Instead of the mixing length ℓ we use the characteristic time t_K, thus transforming the classical assumptions for k^{turb} and ψ in (9) into the assumptions (11). This is the cornerstone of the proposed model, differentiating it from earlier models (for example, from the model of Balmforth [11]) which proceed from the same initial set of balance equations. According to Eq. (13), in a uniform density field t_K determines the exponential decay time of the homogeneous kinetic energy K. The theory of rotationally anisotropic turbulence suggests that t_K should not depend on kinetic energy K. The value of t_K (assumed to be constant also in time and space) can be considered as the model parameter or determined from additional assumptions while c is determined from the classical approach.

Measuring $K, q_i, \rho, t,$ and z in units of their characteristic values $K_0, q_{i,0}, \rho_0, t_K$, and z_0, we get Eqs. (12) and (13) in the nondimensional form:

$$\frac{\partial q_i}{\partial t} = \frac{\partial}{\partial z}\left[(a_i + bK)\frac{\partial q_i}{\partial z}\right], \quad i = 1, 2, ..., m, \tag{14}$$

$$\frac{\partial K}{\partial t} = b\frac{\partial}{\partial z}\left[K\frac{\partial K}{\partial z}\right] - \left(1 + rb\frac{\partial \rho}{\partial z}\right)K, \tag{15}$$

where

$$a_i = k_i\frac{t_K}{z_0^{\,2}}, \quad b = c\frac{t_K{}^2 K_0}{z_0^{\,2}}, \quad r = g\frac{z_0}{K_0}. \tag{16}$$

3. THE MODEL OF THE STAIRCASE-LIKE VERTICAL STRUCTURE OF HYDROPHYSICAL FIELDS

3.1. Differential form of the model

We now describe the layering process in the environment caused by double diffusive effects of temperature and salinity fields $q_1 = T$, $q_2 = S$, and $q_{i+2} = 0$ for $i = 1, ..., m-2$. Let T_0 and S_0 be the characteristic values of temperature T and salinity S. Within the linear state equation

$$\rho = 1 - \alpha(T-1) + \beta(S-1),$$

where α and β are measured in units ρ_0/T_0 and ρ_0/S_0, the set of Eqs (14) and (15) takes the form $(a = a_1 \succ\succ d = a_2)$:

$$\frac{\partial K}{\partial t} = b\frac{\partial}{\partial z}\left(K\frac{\partial K}{\partial z}\right) - K\left[1 + rb\left(\beta\frac{\partial S}{\partial z} - \alpha\frac{\partial T}{\partial z}\right)\right], \tag{17}$$

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left[(a + bK)\frac{\partial T}{\partial z}\right], \tag{18}$$

$$\frac{\partial S}{\partial t} = \frac{\partial}{\partial z}\left[(d + bK)\frac{\partial S}{\partial z}\right]. \tag{19}$$

To demonstrate the layering process, the following initial and boundary conditions for Eqs. (17)–(19) are appropriate:
at the initial time instant

$$K(z, 0) = \frac{a}{b}O\left(10^{-2}\right)$$

and

$$\beta[S(z,0)-1]-\alpha[T(z,0)-1]=Cz,$$

where $T(z,0)$ and $S(z,0)$ have an interface determined as a thin layer with relatively large temperature and/or salinity gradients in comparison with their gradients outside the interface. Let the integration region be determined as $\tilde{z}-D\le z\le\tilde{z}+D$ and the interface be centred at $z=\tilde{z}$ with thickness 2δ, where $\delta<<D$. Outside the interface the profiles of $T(z,0)$ and $S(z,0)$ are defined as follows:

$$\begin{aligned}&\text{at } \tilde{z}-D\le z\le\tilde{z}-\delta,\ S\equiv S_{-\delta},\ T=T_{-\delta}-\frac{C}{\alpha}(z+\delta),\\&\text{at } \tilde{z}+D\ge z\ge\tilde{z}+\delta,\ S\equiv S_{\delta},\ T=T_{\delta}-\frac{C}{\alpha}(z-\delta)\end{aligned}\tag{20}$$

or

$$\begin{aligned}&\text{at } \tilde{z}-D\le z\le\tilde{z}-\delta,\ T\equiv T_{-\delta},\ S=S_{-\delta}+\frac{C}{\alpha}(z+\delta),\\&\text{at } \tilde{z}+D\ge z\ge\tilde{z}+\delta,\ T\equiv T_{\delta},\ S=S_{\delta}+\frac{C}{\alpha}(z-\delta).\end{aligned}\tag{21}$$

Boundary conditions at $z=\tilde{z}\pm D$ are

$$\frac{\partial S}{\partial z}=0,\quad \frac{\partial T}{\partial z}=-\frac{C}{\alpha}\tag{22}$$

or

$$\frac{\partial S}{\partial z}=\frac{C}{\beta},\quad \frac{\partial T}{\partial z}=0.\tag{23}$$

Physical situations at the interfaces corresponding to (20) and (21), are classified as salt-fingering and diffusive layering regimes [6–8].

3.2. Discrete model

For the discretization of the model described by Eqs. (17)–(19) we introduce the rectangular time-space grid, with the time step τ and vertical step h. Let (t_i, z_j) denote the grid points and K_{ij}, T_{ij}, and S_{ij} the approximate values of K, T, and S, respectively, at these points. Here $t_i=i\tau$, $z_j=z_0+jh$, $i=0,1,2,...,$ $j=0,1,2,...,n$.

If the approximate values of the system variables are known at $t=t_i$, then discretization of the initial set of equations leads to the following system of algebraic equations for values at $t=t_{i+1}$:

$$f_j^{(1)} \equiv \frac{K_{i+1,j} - K_{i,j}}{\tau} - b\left(\frac{K_{i+1,j+1} - K_{i+1,j-1}}{2h}\right)^2 - bK_{i+1,j}\frac{K_{i+1,j+1} - 2K_{i+1,j} + K_{i+1,j-1}}{h^2}$$

$$+ K_{i+1,j}\left[1 + rb\left(\beta\frac{S_{i+1,j+1} - S_{i+1,j-1}}{2h} - \alpha\frac{T_{i+1,j+1} - T_{i+1,j-1}}{2h}\right)\right] = 0;$$

$$f_j^{(2)} \equiv \frac{T_{i+1,j} - T_{i,j}}{\tau} - \left(a + b\frac{K_{i+1,j+1} - K_{i+1,j-1}}{2h}\right)\frac{T_{i+1,j+1} - T_{i+1,j-1}}{2h} \tag{24}$$

$$- (a + bK_{i+1,j})\frac{T_{i+1,j+1} - 2T_{i+1,j} + T_{i+1,j-1}}{h^2} = 0;$$

$$f_j^{(3)} \equiv \frac{S_{i+1,j} - S_{i,j}}{\tau} - \left(d + b\frac{K_{i+1,j+1} - K_{i+1,j-1}}{2h}\right)\frac{S_{i+1,j+1} - S_{i+1,j-1}}{2h}$$

$$- (d + bK_{i+1,j})\frac{S_{i+1,j+1} - 2S_{i+1,j} + S_{i+1,j-1}}{h^2} = 0.$$

Equations (24), where $j = 0, 1, ..., n,$ present the nonlinear algebraic system accompanied by the following set of linear equations representing boundary conditions and allowing us to eliminate values of unknown functions beyond the grid:

$$\frac{K_{i+1,1} - K_{i+1,-1}}{2h} = 0, \qquad \frac{T_{i+1,1} - T_{i+1,-1}}{2h} = \frac{A}{2h}, \qquad \frac{S_{i+1,1} - S_{i+1,-1}}{2h} = \frac{B}{2h},$$

$$\frac{K_{i+1,n+1} - K_{i+1,n-1}}{2h} = 0, \qquad \frac{T_{i+1,n+1} - T_{i+1,n-1}}{2h} = \frac{A}{2h}, \qquad \frac{S_{i+1,n+1} - S_{i+1,n-1}}{2h} = \frac{B}{2h}.$$

For the case (22), $A = -\frac{C}{\alpha}h,\ B = 0$ and for the case (23), $A = 0,\ B = \frac{C}{\beta}h,$ or simply

$$K_{i+1,-1} = K_{i+1,1}, \qquad T_{i+1,-1} = T_{i+1,1} - A, \qquad S_{i+1,-1} = S_{i+1,1} - B, \tag{25}$$

$$K_{i+1,n+1} = K_{i+1,n-1}, \qquad T_{i+1,n+1} = T_{i+1,n-1} + A, \qquad S_{i+1,n+1} = S_{i+1,n-1} + B. \tag{26}$$

For $t = 0,$ those approximate values are considered to be exact as the initial conditions of the problem.

For fixed $i,$ the system (24) presents the system of model equations (17)–(19) in the following discrete form:

$$\boldsymbol{F}(x) = \vec{0}, \tag{27}$$

where x denotes the vector of dimension $N = 3(n+3)$ of unknown approximate values, $x = (K_{i+1,0}, \ldots, K_{i+1,n}, T_{i+1,0}, \ldots, T_{i+1,n}, S_{i+1,0}, \ldots, S_{i+1,n})$; $\boldsymbol{F}$ is the vector function determined by the system of equations (24)–(26) with the components $f_j^{(s)} = f_j^{(s)}(x)$, $s = 1, 2, 3$, $j = 0, 1, \ldots, n$; $\vec{0}$ is the vector of zeroes of length N. Below we use also the notation F_p, x_p, $p = 0, 1, \ldots, N-1$, for the components of $\boldsymbol{F}$ or x.

4. STABILITY OF THE NUMERICAL PROCESS

4.1. The numerical process

It is convenient to solve the nonlinear algebraic system (24)–(26) using the Newton iteration method. For given initial values x^0 the iteration process will be:

$$\boldsymbol{F}'(x^{m-1})(x^m - x^{m-1}) = \boldsymbol{F}(x^{m-1}), \tag{28}$$

where m is the iteration index and $\boldsymbol{F}'(x^m)$ denotes the Jacobian of $\boldsymbol{F}$ evaluated at x^m, $m = 1, 2, \ldots$ In our case the Jacobian has the 3×3 block structure as shown below:

$$\begin{pmatrix} A & B & B \\ A & A & 0 \\ A & 0 & A \end{pmatrix}, \tag{29}$$

where A, B, and 0 denote the 3-diagonal, 2-diagonal, and zero matrixes.

The sparse structure of the Jacobian (29) makes it possible to apply a modification of the elimination method for solving this system to reduce the number of arithmetical operations. The method will be discussed elsewhere.

In the following we investigate the conditions under which the Jacobian $\boldsymbol{F}'(x^m)$, (29), has a dominant main diagonal, i.e. the problem of solvability of the system (28). It is well known (see, e.g., [17]) that such domination ensures the uniqueness of the solution of the system (24) and also stability of the elimination process.

4.2. Nonzero components of the Jacobian

Let us consider Eq. (27), $\boldsymbol{F}(x) = \vec{0}$ (here and afterwards we omit the upper iteration index). Equation (27) in component form is seen in (24).

According to (24), we find the nonzero components (see (29)) of the Jacobian $f_{p,q} \equiv \dfrac{\partial F_p}{\partial x_q}$, $j = 0, 1, \ldots, n$ as follows:

$$f_{j,j} = \frac{1}{\tau} - b\frac{K_{i+1,j+1} - 2K_{i+1,j} + K_{i+1,j-1}}{h^2} + \frac{2b}{h^2}K_{i+1,j} + \left[1 + rb\left(-\alpha\frac{T_{i+1,j+1} - T_{i+1,j-1}}{2h} + \beta\frac{S_{i+1,j+1} - S_{i+1,j-1}}{2h}\right)\right], \tag{30}$$

$$f_{j,j\pm1} = -b\frac{1}{h^2}K_{i+1,j} \mp b\frac{K_{i+1,j+1} - K_{i+1,j-1}}{2h^2}, \tag{31}$$

$$f_{j,j+n+3} = -f_{j,j+n+1} = \beta rbK_{i+1,j}\frac{1}{2h}, \tag{32}$$

$$f_{j,j+2n+4} = -f_{j,j+2n+2} = -\alpha rbK_{i+1,j}\frac{1}{2h}, \tag{33}$$

$$f_{j+n+2,j+1} = -f_{j+n+2,j-1} = -\frac{b}{2h}\frac{T_{i+1,j+1} - T_{i+1,j-1}}{2h}, \tag{34}$$

$$f_{j+n+2,j} = -b\frac{T_{i+1,j+1} - 2T_{i+1,j} + T_{i+1,j-1}}{h^2}, \tag{35}$$

$$f_{j+n+2,j+n+2\pm1} = -\frac{1}{h^2}(a + bK_{i+1,j}) \pm \frac{1}{2h}\left(a + b\frac{K_{i+1,j+1} - K_{i+1,j-1}}{2h}\right), \tag{36}$$

$$f_{j+n+2,j+n+2} = \frac{1}{\tau} + \frac{2}{h^2}(a + bK_{i+1,j}), \tag{37}$$

$$f_{j+2n+3,j+1} = -f_{j+2n+3,j-1} = -\frac{b}{2h}\frac{S_{i+1,j+1} - S_{i+1,j-1}}{2h}, \tag{38}$$

$$f_{j+2n+3,j} = -b\frac{S_{i+1,j+1} - 2S_{i+1,j} + S_{i+1,j-1}}{h^2}, \tag{39}$$

$$f_{j+2n+3,j+2n+3\pm1} = -\frac{1}{h^2}(d + bK_{i+1,j}) \pm \frac{1}{2h}\left(d + b\frac{K_{i+1,j+1} - K_{i+1,j-1}}{2h}\right), \tag{40}$$

$$f_{j+2n+3,j+2n+3} = \frac{1}{\tau} + \frac{2}{h^2}(d + bK_{i+1,j}). \tag{41}$$

4.3. Stability criterion of the numerical process

In the following we show that by the choice of the size of grid steps h and τ it is possible to reach the domination of the main diagonal of the Jacobian $\boldsymbol{F}'(x)$. This in turn guarantees the uniqueness of the solution of the system (24) and the stability of the elimination process in solving that system. Denote also

$$\boldsymbol{K} = \max_j \left|K_{i+1,j}\right|, \quad \boldsymbol{T} = \max_j \left|T_{i+1,j}\right|, \text{ and } \boldsymbol{S} = \max_j \left|S_{i+1,j}\right|,$$

where $j = 0, 1, \ldots, n$.

According to the formulae for the components of the Jacobian, we analyse the estimations

$$\sum_{q \neq j} \left|f_{j,q}\right| < \left|f_{j,j}\right|, \quad j = 0, 1, \ldots, N-1. \tag{42}$$

We consider the estimations (42) separately in three cases: for $j = 0, 1, \ldots, n$ (case A), $j = n+1, n+2, \ldots, 2n+2$ (case B), and for $j = 2n+3, 2n+4, \ldots, N-1$ (case C).

For case A, from (30) we find

$$\begin{aligned}
\left|f_{j,j}\right| &= \left|\frac{1}{\tau} - \frac{b}{h^2}\left(K_{i+1,j+1} - 2K_{i+1,j} + K_{i+1,j-1}\right) + \frac{2b}{h^2}K_{i+1,j} + \frac{c}{b}\right. \\
&\quad \left. + rb\beta\frac{1}{2h}\left(S_{i+1,j+1} - S_{i+1,j-1}\right) - rb\alpha\frac{1}{2h}\left(T_{i+1,j+1} - T_{i+1,j-1}\right)\right| \\
&\geq \left|\frac{1}{\tau} - \frac{b}{h^2}\left(K_{i+1,j+1} - 4K_{i+1,j} + K_{i+1,j-1}\right)\right. \\
&\quad \left. + rb\left(\beta\frac{S_{i+1,j+1} - S_{i+1,j-1}}{2h} - \alpha\frac{T_{i+1,j+1} - T_{i+1,j-1}}{2h}\right)\right| \\
&\geq \frac{1}{\tau} - \frac{6b}{h^2}\boldsymbol{K} - \frac{rb\beta}{h}\boldsymbol{S} - \frac{rb\alpha}{h}\boldsymbol{T}.
\end{aligned} \tag{43}$$

For case A, from (31)–(33) we find

$$\begin{aligned}
\sum_{q \neq j}\left|f_{j,q}\right| &= \frac{b}{2h^2}\left|K_{i+1,j+1} + 2K_{i+1,j} - K_{i+1,j-1}\right| + \frac{b}{2h^2}\left|K_{i+1,j+1} - 2K_{i+1,j} - K_{i+1,j-1}\right| \\
&\quad + \frac{rb(\alpha+\beta)}{h}\left|K_{i+1,j}\right| \\
&= \frac{b}{h^2}\max\left\{\left|K_{i+1,j+1} - K_{i+1,j-1}\right|, 2\left|K_{i+1,j}\right|\right\} + \frac{rb(\alpha+\beta)}{h}\left|K_{i+1,j}\right| \\
&\leq 2\boldsymbol{K}\left[\frac{b}{h^2} + \frac{rb(\alpha+\beta)}{2h}\right].
\end{aligned}$$

So

$$\sum_{q\neq j}\left|f_{j,q}\right| \le 2\boldsymbol{K}\left[\frac{b}{h^2}+\frac{\rho b(\alpha+\beta)}{2h}\right]. \tag{44}$$

For case B, from (37) we find

$$\left|f_{j+n+2,\,j+n+2}\right| = \left|\frac{1}{\tau}+\frac{2}{h^2}(a+bK_{i+1,j})\right| \ge \frac{1}{\tau}+\frac{2a}{h^2}$$

or

$$\left|f_{j+n+2,\,j+n+2}\right| = \left|\frac{1}{\tau}+\frac{2}{h^2}(a+bK_{i+1,j})\right| \tag{45}$$

and from (34)–(36) it follows that

$$\sum_{q\neq j+n+2}\left|f_{j+n+2,q}\right| = \frac{b}{2h^2}\left|T_{i+1,j+1}-T_{i+1,j-1}\right| + \frac{b}{h^2}\left|T_{i+1,j+1}-2T_{i+1,j}+T_{i+1,j-1}\right|$$
$$+ 2\max\left\{\frac{1}{h^2}\left|a+bK_{i+1,j}\right|, \frac{1}{2h}\left|a+b\frac{K_{i+1,j+1}-K_{i+1,j-1}}{2h}\right|\right\}.$$

Using

$$\frac{b}{2h^2}\left|T_{i+1,j+1}-T_{i+1,j-1}\right| + \frac{b}{h^2}\left|T_{i+1,j+1}-2T_{i+1,j}+T_{i+1,j-1}\right| \le \frac{b}{h^2}\boldsymbol{T}+\frac{4b}{h^2}\boldsymbol{T} = \frac{5b}{h^2}\boldsymbol{T}$$

and

$$2\max\left\{\frac{1}{h^2}\left|a+bK_{i+1,j}\right|, \frac{1}{2h}\left|a+b\frac{K_{i+1,j+1}-K_{i+1,j-1}}{2h}\right|\right\}$$
$$= \frac{2}{h^2}\max\left\{\left|a+bK_{i+1,j}\right|, \frac{h}{2}\left|a+b\frac{K_{i+1,j+1}-K_{i+1,j-1}}{2h}\right|\right\} < \frac{2}{h^2}(a+b\boldsymbol{K}),$$

we have for nondiagonal elements the estimation

$$\sum_{q\neq j+n+2}\left|f_{j+n+2,q}\right| \le \frac{5b}{h^2}\boldsymbol{T}+\frac{2}{h^2}(a+b\boldsymbol{K}). \tag{46}$$

For case C, from (41) we find

$$\left|f_{j+2n+3,\,j+2n+3}\right| = \left|\frac{1}{\tau}+\frac{2}{h^2}(d+bK_{i+1,j})\right| \tag{47}$$

and from (38)–(40) we get

$$\sum_{q\neq j+2n+3}\left|f_{j+2n+3,q}\right| = \frac{b}{2h^2}\left|S_{i+1,j+1}-S_{i+1,j-1}\right| + \frac{b}{h^2}\left|S_{i+1,j+1}-2S_{i+1,j}+S_{i+1,j-1}\right|$$

$$+2\max\left\{\frac{1}{h^2}\left|d+bK_{i+1,j}\right|, \frac{1}{2h}\left|d+b\frac{K_{i+1,j+1}-K_{i+1,j-1}}{2h}\right|\right\} \quad (48)$$

$$\leq \frac{5b}{h^2}\boldsymbol{S}+\frac{2}{h^2}(d+b\boldsymbol{K}).$$

The estimations for nondiagonal elements hold due to some elementary relations for real numbers. For cases B and C, the condition (42) leads to the following inequalities:

$$\frac{1}{\tau} > \frac{5b}{h^2}\boldsymbol{T}+\frac{2b}{h^2}\boldsymbol{K}, \qquad \frac{1}{\tau} > \frac{5b}{h^2}\boldsymbol{S}+\frac{2b}{h^2}\boldsymbol{K}.$$

Both of these are satisfied if

$$\frac{1}{\tau} > \frac{5b}{h^2}\boldsymbol{T}+\frac{5b}{h^2}\boldsymbol{S}+\frac{2b}{h^2}\boldsymbol{K}. \quad (49)$$

In case A the condition (42) holds if

$$\frac{1}{\tau}-\frac{6b}{h^2}\boldsymbol{K}-\frac{rb\beta}{h}\boldsymbol{S}-\frac{rb\alpha}{h}\boldsymbol{T} > 2\boldsymbol{K}\left[\frac{b}{h^2}+\frac{rb(\alpha+\beta)}{2h}\right]$$

or

$$\frac{1}{\tau} > \frac{rb\beta}{h}\boldsymbol{S}+\frac{rb\alpha}{h}\boldsymbol{T}+\left[\frac{8b}{h^2}+\frac{rb(\alpha+\beta)}{h}\right]\boldsymbol{K}. \quad (50)$$

For $h<5/r\alpha$ and $h<5/r\beta$, the inequalities (49) and (50) are true if

$$\frac{h^2}{\tau} > 5b\boldsymbol{S}+5b\boldsymbol{T}+[8b+hrb(\alpha+\beta)]\boldsymbol{K}, \quad (51)$$

which is the final estimation to the size of grid steps to ensure the numerical stability of the discrete model.

It is easy to see that for boundary equations the established estimations hold because of the conditions (25) and (26).

From the right-hand side of inequality (51) it follows that variability in parameters *S*, *T*, *K* cannot introduce bifurcation in the stability criterion, as their estimations depend on the essential parameters *b*, *r* linearly.

5. CONCLUSIONS

The set-up and the stability criteria for numerical realization of the model of the vertical structure of thermohaline fields caused by double diffusion effects were discussed. It was shown that the numerical stability of a discrete model based on the set of nonlinear differential equations (17)–(19) can be guaranteed by choosing suitable space and time steps on the time-space grids. The estimated criterion is sufficient for numerical stability of the formulated discrete model.

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TERMOHALIINSETE VÄLJADE KIHILISE STRUKTUURI MUDELI NUMBRILINE STABIILSUS

Peep MIIDLA, Kalev RANNAT ja Jaak HEINLOO

On formuleeritud temperatuuri ja soolsuse topeltdifusioonist tingitud keskkonna vertikaalse kihistumise diferentsiaalmudel kolmest mittelineaarsest võrrandist koosneva võrrandite süsteemina keskkonna turbulentse energia, temperatuuri ja soolsuse vertikaalse jaotuse määramiseks, mudeli võrrandite lähislahendi leidmise algoritm ja sellel põhineva numbrilise lahenduse stabiilsuse kriteerium.