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GENERAL SOLUTION OF A SYSTEM OF DIFFERENTIAL EQUATIONS MODELLING A CLASS OF EXACTLY-SOLVABLE POTENTIALS

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Abstract. We obtain the general solution of a system of differential equations introduced by Ge et al. (*Phys. Rev*. A, 2000, **62**, 052110–052117). This solution yields a class of exactly-solvable potentials and can be used to calculate the ground state for the class of these potentials.

Key words: Schrödinger equation, exactly-solvable potential.

1. INTRODUCTION

In recent times exactly-solvable models in quantum mechanics have been extensively investigated . Especially for the nonrelativistic Schrödinger equation numerous methods have been elaborated to determine potentials for which at least part of the energy spectrum and the corresponding exact solution functions can be displayed in closed form. Such methods include intertwining techniques [1–3], transformations of the Schrödinger equation [4,5], and others [6].

Recently a new method for generating solvable potentials was introduced in [7]. Given a general Hamiltonian

$$H = X(x)\frac{d^2}{dx^2} + V(x),$$

where X is an arbitrary function and V denotes an arbitrary potential, one defines a momentum operator P via

$$P(x) = Y(x)\frac{d}{dx} + Z(x),$$

with the arbitrary functions Y and Z. Furthermore, define a coordinate operator Q by

$$Q(x) = \frac{1}{1+\beta V(x)} \big(X(x)Z''(x) - \gamma V(x) - \alpha Z(x) - Y(x)V'(x)\big)$$

with free constants α and γ. Provided some constrained relations between the functions $X,\ Y,\ Z,\ V,\ Q$ hold which determine Q, Z, and V in terms of X and Y, one can write the commutators between H and P, resp. H and Q, in the form

$$\begin{aligned} [H,P] &= Q\,\Theta_1 + P\,\Pi_1, \\ [H,Q] &= Q\,\Theta_2 + P\,\Pi_2, \end{aligned}$$

which in turn leads to shift operators allowing computation of the ground state for the potential V determined by the constraints. In other words, given X and Y, one has to solve the constraints for Q, Z and the potential V to obtain the ground state for this potential.

These constraints – a set of coupled differential equations – are given in [7] in explicit terms, but only solved exemplarily: their general solution for Q, Z, and V is not given. Aiming at a particular potential V, it is therefore difficult to choose X and Y in such a way that the constraints yield exactly the desired V.

In this note we compute the general solution of the constraints for $Q,\ Z$, and V in explicit terms of X and Y. To illustrate our computations, we give some examples.

2. SOLUTION OF THE CONSTRAINED EQUATIONS

The set of constrained equations we have to solve reads explicitly [7]:

$$-\gamma X(x) - \beta Q(x)X(x) - Y(x)X'(x) + 2X(x)Y'(x) = 0, \quad (1)$$

$$-\lambda Y(x) + X(x)Q'(x) = 0, \quad (2)$$

$$-\alpha Y(x) + 2X(x)Z'(x) + X(x)Y''(x) = 0, \quad (3)$$

$$\tau - \nu Q(x) - 2\lambda Z(x) + X(x)Q''(x) = 0, \quad (4)$$

$$Q(x) + \gamma V(x) + \beta Q(x)V(x) + \alpha Z(x) + Y(x)V'(x) - X(x)Z''(x) = 0. \quad (5)$$

These equations are to be solved for the functions $Q,\ Z$, and V in terms of the functions X, Y and the constants α, β, γ, λ, ν, and τ.

2.1. The case $X'(x) \neq 0$ and $\beta \neq 0$

In order to determine Q, let us consider Eqs. (1) and (2). Equation (1) yields

$$Q(x) = \frac{-\gamma + 2Y'(x)}{\beta} - \frac{Y(x)X'(x)}{\beta X(x)}, \tag{6}$$

whereas (2) leads to

$$\frac{Y(x)}{X(x)} = \frac{Q'(x)}{\lambda}. \tag{7}$$

Inserting the latter into (6), we come to the following equation for Q:

$$\begin{aligned} Q(x) &= \frac{-\gamma + 2Y'(x)}{\beta} - \frac{X'(x)Q'(x)}{\beta\lambda} \\ \Rightarrow Q'(x) &= -\frac{\beta\lambda}{X'(x)}\, Q(x) - \frac{\lambda\gamma - 2Y'(x)}{X'(x)}, \end{aligned} \tag{8}$$

with the general solution

$$Q(x) = \exp\left(-\int\limits_0^x \frac{\beta\lambda}{X'(k)}\, dk\right)\left(Q_0 - \int \exp\left(\int\limits_0^x \frac{\beta\lambda}{X'(k)}\, dk\right) \frac{\lambda\gamma - 2Y'(x)}{X'(x)}\, dx\right), \tag{9}$$

where Q_0 denotes a free constant.

Equation (3) can be solved for Z by integration:

$$\begin{aligned} Z'(x) &= -\frac{Y''(x)}{2} + \frac{\alpha Y(x)}{2X(x)} \\ \Rightarrow Z(x) &= -\frac{Y'(x)}{2} + \frac{\alpha}{2} \int \frac{Y(x)}{X(x)}\, dx. \end{aligned} \tag{10}$$

Simultaneously, the function Z has to fulfill Eq. (4), that is

$$Z(x) = \frac{-\tau - \nu Q(x) + X(x)Q''(x)}{2\lambda}. \tag{11}$$

Adding the results (10) and (11), we arrive at the following expression for Z:

$$Z(x) = \frac{\alpha}{4} \int \frac{Y(x)}{X(x)}\, dx + \frac{-\lambda Y'(x) - \tau - \nu Q(x) + X(x)Q''(x)}{4\lambda}. \tag{12}$$

The expressions Q and Q'' appearing on the right-hand side of the last equality can be removed by inserting (9) for Q and its second derivative for Q''. We do not give the resulting expression for Z here because it is rather lengthy.

At last, we use Eq. (5) to determine V:

$$\begin{aligned} V'(x) &= -\left(\frac{\gamma+\beta Q(x)}{Y(x)}\right) V(x) - \frac{Q(x)+\alpha Z(x)-X(x)Z''(x)}{Y(x)} \\ \Rightarrow V(x) &= \exp\left(-\int_0^x \frac{\gamma+\beta Q(k)}{Y(k)}\,dk\right)\left(V_0 - \int \exp\left(\int_0^x \frac{\gamma+\beta Q(k)}{Y(k)}\,dk\right)\right. \\ &\qquad \left. \times\ \frac{Q(x)+\alpha Z(x)-X(x)Z''(x)}{Y(x)}\,dx\right). \end{aligned} \tag{13}$$

Again, we can insert the expressions (9) and (12) in order to express V in terms of X and Y only. We omit to give examples to this general case, because even for the simplest functions X and Y we obtain very long and involved expressions. Examples will be given in special cases below.

2.2. The case $\beta = 0$

In the case $\beta = 0$ the constrained equations (1) and (5) simplify as follows:

$$-\gamma X(x) - Y(x)X'(x) + 2X(x)Y'(x) = 0, \tag{14}$$

$$Q(x) + \gamma V(x) + \alpha Z(x) + Y(x)V'(x) - X(x)Z''(x) = 0. \tag{15}$$

We see from (14) that the term containing Q has vanished. Therefore Q is determined only by (2), that is

$$Q(x) = \lambda \int \frac{Y(x)}{X(x)}\,dx. \tag{16}$$

Equation (14) therefore represents an interrelation between X and Y. Solved for Y we obtain

$$\begin{aligned} Y'(x) &= \frac{\gamma}{2} + \frac{X'(x)Y(x)}{2X(x)} \\ \Rightarrow Y(x) &= \sqrt{X(x)}\left(Y_0 + \int \frac{\gamma}{2\sqrt{X(x)}}\,dx\right), \end{aligned} \tag{17}$$

where Y_0 is a free constant.

Since the other constrained equations (2)–(4) remain invariant in case $\beta = 0$, the functions Z and V are again given by (12) and (13) with $\beta = 0$.

Example: $X(x) = -\exp(cx),\ Y(x) = 1$. Let us first check the interrelation (14) to make sure that we are able to fulfill it. We get from (14)

$$\begin{aligned} \gamma \exp(cx) + c\exp(cx) &= 0 \\ \Rightarrow \gamma &= -c. \end{aligned}$$

We see that the interrelation (14) is fulfilled provided the latter setting holds. Let us now compute Q via (16):

$$Q(x) = -\lambda \int \frac{1}{\exp(cx)}\, dx = \frac{\lambda}{c} \exp(-cx) + Q_0. \tag{18}$$

Next, we use (12) to get Z, that is

$$\begin{aligned} Z(x) &= -\frac{\alpha}{4} \int \exp(-cx)\, dx \\ &\quad + \frac{-\tau - \nu(\frac{\lambda}{c}\exp(-cx) + Q_0) - \exp(cx)\,(\lambda c \exp(-cx))}{4\lambda} \\ &= \exp(-cx)\left(\frac{\alpha}{4c} - \frac{\nu}{2c}\right) + \frac{-c\lambda - \nu Q_0 - \tau}{4\lambda} + \frac{Z_0}{2}, \end{aligned} \tag{19}$$

where Z_0 is an integration constant. Now we can calculate V via (13); note that $\gamma = -c$. We obtain

$$\begin{aligned} V(x) &= \frac{\alpha^2\lambda + 4\lambda^2 - \alpha\lambda\nu}{8c^2\lambda}\, e^{-cx} + V_0 e^{cx} \\ &\quad + \frac{-2c^2\lambda\nu + 8c\lambda Q_0 - 2\alpha c\nu Q_0 - 2\alpha c\tau + 4\alpha c\lambda Z_0}{8c^2\lambda}. \end{aligned} \tag{20}$$

The results (18), (19), and (20) agree with those obtained in [7].

2.3. The case $X'(x) = 0$

We see that in the case $X'(x) = 0$ the expression (9) is not defined. Let us thus reformulate the constrained equations (1) and (2) that determine Q. Setting $X(x) = X_0 =$ constant, we come to

$$-\gamma X_0 - \beta X_0 Q(x) + 2X_0 Y'(x) = 0, \tag{21}$$

$$-\lambda Y(x) + X_0 Q'(x) = 0. \tag{22}$$

The second equation gives

$$\begin{aligned} Y(x) &= \frac{X_0}{\lambda}\, Q'(x) \\ \Rightarrow Y'(x) &= \frac{X_0}{\lambda}\, Q''(x), \end{aligned}$$

which we insert into (21):

$$\begin{aligned} -\gamma X_0 - \beta X_0 Q(x) + \frac{2X_0^2}{\lambda} Q''(x) &= 0 \\ \Rightarrow Q''(x) - \frac{\beta\lambda}{2X_0}\, Q(x) - \frac{\gamma\lambda}{2X_0} &= 0. \end{aligned}$$

The latter equation can be solved for Q; its general solution reads

$$Q(x) = Q_1 \exp\left(\sqrt{\frac{\beta\lambda}{2X_0}}\, x\right) + Q_2 \exp\left(-\sqrt{\frac{\beta\lambda}{2X_0}}\, x\right) - \frac{\gamma}{\beta}, \tag{23}$$

where Q_1 and Q_2 denote free constants. Note that this solution for Q does not depend on the function Y!

Since the other constrained equations (3)–(5) do not change, Z and V are given by (12) and (13) with $X(x) = X_0$.

Example: $\boldsymbol{X(x) = -1}$. We see that now $X'(x) = 0$. In order to calculate Q, we have to use the relation (23) with $X_0 = -1$. We first find for the square root in (23):

$$\sqrt{\frac{\beta\lambda}{2X_0}} = \frac{1}{i}\sqrt{\frac{\beta\lambda}{2}} = -i\sqrt{\frac{\beta\lambda}{2}},$$

which gives after insertion into (23)

$$\begin{aligned} Q(x) &= Q_1 \left(\sin\left(\sqrt{\frac{\beta\lambda}{2}}\, x\right) - \cos\left(\sqrt{\frac{\beta\lambda}{2}}\, x\right)\right) \\ &\quad + Q_2 \left(\sin\left(\sqrt{\frac{\beta\lambda}{2}}\, x\right) + \cos\left(\sqrt{\frac{\beta\lambda}{2}}\, x\right)\right) - \frac{\gamma}{\beta} \\ &= \sin\left(\sqrt{\frac{\beta\lambda}{2}}\, x\right)(Q_1 + Q_2) + \cos\left(\sqrt{\frac{\beta\lambda}{2}}\, x\right)(Q_2 - Q_1) - \frac{\gamma}{\beta}. \end{aligned}$$

The latter result agrees with the one obtained in [7] for $Y(x) = a\exp(cx) + b\exp(-cx)$. Observe again that the result does not depend on Y. The other functions Z and V are computed via (12) and (13).

3. CONCLUDING REMARKS

In summary, we have calculated the general solution of the constraints (1)–(5) for Q, Z, and V in explicit terms. The advantage of these solutions is that they allow a first insight into the structure of V. Regarding V as a candidate for a solvable potential [7], the search for a particular potential and its ground state can thus be done more systematically.

Clearly, in case of involved functions X and Y it is not possible to solve the integrals occurring in (9) and (13) symbolically. The calculations should be carried out on a computer algebra system.

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TÄPSELT LAHENDUVATE POTENTSIAALIDE KLASSI MODELLEERIV DIFERENTSIAALVÕRRANDISÜSTEEMI ÜLDLAHEND

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On tuletatud töös [7] sissetoodud diferentsiaalvõrrandisüsteemi üldlahend. See lahend viib täpselt lahenduvate potentsiaalide klassini ja seda võib kasutada selle potentsiaalide klassi põhioleku arvutamiseks.