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PARALLEL AND SEMIPARALLEL SPACE-LIKE SURFACES IN PSEUDO-EUCLIDEAN SPACES

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Abstract. Parallel submanifolds in pseudo-Euclidean spaces are characterized locally by the system $\overline{\nabla} h = 0$. Submanifolds satisfying the integrability condition $\overline{R} \circ h = 0$ of this system are called semiparallel; geometrically they are 2nd-order envelopes of the parallel submanifolds. The existence and geometry of such two-dimensional Riemannian submanifolds (surfaces) are investigated and their complete classification is given. Moreover, it is shown that in E^n_s with $s > 0$ do exist not totally geodesic minimal semiparallel space-like surfaces.

Key words: parallel surfaces, semiparallel surfaces, space-like surfaces, minimal surfaces.

1. INTRODUCTION

Let $N^n_s(c)$ be a space form of constant curvature c. If $s = 0$ (or $s = n$), it is Riemannian and if $0 < s < n$, it is pseudo-Riemannian [1]. A submanifold M^m in $N^n_s(c)$ is called *semiparallel* (or *semisymmetric*, extrinsically) if $\overline{R}(X, Y)h = 0$ (this is the integrability condition of the system $\overline{\nabla} h = 0$ which characterizes a parallel (or locally symmetric, extrinsically) submanifold). Here $\overline{R}$ is the curvature operator of the van der Waerden–Bortolotti connection $\overline{\nabla}$ $(\overline{\nabla} = \nabla \oplus \nabla^{\perp})$ and h is the second fundamental form.

Parallel submanifolds in the case $s = 0,\ c = 0$ are classified by Ferus [2]. His results have been extended to the case $s = 0,\ c \neq 0$ in [3] and [4], and to the case of pseudo-Riemannian parallel submanifolds in $N^n_s(c),\ s > 0$, by Blomstrom [5] and Naitoh [6]. Some special classes of parallel submanifolds in E^n_1 and E^n_2 are described by Magid [7].

Semiparallel submanifolds M^m in $N^n_s(c)$ by $s = 0$ have been classified and described in the following cases:

surfaces $(m = 2)$ if $c = 0$ by Deprez [8];

surfaces $(m = 2)$ if $c > 0$ by Mercuri [9] (see also [10]);

three-dimensional submanifolds, two-codimensional submanifolds (i.e. $m = n - 2$), and hypersurfaces $(m = n - 1)$ if $c = 0$ in [11], [12], and [13], respectively;

submanifolds M^m with flat normal connection $\nabla^{\perp}$ if $c = 0$ in [14] and if $c > 0$ in [15].

Note that semiparallel time-like surfaces in a Lorentzian spacetime form are classified in [16]. A survey on parallel submanifolds and their generalizations is given in [17].

The present paper deals with the classification and description of the semiparallel surfaces which are space-like (i.e. have positive definite inner metric) in pseudo-Euclidean spaces E^n_s. In the first part of the paper all parallel surfaces are determined, in the second part their 2nd-order envelopes are found.

Here the result by Lumiste [18] that every semiparallel submanifold is a 2nd-order envelope of parallel submanifolds can be used. The main result of the present paper is as follows.

Theorem. *Let M^2 be a semiparallel space-like surface in E^n_s. There exists an open and dense part U of M^2 such that the connected components of U are of the following types*:

(i) *open parts of totally umbilical M^2* (*in particular, of totally geodesic M^2*) *in E^n_s;*

(ii) *surfaces with flat $\overline{\nabla}$;*

(iii) *isotropic surfaces with nonflat $\nabla^{\perp}$ satisfying $\| H \|^2 = 3K$, where K is the Gaussian curvature and H is the mean curvature vector.*

This Theorem has some analogy with the classification of the semiparallel surfaces in Euclidean space (see [8]).

Geometric description of these semiparallel space-like surfaces M^2 in E^n_s will be given first for the particular case of parallel space-like surfaces, bearing in mind that every semiparallel surface is a 2nd-order envelope of the parallel ones (according to the result of [18]). Here every semiparallel M^2 of type (i) is parallel itself. The semiparallel M^2 of type (iii) is a 2nd-order envelope of the Veronese surfaces studied in [19]. Therefore only the semiparallel M^2 of type (ii) are to be investigated more thoroughly.

The results for parallel space-like M^2 are formulated in Proposition 1, where more detailed classification and description are given. Mostly this M^2 lies in a semipseudo-Euclidean subspace $E^k_{l,d}$ of E^n_s, which has an orthogonal frame consisting of l vectors of imaginary length, d vectors of zero length, and $k - l - d$ vectors of real length.

The problem of whether there exist semiparallel space-like M^2 in E^n_s, which are not parallel ones, is solved affirmatively in Proposition 2. Also the arbitrariness of their existence is established.

It is known that in Euclidean space E^n the semiparallel minimal surfaces M^2 are trivial: planes or their open parts (see [8,17]). The existence of minimal semiparallel time-like surfaces in E^n_1 different from planes is shown in [16]. In Proposition 3 it will be established that nontrivial minimal semiparallel space-like surfaces do exist in E^n_s with $s > 0$.

2. APPARATUS

Let $\{x, e_I\}$ $(I = 1, 2, ..., n)$ be the moving frame adapted to a space-like submanifold M^2 in E^n_s, i.e. $x \in M^2$, $e_i \in T_x M^2$, $e_\alpha \in T^\perp_x M^2$, $i, j = 1, 2$; $\alpha, \beta = 3, ..., n$. Then, denoting as usual $\langle e_I, e_J\rangle = g_{IJ}$, one obtains $g_{ia} = 0$ and $g_{ij} = \delta_{ij}$ for orthonormal $e_1,\ e_2$; moreover, notation $\langle e_\alpha, e_\alpha\rangle = \varepsilon_\alpha$ can be used. In the well-known formulae $de_I = e_J \omega^J_I,\ \ d\omega^I = \omega^J \wedge \omega^I_J,\ \ d\omega^I_J = \omega^K_J \wedge \omega^I_K$ there hold

$$\omega^1_1 = \omega^2_2 = 0,\ \ \omega^2_1 = -\omega^1_2,\ \ g_{\alpha\beta}\omega^\beta_i + \omega^i_\alpha = 0, \tag{2.1}$$

$$dg_{\alpha\beta} = g_{\gamma\beta}\omega^\gamma_\alpha + g_{\alpha\gamma}\omega^\gamma_\beta. \tag{2.2}$$

Identifying the point x with its radius vector, in the derivation formulae above and in $dx = e_I\omega^I$ one has (see [17])

$$\omega^\alpha = 0,$$

$$\omega^\alpha_i = h^\alpha_{ij}\omega^j, \tag{2.3}$$

$$\overline{\nabla} h^\alpha_{ij} (\equiv dh^\alpha_{ij} - h^\alpha_{kj}\omega^k_i - h^\alpha_{ik}\omega^k_j + h^\beta_{ij}\omega^\alpha_\beta) = h^\alpha_{ijk}\omega^k, \tag{2.4}$$

$$h^\alpha_{ij} = h^\alpha_{ji},\ \ h^\alpha_{ijk} = h^\alpha_{ikj}\ ,$$

$$\overline{\nabla} h^\alpha_{ijk} \wedge \omega^k = -h^\alpha_{kj}\Omega^k_i - h^\alpha_{ik}\Omega^k_j + h^\beta_{ij}\Omega^\alpha_\beta\ \ , \tag{2.5}$$

where each of the formulae (2.3)–(2.5) can be obtained from the previous equations by exterior differentiation and Cartan's lemma. In (2.5)

$$\Omega^j_i = d\omega^j_i - \omega^k_i \wedge \omega^j_k = -g_{\alpha\beta}\omega^\alpha_i \wedge \omega^\beta_j, \tag{2.6}$$

$$\Omega^\alpha_\beta = d\omega^\alpha_\beta - \omega^\gamma_\beta \wedge \omega^\alpha_\gamma = -\sum_i g_{\beta\gamma}\omega^\gamma_i \wedge \omega^\alpha_i \tag{2.7}$$

are the curvature 2-forms of the Levi–Civita connection ∇ and the normal connection $\nabla^\perp$, respectively. Together they represent the curvature 2-forms of the van der Waerden–Bortolotti connection $\overline{\nabla}$. If $\Omega^j_i = 0$, or $\Omega^\beta_\alpha = 0$, or $\Omega^j_i = \Omega^\beta_\alpha = 0$, we have the flat connection ∇, or $\nabla^\perp$, or $\overline{\nabla}$, respectively.

Due to (2.4) and (2.5) the parallelity and semiparallelity conditions are, respectively (see [17]),

$$dh^\alpha_{ij} - h^\alpha_{kj}\omega^k_i - h^\alpha_{ik}\omega^k_j + h^\beta_{ij}\omega^\alpha_\beta = 0\ , \tag{2.8}$$

$$h^\alpha_{kj}\Omega^k_i + h^\alpha_{ik}\Omega^k_j - h^\beta_{ij}\Omega^\alpha_\beta = 0\ . \tag{2.9}$$

If we denote $H_{ij,kl} = g_{\alpha\beta} h^{\alpha}_{ij} h^{\beta}_{kl}$ and $h_{ij} = h^{\alpha}_{ij} e_{\alpha}$, then (2.9) is equivalent to

$$\sum_k (h_{kj} H_{i[p,q]k} + h_{ik} H_{j[p,q]k} - H_{ij,k[p} h_{q]k}) = 0. \tag{2.10}$$

3. TRANSFORMATION FORMULAE

The tangent part $\{e_1, e_2\}$ of the adapted frame can be transformed according to

$$e_1^{'} = e_1 \cos\phi + e_2 \sin\phi, \tag{3.1}$$

$$e_2^{'} = -e_1 \sin\phi + e_2 \cos\phi. \tag{3.2}$$

Then

$$\omega^{2'}_{1'} = \omega^2_1 + d\phi, \tag{3.3}$$

$$\omega^1 = \omega^{1'} \cos\phi - \omega^{2'} \sin\phi \ , \tag{3.4}$$

$$\omega^2 = \omega^{1'} \sin\phi + \omega^{2'} \cos\phi \tag{3.5}$$

and for $h = h_{ij}\omega^i\omega^j$ one obtains

$$h_{11}^{'} = \frac{1}{2}(h_{11} + h_{22}) + \frac{1}{2}(h_{11} - h_{22})\cos 2\phi + h_{12}\sin 2\phi \ ,$$

$$h_{12}^{'} = \frac{1}{2}(h_{22} - h_{11})\sin 2\phi + h_{12}\cos 2\phi \ ,$$

$$h_{22}^{'} = \frac{1}{2}(h_{11} + h_{22}) + \frac{1}{2}(h_{22} - h_{11})\cos 2\phi - h_{12}\sin 2\phi \ .$$

Therefore span$\{h_{11}, h_{22}, h_{12}\}$ is an invariant vector subspace of $T^{\perp}_x M^2$ at an arbitrary fixed point $x \in M^2$, called the first normal subspace of M^2 at x and denoted by $N^{(1)}_x M^2$. Let us denote $\frac{1}{2}(h_{11} - h_{22}) = A$, $h_{12} = B$, and $\frac{1}{2}(h_{11} + h_{22}) = H$; then $A' = A\cos 2\phi + B\sin 2\phi$, $B' = -A\sin 2\phi + B\cos 2\phi$, $H' = H$. It is seen that H is an invariant vector, called the mean curvature vector, and that span$\{A, B\}$ is an invariant vector subspace at x, denoted by $I_x M^2$; the latter is the plane of the indicatrix of normal curvature determined as $\{y : y - x = h_{ij}X^iX^j, X \in T_xM^2, \| X \| = 1\}$. Since

$$\langle A', B'\rangle = \langle A, B\rangle \cos 4\phi + \frac{1}{2}(B^2 - A^2)\sin 4\phi,$$

there exists ϕ_0 such that $\langle A', B'\rangle = 0$. So it can be made that $\langle A, B\rangle = 0$. Taking now $\phi = \frac{\pi}{4}$ gives $A' = B$, $B' = -A$, and $\langle A', B'\rangle = 0$, so that the roles of A and B can be interchanged, if this is not obstructed by the metric.

4. PROOF OF THEOREM

4.1. The case $\mathbf{dim} I_x M^2 = 0$

In the case of dim$I_xM^2 = 0$, the indicatrix of normal curvature degenerates into a point, i.e. $A = B = 0$. If here dim$N_x^{(1)}M^2 = 1$, then $H \neq 0$ and e_3 can be taken so that $H = \delta e_3$ and the components of the second fundamental form h_{ij} can be written as follows: $h_{11} = h_{22} = \delta e_3,\ h_{12} = 0$. Thus M^2 is totally umbilic. For the case dim$N_x^{(1)}M^2 = 0$ one has $\delta = 0$ and the considered surface is totally geodesic. This leads to case (i) of the Theorem.

4.2. The case $\mathbf{dim} I_x M^2 = 1$

Here the mutually orthogonal A and B are collinear and at least one of them is nonzero. Since A and B can be interchangeable, it is always assumed that $A \neq 0$. Then the frame vector e_3 can be taken so that

$$A = ae_3,\ a > 0,\ B = be_3.$$

Let dim$N_x^{(1)}M^2 = 2$. The next frame vector e_4 can be taken so that $H = \delta e_3 + \sigma e_4$ (if dim$N_x^{(1)}M^2 = 1$, one has $\sigma = 0$). The Pfaff system (2.3) can be written as

$$\omega_1^3 = (\delta + a)\omega^1 + b\omega^2,\ \ \omega_1^4 = \sigma\omega^1,\ \ \omega_1^\xi = 0, \tag{4.1}$$

$$\omega_2^3 = b\omega^1 + (\delta - a)\omega^2,\ \ \omega_2^4 = \sigma\omega^2,\ \ \omega_2^\xi = 0, \tag{4.2}$$

where $\xi = 5, ..., n$. Hence

$$\Omega_1^2 = -\Omega_2^1 = [\varepsilon_3 a^2 + \varepsilon_3 b^2 - H^2]\omega^1 \wedge \omega^2, \tag{4.3}$$

where $H^2 = \varepsilon_3\delta^2 + \varepsilon_4\sigma^2 + 2g_{34}\delta\sigma$, and all Ω_α^β are zero.

With respect to the metric in subspaces I_xM^2 and $N_x^{(1)}M^2$ there are the following possibilities.

If the metric of I_xM^2 is regular, then $\varepsilon_3 = \pm 1$, and $b = 0$. Thus the vector e_4 can be taken so that either

$$\varepsilon_4 \neq 0,\ \ g_{34} = 0 \tag{4.4}$$

or

$$\varepsilon_4 = 0,\ g_{34} = 0; \tag{4.5}$$

moreover, in the last case the vector e_5 can be taken so that $\varepsilon_5 = 0,\ g_{45} = 1$. So (4.4) and (4.5) mean that the metric of $N_x^{(1)}M^2$ is regular or singular nonvanishing, respectively.

If the metric of I_xM^2 is vanishing, then $\varepsilon_3 = 0$ (and $b \neq 0$ or $b = 0$) and the metric of $N_x^{(1)}M^2$ is either regular, or singular nonvanishing, or vanishing. This means that the vector e_4 can be taken so that either

$$\varepsilon_4 = 0,\ g_{34} = 1, \tag{4.6}$$

or

$$\varepsilon_4 \neq 0,\ g_{34} = 0, \tag{4.7}$$

(moreover, in (4.7) the vector e_5 can be taken so that $\varepsilon_5 = 0,\ g_{35} = 1$), or

$$\varepsilon_4 = 0,\ \ g_{34} = 0. \tag{4.8}$$

In the last case the frame vectors $e_5,\ e_6$ can be chosen so that $\varepsilon_5 = \varepsilon_6 = 0$, $g_{35} = g_{46} = 1,\ g_{36} = g_{45} = g_{56} = 0,\ n \geq 6$.

For all these cases the semiparallelity condition (2.9) reduces to

$$b\Omega_1^2 = 0,\ a\Omega_1^2 = 0.$$

Since $a > 0$, one has $\Omega_1^2 = 0$. This result together with $\Omega_\alpha^\beta = 0$ gives that $\overline{\nabla}$ is flat, i.e. leads to case (ii) of the Theorem.

4.3. The case $\dim I_xM^2 = 2$

In this case the mutually orthogonal vectors A and B are noncollinear. Then the indicatrix of normal curvature $\{y : y - x = H + A\cos 2\psi + B\sin 2\psi\}$ is an ellipse. The orthogonal frame vectors e_3 and e_4 (i.e. with $g_{34} = 0$) in I_xM^2 can be taken so that

$$A = ae_3,\ B = be_4,\ a \geq b > 0.$$

The dimension of the subspace $N_x^{(1)}M^2$ is either 3 or 2.

Let $\dim N_x^{(1)}M^2 = 3$, thus the next frame vector e_5 can be taken so that $H = \delta e_3 + \sigma e_4 + \tau e_5$. Here the components h_{ij} can be written as follows:

$$h_{11} = (\delta + a)e_3 + \sigma e_4,\ h_{22} = (\delta - a)e_3 + \sigma e_4,\ h_{12} = be_4.$$

Thus M^2 is determined by the Pfaff system

$$\omega_1^3 = (\delta + a)\omega^1,\ \ \omega_1^4 = \sigma\omega^1 + b\omega^2,\ \ \omega_1^5 = \tau\omega^1,\ \ \omega_1^\xi = 0,$$

$$\omega_2^3 = (\delta - a)\omega^2,\ \ \omega_2^4 = b\omega^1 + \sigma\omega^2,\ \ \omega_2^5 = \tau\omega^2,\ \ \omega_2^\xi = 0,$$

where $\xi = 6, ..., n$. Hence the curvature 2-forms are: in (2.6)

$$\Omega_1^1 = \Omega_2^2 = 0,\ \Omega_1^2 = -\Omega_2^1 = (\varepsilon_3 a^2 + \varepsilon_4 b^2 - H^2)\omega^1 \wedge \omega^2, \tag{4.9}$$

where

$$H^2 = \varepsilon_3\delta^2 + \varepsilon_4\sigma^2 + \varepsilon_5\tau^2 + 2g_{35}\delta\tau + 2g_{45}\sigma\tau,$$

and in (2.7)

$$\Omega_3^4 = -2\varepsilon_3 ab\omega^1 \wedge \omega^2,\ \Omega_4^3 = 2\varepsilon_4 ab\omega^1 \wedge \omega^2,$$
$$\Omega_5^3 = 2g_{54}ab\omega^1 \wedge \omega^2,\ \Omega_5^4 = -2g_{35}\omega^1 \wedge \omega^2;$$

all other Ω_α^β are zero. Thus the semiparallelity condition (2.9) transforms into

$$\begin{aligned} &ab(\varepsilon_4\sigma + g_{45}\tau) = 0, && b(2\varepsilon_3 a^2 + \varepsilon_4 b^2 - H^2 + \varepsilon_3 a\delta + g_{35}a\tau) = 0,\\ &a(\varepsilon_3 a^2 + 2\varepsilon_4 b^2 - H^2) = 0, && b(2\varepsilon_3 a^2 + \varepsilon_4 b^2 - H^2 - \varepsilon_3 a\delta - g_{35}a\tau) = 0. \end{aligned}$$

Consideration of this system gives, due to $ab\tau \neq 0$, that $\varepsilon_3 = \varepsilon_4 = \varepsilon_5$, having the values $1,\ -1$ or 0.

In the last case the metric of $N_x^{(1)}M^2$ vanishes completely and the frame vectors can be taken so that

$$\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0,\ \ \varepsilon_6 = \varepsilon_7 = \varepsilon_8 = 0,\ \ g_{36} = g_{47} = g_{58} = 1,\ \ n \geq 8. \quad (4.10)$$

It can be obtained by the appropriate choice of remaining frame vectors so that all other $g_{\alpha\beta},\ \alpha \neq \beta$, are zero.

If $\dim N_x^{(1)}M^2 = 2$, then $\tau = 0$ and the semiparallelity condition leads to $\varepsilon_3 = \varepsilon_4 = 0$. Here the vectors $e_5,\ e_6$ can be taken so that

$$\varepsilon_5 = \varepsilon_6 = 0,\ g_{35} = g_{46} = 1,\ g_{56} = 0,\ n \geq 6. \quad (4.11)$$

Note that in the cases of this section the choice of all other e_α depends on the value of s in E_s^n.

Let us consider the cases (4.10), (4.11) in more detail, denoting $\dim N_x^{(1)}M^2 = n_1\ (n_1 = 3, 2)$ and $a, b = \{3, ..., n_1 + 2\},\ \overline{a}, \overline{b} = \{n_1 + 3, ..., 2n_1 + 2\}$. Then in (2.1), (2.2) one has

$$\omega_i^{a+n_1} + \omega_a^i = 0, \quad \omega_i^{\overline{a}-n_1} + \omega_{\overline{a}}^i = 0, \quad (4.12)$$

$$\omega_a^a + \omega_{\overline{a}}^{\overline{a}} = 0, \quad \omega_a^{\overline{a}} = \omega_{\overline{a}}^a = 0, \quad (4.13)$$

$$\omega_a^{\overline{b}} + \omega_b^{\overline{a}} = 0, \quad \omega_{\overline{a}}^b + \omega_{\overline{b}}^a = 0, \quad \omega_a^b + \omega_{\overline{b}}^{\overline{a}} = 0. \quad (4.14)$$

Substitution from (4.12)–(4.14) into (2.6), (2.7) gives that $\Omega_i^j = \Omega_\alpha^\beta = 0$, i.e. $\overline{\nabla}$ is flat. This leads to case (ii) of the Theorem.

In the case where the first normal subspace has a regular metric (i.e. $\varepsilon_3 = \varepsilon_4 = \varepsilon_5$, equalling to 1 or -1), the semiparallelity condition leads to $\delta = \sigma = 0,\ \ a = b,\ \ \tau = a\sqrt{3}$. It follows that the normal curvature vector has a constant scalar square at every point $x \in M^2$, i.e. the considered surface is an isotropic surface and $H^2 = 3K$. It gives case (iii) of the Theorem.

5. DESCRIPTION OF PARALLEL CURVES AND SURFACES

5.1. Parallel curves

For the principal normal of the curve M^1 in pseudo-Euclidean spaces there are three possibilities: it can be space-like, time-like, or light-like.

In the first two cases one has $\varepsilon_2 = \pm 1$. The parallel curve can be treated like in [17]. It is a straight line or a circle; the latter can be of either real or imaginary radius.

If the principal normal is light-like, then $\varepsilon_2 = 0$ and the next frame vector e_3 can be taken so that $\varepsilon_3 = 0,\ g_{23} = 1$. Thus the Bartels–Frenet formulae can be written as

$$dx = e_1 ds,\ de_1 = k_1 e_2 ds,\ de_2 = -d\ln k_1 e_2 ds.$$

The parallelity condition leads to $k_1 =$ const, thus $de_2 = 0$. After integration it gives $x = \frac{1}{2}cs^2 + c_1 s + c_2$, where all coefficients are constant vectors. Therefore the parallel curve of this case is a parabola.

5.2. Parallel surfaces

As is noted in Introduction, for the geometric description of the surfaces of Theorem more detailed classification and characterization of the surfaces of type (ii) are needed. First, the same must be done for the corresponding parallel surfaces.

Proposition 1. *Let M^2 be a space-like parallel surface in E^n_s with flat $\overline{\nabla}$, which lies essentially in an affine subspace of the E^n_s. Such an M^2 is either*

(ii_1) *translation surface of two parallel curves, or*

(ii_2) *surface in E^4_1 on its isotropic cone C^3, with a fixed vertex*; *the mean curvature vector of this surface is isotropic and $T^\perp_x M^2$ goes through the generator of the cone, or*

(ii_3) *surface in $E^3_{0,1}$, $E^4_{0,2}$, or $E^5_{0,3}$ with two families of parabola generators* (*one of them can degenerate into a family of straight lines*). *This surface can be represented by the equation $x = \frac{1}{2}h_{11}(u)^2 + \frac{1}{2}h_{22}(v)^2 + h_{12}uv + c_1 u + c_2 v$, where all coefficients are some constant vectors and the first three of them are isotropic* (*no matter whether this degeneration occurs or not*).

Proof. For the full classification of parallel surfaces with flat $\overline{\nabla}$ there must be considered the frame possibilities (4.4)–(4.8), (4.10), (4.11).

The subspace $I_x M^2$ has dimension 1. Let, at first, the frame vectors be taken as shown in (4.4). If here $\sigma \neq 0$, then from the parallelity condition one has $\omega^4_3 = \omega^2_1 = \omega^\xi_3 = \omega^\xi_4 = da = d\sigma = d\delta = 0$. Thus the derivation formulae are

$$dx = e_1\omega^1 + e_2\omega^2,$$
$$de_1 = [(\delta + a)e_3 + \sigma e_4]\omega^1,$$

$$de_2 = [(\delta - a)e_3 + \sigma e_4]\omega^2,$$
$$de_3 = -\varepsilon_3[(\delta + a)e_1\omega^1 + (\delta - a)e_2\omega^2],$$
$$de_4 = -\varepsilon_4[\sigma e_1\omega^1 + \sigma e_2\omega^2].$$

If $\varepsilon_3 = \varepsilon_4$, then the considered parallel surface lies in an $E^4_s,\ s$ is 0 or 2 (it depends on the signature of the metric). Since $d\omega^1 = 0, \quad d\omega^2 = 0$, at least locally $\omega^1 = du,\ \ \omega^2 = dv$. The geodesic lines $v =$ const and $u =$ const are circles. Hence the parallel surface is a translation surface of two circles on the totally orthogonal $E^2_s,\ s$ is 0 or 1.

If $\varepsilon_3 \neq \varepsilon_4$, the considered parallel surface is a translation surface of two lines; one of them is a circle with real radius on E^2 and the other is a circle with imaginary radius on E^2_1.

On supposition $\sigma = 0$, from (4.3) one has that $a^2 = \delta^2$ and the vector e_3 can be directed so that $a = \delta$, thus one of the geodesic lines degenerates into a straight line.

In the case where $N^{(1)}_x M^2$ has the metric (4.5), the equality (4.3) leads to $a^2 = \delta^2$. Then the vector e_3 can be taken so that $a = \delta$ and the parallelity condition gives $\omega^2_1 = \omega^3_4 = \omega^4_3 = \omega^\xi_3 = \omega^\xi_4 = 0$, $\omega^4_4 = -\frac{d\sigma}{\sigma}$, $a =$ const and the derivation formulae can be written as $dx = e_1\omega^1 + e_2\omega^2,\ de_1 = (2ae_3 + \sigma e_4)\omega^1,\ de_2 = \sigma e_4\omega^2,\ de_3 = -2\varepsilon_3 a e_1\omega^1,\ d(\sigma e_4) = 0$. So the considered surface lies in an $E^4_{0,1}$ (or in $E^4_{1,1}$) if $\varepsilon_3 = 1$ (or $\varepsilon_3 = -1$, respectively), which is spanned by the point x and mutually orthogonal vectors $e_1,\ e_2,\ e_3,\ \sigma e_4$. Investigation of geodesic lines gives that parallel M^2 is a translation surface of circles and parabolas.

On supposition $\sigma = 0$, it is easy to see that geometry of the corresponding parallel surface is the same as in the previous case on analogous supposition.

In the case where $I_x M^2$ has a vanishing metric and the frame is described by the equalities (4.6), one has $\Omega^2_1 = 0$; then $d\omega^2_1 = 0$, i.e. $\omega^2_1 = d\psi$. Thus the formulae (3.1)–(3.5) lead to $\omega^{2'}_{1'} = 0$ and h_{ij} transform into

$$h'_{11} = (\delta + a')e_3 + \sigma e_4,\ h'_{22} = (\delta - a')e_3 + \sigma e_4,\ h'_{12} = b'e_3,$$

where $a' = a\cos 2\psi + b\sin 2\psi$ and $b' = -a\sin 2\psi + b\cos 2\psi$. Thus $A = a'e_3,\ B = b'e_3,\ H = \delta e_3 + \sigma e_4$.

The parallelity condition leads to $\omega^\xi_3 = \omega^\xi_4 = 0$, $\omega^3_3 = -\frac{da'}{a'} = -\frac{db'}{b'} = -\frac{d\delta}{\delta} = \frac{d\sigma}{\sigma}$, i.e. $b' = k_1 a',\ \delta = k_2 a',\ \sigma = \frac{k_3}{a'}$, where $k_1,\ k_2,\ k_3$ are some constants. Moreover, from (4.3) and the semiparallelity condition one has $\delta\sigma = 0$ and either

$$1)\ \delta = 0,\ \sigma \neq 0, (k_2 = 0), \text{or}$$
$$2)\ \delta \neq 0,\ \sigma = 0, (k_3 = 0), \text{or}$$
$$3)\ \delta = \sigma = 0, (k_2 = k_3 = 0).$$

Note that in the last case M^2 is a minimal surface and will be considered in Section 7.

Since $d\omega^{1'} = 0,\ d\omega^{2'} = 0,$ at least locally $\omega^{1'} = du,\ \omega^{2'} = dv,$ and the derivation formulae in subcase 1) by $b' \neq 0$ can be written so that

$$\begin{aligned}
&dx = e_1' du + e_2' dv,\\
&de_1' = (a'e_3 + \tfrac{k_3}{a'}e_4)du + k_1 a' e_3 dv,\\
&de_2' = k_1 a' e_3 du + (-a'e_3 + \tfrac{k_3}{a'}e_4)dv,\\
&d(a'e_3) = -k_3 dx,\\
&d(\tfrac{k_3}{a'}e_4) = -k_3[(e_1' + k_1 e_2')du + (k_1 e_1' - e_2')dv].
\end{aligned}$$

It is seen that this surface lies in an E_1^4, spanned by the point x and vectors $e_1',\ e_2',\ a'e_3,\ \frac{k_3}{a'}e_4$. The point $z \in E_1^4$ with the radius vector $z = x + \frac{1}{k_3}a'e_3$ is fixed for the surface since $dz = 0$, thus there is an isotropic cone C^3 with a vertex at the point z. The considered surface lies on the cone, its normal plane $T_x^{\perp}M^2$ goes through the generator of the cone (collinear to e_3) and the mean curvature vector is isotropic (noncollinear to e_3).

On supposition $b' = 0$, the derivation formulae for the considered parallel M^2 from subcase 1) can be written as

$$\begin{aligned}
&dx = e_1' du + e_2' dv,\\
&de_1' = (a'e_3 + \tfrac{k_3}{a'}e_4)du,\\
&de_2' = (-a'e_3 + \tfrac{k_3}{a'}e_4)dv,\\
&d(a'e_3 + \tfrac{k_3}{a'}e_4) = -2k_3 e_1' du,\\
&d(-a'e_3 + \tfrac{k_3}{a'}e_4) = 2k_3 e_2' dv.
\end{aligned}$$

Since $(a'e_3 + \frac{k_3}{a'}e_4)^2 = 2k_3$ and $(-a'e_3 + \frac{k_3}{a'}e_4)^2 = -2k_3$, this surface lies in an E_1^4, spanned by the point x and mutually orthogonal vectors $e_1',\ e_2',\ a'e_3 + \frac{k_3}{a'}e_4,\ -a'e_3 + \frac{k_3}{a'}e_4$. Investigation of its geodesics gives that the considered parallel surface is a translation surface of two plane lines of constant curvature.

In subcase 2), when $k_3 = 0$ on supposition $k_1 \neq 0$ (i.e. $b' \neq 0$), in the derivation formulae one has

$$\begin{aligned}
&dx = e_1' du + e_2' dv,\\
&de_1' = [(k_2 + 1)du + k_1 dv]a'e_3,\\
&de_2' = [k_1 du + (k_2 - 1)dv]a'e_3,\\
&d(a'e_3) = 0.
\end{aligned}$$

The considered surface lies in an $E_{0,1}^3 \subset E_1^4$, spanned by the point x and mutually orthogonal vectors $e_1',\ e_2',\ a'e_3$. Denoting the partial derivatives of x by x_u, x_v, etc., one has

$$x_u = e_1,\ x_v = e_2,$$
$$x_{uu} = (k_2+1)a'e_3,\ x_{uv} = k_1 a'e_3,\ x_{vv} = (k_2-1)a'e_3,$$
$$x_{uuu} = x_{uuv} = x_{vvu} = x_{vvv} = 0.$$

Since for this case $(k_2+1)a'e_3 = h_{11},\ (k_2-1)a'e_3 = h_{22},\ k_1a'e_3 = h_{12}$, then parallel M^2 can be represented by the equation $x = \frac{1}{2}h_{11}(u)^2 + \frac{1}{2}h_{22}(v)^2 + h_{12}uv + c_1u + c_2v$, where all coefficients are some constant vectors; the absolute term can be made zero by exchanging the initial point. It is seen that the geodesic lines on this parallel surface are parabolas (one of them can degenerate into a straight line).

On supposition $k_1 = 0$, the considered surface lies in an $E^3_{0,1} \subset E^4_1$ and is a translation surface of either two parabolas, or of a parabola and a straight line.

If the frame vectors are taken as shown in (4.7), then geometry of the corresponding parallel surface is the same as in the previous case, subcase 2).

Finally, if (4.8) holds, then the parallelity condition implies $\omega^3_3 = -\frac{db'}{b'} = -\frac{da'}{a'} = -\frac{d\delta}{\delta}$ (i.e. $b' = k_1a',\ \delta = k_2a'$, where $k_1,\ k_2$ are some constants), $\omega^4_4 = -\frac{d\sigma}{\sigma}$, and $\omega^4_3 = \omega^3_4 = \omega^\xi_3 = \omega^\xi_4 = 0$.

Since $d\omega^1 = d\omega^2 = 0$, then at least locally $\omega^1 = du,\ \omega^2 = dv$ and the derivation formulae can be written so that

$$dx = e'_1du + e'_2dv,$$
$$de'_1 = [(k_2+1)a'e_3 + \sigma e_4]du + k_1a'e_3dv,$$
$$de'_2 = k_1a'e_3du + [(k_2-1)a'e_3 + \sigma e_4]dv,$$
$$d(a'e_3) = 0,\ \ d(\sigma e_4) = 0.$$

On supposition $\sigma = 0$, the considered surface lies in $E^3_{0,1}$ spanned by the point x and mutually orthogonal vectors $e'_1,\ e'_2,\ a'e_3$, and geometry of this M^2 coincides with geometry in the case with (4.6), subcase 2).

If $\sigma \neq 0$ and $b' \neq 0$, then M^2 lies in $E^4_{0,2}$ spanned by the point x and mutually orthogonal vectors $e'_1,\ e'_2,\ a'e_3,\ \sigma e_4$ and is determined by the equation $x = \frac{1}{2}h_{11}(u)^2 + \frac{1}{2}h_{22}(v)^2 + h_{12}uv + c_1u + c_2v$, where $h_{11} = (k_2+1)a'e_3 + \sigma e_4$, $h_{12} = k_1a'e_3,\ \ h_{22} = (k_2-1)a'e_3 + \sigma e_4$ (all coefficients are constant vectors), and has two families of parabola generators.

Finally, if $\sigma \neq 0$, but $b' = 0$, then the considered parallel surface lies in $E^4_{0,2}$ and is a translation surface of two geodesic lines, each of which is a parabola.

The subspace I_xM^2 has dimension 2. Here the metric in $N^{(1)}_xM^2$ vanishes completely and is described either by (4.10) or (4.11).

In the first case the subspace $N^{(1)}_xM^2$ has a maximal dimension. Then the frame can be adapted to the considered surface so that $A = e_3,\ B = e_4,\ H = e_5$, i.e. $a = b = \tau = 1$ and $\delta = \sigma = 0$. Thus the Pfaff system can be written as follows:

$$\omega_1^3 = \omega^1,\ \omega_1^4 = \omega^2,\ \omega_1^5 = \omega^1,\ \omega_1^{\overline{a}} = \omega_1^{\xi} = 0, \tag{5.1}$$

$$\omega_2^3 = \omega^2,\ \omega_2^4 = \omega^1,\ \omega_2^5 = \omega^2,\ \omega_2^{\overline{a}} = \omega_2^{\xi} = 0. \tag{5.2}$$

Here $\overline{a} = 6, 7, 8;\ \xi = 9, ..., n$, and substitution into the parallelity condition leads to $\omega_3^3 = \omega_4^4 = \omega_5^5 = \omega_3^5 = \omega_4^5 = \omega_5^3 = \omega_5^4 = 0, \quad 2\omega_1^2 = \omega_3^4 = -\omega_3^4$. Due to $e_3 + e_5 = h_{11},\ \ e_4 = h_{12},\ \ e_5 - e_3 = h_{22}$, the derivation formulae can be written as

$$\begin{aligned}
&dx = e_1\omega^1 + e_2\omega^2,\\
&de_1 = \omega_1^2 e_2 + h_{11}\omega^1 + h_{12}\omega^2,\\
&de_2 = -\omega_1^2 e_1 + h_{12}\omega^1 + h_{22}\omega^2,\\
&dh_{11} = \omega_1^2 h_{12},\\
&dh_{22} = -\omega_1^2 h_{12},\\
&dh_{12} = -\omega_1^2 (h_{11} - h_{22}).
\end{aligned}$$

Since $\Omega_1^2 = 0$, due to (2.6) here $d\omega_1^2 = 0$, i.e. $\omega_1^2 = d\psi$. Using transformation formulae (3.1)–(3.5), one has $\omega_{1'}^{2'} = 0$, $d\omega^{1'} = 0$, $d\omega^{2'} = 0$. The last two equalities imply, at least locally, that $\omega^{1'} = du,\ \ \omega^{2'} = dv$, thus

$$\begin{aligned}
&dx = e_1' du + e_2' dv,\\
&de_1' = h_{11}' du + h_{12}' dv,\\
&de_2' = h_{12}' du + h_{22}' dv,\\
&dh_{11}' = dh_{12}' = dh_{22}' = 0.
\end{aligned}$$

So the considered parallel space-like M^2 lies in an $E_{0,3}^5$ spanned by the point x and mutually orthogonal vectors $e_1,\ e_2,\ h_{11}',\ h_{22}',\ h_{12}'$, the last three of which are light-like, two others space-like. This surface can be represented by the equation $x = \frac{1}{2}h_{11}'(u)^2 + \frac{1}{2}h_{22}'(v)^2 + h_{12}'uv + h_{01}'u + h_{02}'v$, where all coefficients are some constant vectors. It is seen that the geodesic lines $u =$ const and $v =$ const on this parallel surface are parabolas.

For the parallel space-like surface M^2, with dim$N_x^{(1)}M^2 = 2$, i.e. when the frame vectors are taken as shown in (4.11) and the frame can be adapted to M^2 so that $A = e_3,\ B = e_4$, i.e. $a = b = 1$. Moreover, if the mean curvature vector H is nonzero, then A and B can be taken so that $A \parallel H$ and thus $H = \delta e_3$. Now the Pfaff system can be written as

$$\omega_1^3 = (\delta + 1)\omega^1,\ \ \omega_1^4 = \omega^2,\ \ \omega_1^{\overline{a}} = \omega_1^{\xi} = 0, \tag{5.3}$$

$$\omega_2^3 = (\delta - 1)\omega^2,\ \ \omega_2^4 = \omega^1,\ \ \omega_2^{\overline{a}} = \omega_2^{\xi} = 0. \tag{5.4}$$

Here $\overline{a} = 5, 6;\ \ \xi = 7, ..., n$ and from the parallelity condition (2.8) one has $\omega_3^3 = \omega_4^4 = \omega_4^5 = \omega_3^6 = \omega_1^2 = \omega_3^4 = \omega_4^3 = d\delta = 0$. Thus the derivation formulae

$$
\begin{aligned}
&dx = e_1\omega^1 + e_2\omega^2,\\
&de_1 = (\delta+1)e_3\omega^1 + e_4\omega^2,\\
&de_2 = e_4\omega^1 + (\delta-1)e_3\omega^2,\\
&de_3 = de_4 = 0
\end{aligned}
$$

give that the considered surface lies in an $E^4_{0,2}$ spanned by the point x and mutually orthogonal vectors $e_1,\ e_2,\ e_3,\ e_4$.

Since $h_{11} = (\delta+1)e_3,\ h_{22} = (\delta-1)e_3,\ h_{12} = e_4$, the considered surface can be represented by the equation $x = \frac{1}{2}h_{11}(u)^2 + \frac{1}{2}h_{22}(v)^2 + h_{12}uv + c_1u + c_2v$. Here $\omega^1 = du,\ \omega^2 = dv$; the u- and v-lines are geodesics of this surface and if $\delta^2 - 1 \neq 0$, then they are parabolas, but if $\delta^2 - 1 = 0$, then one of them degenerates into a straight line.

This completes the proof of Proposition 1.

6. EXISTENCE OF SEMIPARALLEL SURFACES

Now there arises the problem of the existence of a nontrivial 2nd-order envelope of parallel surfaces from (i)–(iii) of the Theorem.

It is known that parallel surfaces of (i) of the Theorem have only trivial 2nd-order envelopes (i.e. they are umbilic-like in the sense of [17]).

For case (iii) it is established in [19] that in E^6_s, (s is 0, 3, or 4) there exists the most general semiparallel surface M^2 with some arbitrariness and it is the 2nd-order envelope of a 2-parameter family of mutually noncongruent Veronese orbits. Moreover, these results can be used in E^6_s by $s = 1$, taking e_α so that $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 1,\ \ \varepsilon_6 = -1,\ \ \varepsilon_7 = ... = \varepsilon_n = 1$. Thus the difference from [19] will be in (2.1), where $\omega^6_K = -\varepsilon_6\omega^K_6,\ K = 1, ..., 5$ but it does not influence the final result.

In case (ii) Proposition 1 can be used. In the latter for subcase (ii_1), when the parallel surface M^2 is a translation surface, the existence of the nontrivial 2nd-order envelope of these surfaces is obvious. Thus it remains to consider subcases (ii_2) and (ii_3) of Proposition 1.

Proposition 2. *Let M^2 be a nonminimal parallel surface of subcase (ii_2) or (ii_3). There exists their nontrivial 2nd-order envelope with some arbitrariness.*

Proof. Without a loss of generality only the frame possibilities (4.6) with $b \neq 0$, (4.10), (4.11) can be considered.

In the first of them, taking into account that $\omega_3^4 = \omega_4^3 = 0$ and $\omega_3^3 = -\omega_4^4$, the Pfaff system (4.1), (4.2) after exterior differentiation gives

$$
\begin{aligned}
&[d(\delta+a)+(\delta+a)\omega_3^3-2b\omega_1^2]\wedge\omega^1+(db+b\omega_3^3+2a\omega_1^2)\wedge\omega^2=0,\\
&(db+b\omega_3^3+2a\omega_1^2)\wedge\omega^1+[d(\delta-a)+(\delta-a)\omega_3^3+2b\omega_1^2]\wedge\omega^2=0,\\
&[d\sigma-\sigma\omega_3^3]\wedge\omega^1=0,\\
&[d\sigma-\sigma\omega_3^3]\wedge\omega^2=0,\\
&[(\delta+a)\omega_3^\xi+\sigma\omega_4^\xi]\wedge\omega^1+b\omega_3^\xi\wedge\omega^2=0,\\
&b\omega_3^\xi\wedge\omega^1+[(\delta-a)\omega_3^\xi+\sigma\omega_4^\xi]\wedge\omega^2=0.
\end{aligned}
$$

1) Let $\sigma\neq 0, \delta=0$. It is easy to see that then $d\sigma=\sigma\omega_3^3$. Since $b\neq 0$, the basis of secondary forms consists of $da,\ db,\ 2\omega_1^2,\ \omega_3^3,\ \omega_3^\xi,\ \omega_4^\xi$ and the ranks of the polar systems $s_1=2+2(n-4)$ and $s_2=2$. Thus Cartan's number $Q=6+2(n-4)$. On the other hand, due to Cartan's lemma the number of the independent coefficients is $6+2(n-4)$. Thus Cartan's criterion is satisfied and this Pfaff system is compatible and determines the considered M^2 for subcase (ii_2) with arbitrariness of two real holomorphic functions of two variables.

2) Let $\delta\neq 0,\ \sigma=0$. If here $\delta^2-a^2\neq 0$, then due to Cartan's lemma for the first two equalities one has 8 independent coefficients. Consideration of the last two equalities gives $\omega_3^\xi=r_1^\xi\omega^1+r_2^\xi\omega^2$, whereas $r_1=r_1^\xi e_\xi$ and $r_2=r_2^\xi e_\xi$ are either both zero or linearly dependent vectors. The common number of independent coefficients is either 8 or $8+(n-4)$, respectively. In both cases $Q=N$ and $s_2=3$.

If $\delta^2-a^2=0$ (for example $\delta=a$), then $r_1^\xi=r_2^\xi=0$, $N=Q=6$, where $s_1=2,\ s_2=2$.

Thus the semiparallel surface of subcase (ii_3) in $E^3_{0,1}$ exists either with arbitrariness of three real holomorphic functions of two variables (it has two families of parabola generators), or with arbitrariness of two real holomorphic functions of two variables (this occurs when parabola degenerates into a straight line).

In the case (4.10) the Pfaff system (5.1), (5.2) gives by exterior differentiation

$$
\begin{aligned}
&(\omega_3^3+\omega_5^3)\wedge\omega^1+(2\omega_1^2+\omega_4^3)\wedge\omega^2=0, && (2\omega_1^2+\omega_4^3)\wedge\omega^1+(\omega_5^3-\omega_3^3)\wedge\omega^2=0,\\
&(2\omega_1^2-\omega_3^4-\omega_5^4)\wedge\omega^1-\omega_4^4\wedge\omega^2=0, && \omega_4^4\wedge\omega^2+(2\omega_1^2-\omega_3^4+\omega_5^4)\wedge\omega^2=0,\\
&(\omega_3^5+\omega_5^5)\wedge\omega^1+\omega_4^5\wedge\omega^2=0, && \omega_4^5\wedge\omega^1+(\omega_5^5-\omega_3^5)\wedge\omega^2=0,\\
&\omega_5^6\wedge\omega^1+\omega_4^6\wedge\omega^2=0, && \omega_4^6\wedge\omega^1+\omega_5^6\wedge\omega^2=0,\\
&(\omega_3^7+\omega_5^7)\wedge\omega^1=0, && (\omega_5^7-\omega_3^7)\wedge\omega^2=0,\\
&\omega_3^8\wedge\omega^1+\omega_4^8\wedge\omega^2=0, && \omega_4^8\wedge\omega^1-\omega_3^8\wedge\omega^2=0,\\
&(\omega_3^\xi+\omega_5^\xi)\wedge\omega^1+\omega_4^\xi\wedge\omega^2=0, && \omega_4^\xi\wedge\omega^1+(\omega_5^\xi-\omega_3^\xi)\wedge\omega^2=0,
\end{aligned}
$$

where $\xi=9,...,n$. Using here Cartan's lemma and also relations (4.12)–(4.14), one has $\omega_3^8=\omega_3^7=\omega_4^8=\omega_4^6=\omega_5^7=\omega_5^6=0$. The common number of coefficients on the right sides $N = 14 + 4(n-8) = 4n-18$. On the other hand, first, the basis of the secondary forms consists of $2\omega_1^2,\ \omega_3^3,\ \omega_4^3, \omega_5^3,\ \omega_3^4,\ \omega_4^4,$ $\omega_5^4,\ \omega_3^5,\ \omega_4^5, \omega_5^5,\ \omega_3^\xi,\ \omega_4^\xi, \omega_5^\xi$; second, the ranks of the polar systems are: $s_1 = 6 + 2(n-8)$ and $s_1 + s_2$, where $s_2 = n-4$, thus Cartan's number $Q = s_1 + 2s_2 = 4n - 18$. Hence Cartan's criterion is satisfied and the semiparallel surface of subcase (ii_3) in $E^5_{0,3}$ exists with arbitrariness of $n-4$ real holomorphic functions of two variables.

For the case (4.11) the first eight equations of the Pfaff system (5.3), (5.4) lead by exterior differentiation to

$$\begin{aligned}
&[d\delta + (\delta+1)\omega_3^3 + \sigma\omega_4^3] \wedge \omega^1 + [2\omega_1^2 + \omega_4^3] \wedge \omega^2 = 0, \quad && \omega_4^5 \wedge (\sigma\omega^1 + \omega^2) = 0,\\
&[2\omega_1^2 + \omega_4^4] \wedge \omega^1 + [d\delta + (\delta-1)\omega_3^3 + \sigma\omega_4^3] \wedge \omega^2 = 0, \quad && \omega_4^5 \wedge (\omega^1 + \sigma\omega^2) = 0,\\
&[d\sigma - 2\omega_1^2 + (\delta+1)\omega_3^4 + \sigma\omega_4^4] \wedge \omega^1 + \omega_4^4 \wedge \omega^2 = 0, \quad && (\delta+1)\omega_3^6 \wedge \omega^1 = 0,\\
&\omega_4^4 \wedge \omega^1 + [d\sigma + 2\omega_1^2 + (\delta-1)\omega_3^4 + \sigma\omega_4^4] \wedge \omega^2 = 0, \quad && (\delta-1)\omega_3^6 \wedge \omega^2 = 0.
\end{aligned}$$

This system together with (4.12)–(4.14) gives $\omega_4^5 = \omega_3^6 = 0$. After exterior differentiation the equations $\omega_i^\xi = 0,\ \xi = 7, ..., n$, give

$$[(\delta+1)\omega_3^\xi + \sigma\omega_4^\xi] \wedge \omega^1 + \omega_4^\xi \wedge \omega^2 = 0,$$

$$\omega_4^\xi \wedge \omega^1 + [(\delta-1)\omega_3^\xi + \sigma\omega_4^\xi] \wedge \omega^2 = 0.$$

Now the basis of secondary forms consists of $2\omega_1^2,\ \omega_3^3, d\delta,\ \omega_4^3, \omega_3^4,\ \omega_4^4, \omega_3^\xi,\ \omega_4^\xi$.

Let $\delta^2 - 1 \neq 0$; then $s_1 = 4+2(n-6)$ and $s_2 = 6+2(n-6)-4-2(n-6) = 2$, Cartan's number is $8 + 2(n-6)$ and it is equal to the number of independent coefficients.

In the case $\delta^2 - 1 = 0$ (for example $\delta = 1$), $s_1 = 4 + 2(n-6)$, $s_2 = 1$, and $Q = N = 6 + 2(n-6)$.

Cartan's criterion is satisfied and the semiparallel surface of subcase (ii_3) in $E^4_{0,2}$ exists either with arbitrariness of two real holomorphic functions of two variables, or with arbitrariness of one real holomorphic function of two variables (depending on the occurrence of degeneration).

This completes the proof of Proposition 2.

7. MINIMAL SEMIPARALLEL SPACE-LIKE SURFACES

It is known that in E^n every minimal semiparallel submanifold is totally geodesic (see [8] and [17]), but in E^n_1 there exist minimal semiparallel time-like surfaces (strings), which are not totally geodesic (see [16]). It can be shown that among surfaces of type (ii) in E^n_s with $s > 0$ there do exist not totally geodesic minimal semiparallel space-like surfaces.

Proposition 3. *In E^n_s with $s > 0$ a minimal semiparallel space-like surface M^2, which is not totally geodesic, has flat $\overline{\nabla}$ and is either*

1) *a surface in $E^3_{0,1}$ or in $E^4_{0,2}$, which has two families of parabola generators and can be represented by the equation $x = \frac{1}{2}h_{11}((u)^2-(v)^2)+h_{12}uv+c_1u+c_2v$, where all coefficients are some constant vectors, moreover, the first two of them are isotropic, or*

2) *a hyperbolic paraboloid in $E^3_{0,1}$, or*

3) *a 2nd-order envelope of a family, consisting of the surfaces of one of the previous classes in E^n_s.*

Proof. Due to Proposition 1 here the cases when dimI_xM^2 is either 1 (subcases (4.5)–(4.8)) or 2 (subcase (4.11)) are to be considered with the additional condition $H = 0$ (i.e. $\delta = \sigma = 0$).

Let dim$I_xM^2 = 1$; then Pfaff system (4.1), (4.2) transforms into

$$\omega^3_1 = \omega^1 + b\omega^2,\ \omega^3_2 = b\omega^1 + \omega^2,\ \omega^\xi_1 = \omega^\xi_2 = 0,\ \xi = 4, ..., n.$$

Exterior differentiation leads to

$$(\omega^3_3 - 2b\omega^2_1) \wedge \omega^1 + (db + b\omega^3_3 + 2\omega^2_1) \wedge \omega^2 = 0, \quad \omega^\xi_3 \wedge (\omega^1 + b\omega^2) = 0,$$
$$(db + b\omega^3_3 + 2\omega^2_1) \wedge \omega^1 - (\omega^3_3 - 2b\omega^2_1) \wedge \omega^2 = 0, \quad \omega^\xi_3 \wedge (b\omega^1 - \omega^2) = 0.$$

For all cases (4.5)–(4.8) on supposition $b \neq 0$, due to Cartan's lemma one has $\omega^\xi_3 = 0$ and the number of independent coefficients is 4; since the basis of secondary forms consists of $db,\ \omega^3_3,\ 2\omega^2_1$ and the ranks of the polar systems $s_1 = 2,\ s_2 = 1$, Cartan's number $Q = 4$. The considered minimal surface exists with arbitrariness of one real function of two variables.

If $b = 0$, then it is easy to see that $s_1 = 2,\ s_2 = 0$; here $Q = N = 2$ and the minimal surface for this case exists with arbitrariness of two real functions of one variable.

Investigation of geometry gives the derivation formulae $dx = e'_1du + e'_2dv$, $de'_1 = a'e_3du + k_1a'e_3dv,\ de'_2 = k_1a'e_3du - a'e_3dv,\ d(a'e_3) = 0$. Thus M^2 lies in $E^3_{0,1}$ and due to $h_{22} = -h_{11}$ either is determined by the equation $x = \frac{1}{2}h_{11}((u)^2 - (v)^2) + h_{12}uv + c_1u + c_2v$, where all coefficients are some constant vectors, and thus has two families of parabola generators, or is a hyperbolic paraboloid.

For the case dim$I_xM^2 = 2$, the minimal M^2 occurs in (4.11). The Pfaff system now transforms into

$$\omega^3_1 = \omega^4_2 = \omega^1,\ \omega^3_2 = -\omega^4_1 = -\omega^2,\ \omega^\xi_1 = \omega^\xi_2 = 0,\ \xi = 5, ..., n.$$

After exterior differentiation it gives

$$\omega^3_3 \wedge \omega^1 + [2\omega^2_1 + \omega^3_4] \wedge \omega^2 = 0, \qquad [2\omega^2_1 + \omega^3_4] \wedge \omega^1 - \omega^3_3 \wedge \omega^2 = 0,$$

$$[-2\omega_1^2 + \omega_3^4] \wedge \omega^1 + \omega_4^4 \wedge \omega^2 = 0, \quad \omega_4^4 \wedge \omega^1 + [2\omega_1^2 - \omega_3^4] \wedge \omega^2 = 0,$$
$$\omega_3^\xi \wedge \omega^1 = 0, \qquad \omega_3^\xi \wedge \omega^2 = 0.$$

Due to Cartan's lemma all ω_3^ξ are zero; the other equalities give $N = 4 + 2 = 6$; the basis on the left sides consists of $2\omega_1^2,\ \omega_3^3,\ \omega_4^3,\ \omega_3^4,\ \omega_4^4$ and the ranks of the polar systems $s_1 = 4,\ s_2 = 1$. So Cartan's number is equal to the number of independent coefficients and Cartan's criterion is satisfied. The extended Pfaff system determines M^2 with arbitrariness of one real holomorphic function of two variables. For this surface $dx = e_1\omega^1 + e_2\omega^2,\ de_1 = e_3\omega^1 + e_4\omega^2,\ de_2 = e_4\omega^1 - e_3\omega^2,\ de_3 = de_4 = 0$, thus the considered minimal M^2 lies in $E^4_{0,2}$ and can be represented by the equation $x = \frac{1}{2}h_{11}((u)^2 - (v)^2) + h_{12}uv + c_1u + c_2v$, where all coefficients are constant vectors. This M^2 has two families of parabola generators.

This completes the proof of Proposition 3.

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PARALLEELSED JA SEMIPARALLEELSED RUUMISARNASED PINNAD PSEUDOEUKLEIDILISTES RUUMIDES

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Alammuutkonda M^2 (pinda) ruumis E^n_s nimetatakse *semiparalleelseks*, kui $\overline{R}(X,Y)h = 0$, kus X ja Y on suvalised puutujavektorid, $\overline{R}$ on van der Waerdeni–Bortolotti seostuse $\overline{\nabla}$ ($\overline{\nabla} = \nabla \oplus \nabla^{\perp}$) kõverusoperaator ja h on teine fundamentaalvorm. Semiparalleelsete pindade M^2 klassis on olemas alamklass, mille kõik pinnad on paralleelse vormiga h. Neid pindu iseloomustab tingimus $\overline{\nabla} h = 0$ ja neid nimetatakse *paralleelseteks* pindadeks. Siinses töös on antud semiparalleelsete ruumisarnaste pindade klassifikatsioon ja näidatud nende olemasolu ning suva ruumis E^n_s. On tõestatud, et ruumisarnane pind M^2 on semiparalleelne siis ja ainult siis, kui see on totaalselt ombiline (erijuhul totaalselt geodeetiline) või selle seostus $\overline{\nabla}$ on kõveruseta või see on isotroopne pind, mida iseloomustab tingimus $\| H \|^2 = 3K$, kus H on keskmise kõveruse vektor ja K on Gaussi kõverus. Lisaks sellele on antud vaadeldavate pindade M^2 detailsem klassifikatsioon juhul, kui seostus $\overline{\nabla}$ on kõveruseta, kasutades asjaolu, et igaüks neist on paralleelsete pindade teist järku mähkija, ning tõestatud, et eksisteerivad seda laadi minimaalsed pinnad, mis pole totaalselt geodeetilised.