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SUMMATION METHODS OF TRIGONOMETRIC FOURIER SERIES DEFINED BY THE ZAK TRANSFORM

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Abstract. We investigate the order of approximation by some new-type summation methods of trigonometric Fourier series. These methods are defined by the Zak transform.

Key words: trigonometric Fourier series, Zak transform, sinc function, modulus of continuity, Lanczos' filter.

1. INTRODUCTION

The aim of this paper is to investigate the order of approximation by some new-type summation methods of trigonometric Fourier series. These methods are introduced in [1,2] for constructing some approximation processes for functions in the disc algebra in uniform norm. The definition of the summation methods is based on the Zak transform.

We introduce some notations. Let $\mathbf{N}, \mathbf{Z}, \mathbf{R}, \mathbf{C}$ denote the sets of all naturals, all integers, all real, and all complex numbers, respectively. Let us consider the triangular φ-means (or summation methods)

$$U_n(f,x) := \frac{a_0}{2} + \sum_{k=1}^{n} \varphi\left(\frac{k}{n}\right)(a_k \cos kx + b_k \sin kx) \tag{1}$$

of the real Fourier series of a 2π-periodic continuous function $f \in C_{2\pi}$ with Fourier coefficients a_k, b_k. Several φ-means can be defined by the continuous function $\varphi \in C_{[0,1]}$, which satisfies the boundary conditions $\varphi(0) = 1, \varphi(1) = 0$. Our new summation methods are constructed as follows. The Zak transform $Z\psi : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ (cf. [3], pp. 164, 166; [4], pp. 109–110) of a function $\psi : \mathbf{R} \to \mathbf{C}$ is formally defined by

$$Z\psi(u,v) := \sum_{l \in \mathbf{Z}} \psi(u+l) e^{ilv} \qquad (u, v \in \mathbf{R}).$$

Since $Z\psi$ is a quasi-periodic function, i.e.

$$Z\psi(u+1,v) = e^{-iv} Z\psi(u,v), \quad Z\psi(u, v+2\pi) = Z\psi(u,v) \qquad (u, v \in \mathbf{R}),$$

it is reasonable to consider the variables $(u, v) \in [0, 1] \times \mathbf{R}$ only. We are interested in the Zak transform of a real-valued and even function θ. Then it will be convenient to define

$$Z\theta(u,v) := \sum_{l \in \mathbf{Z}} \theta(u+l) \cos lv \qquad (u \in [0,1],\ v \in \mathbf{R}). \tag{2}$$

Obviously, the (real even) Zak transform (2) of an even function is even in both variables u and v. The simplest condition for the Zak transform $Z\theta$ to be well-defined is

$$\sum_{l \in \mathbf{Z}} |\theta(u+l)| < \infty. \tag{3}$$

Actually, we shall use only one part of the Zak transform defined by

$$Z^{+}\theta(u,v) := \sum_{l=0}^{\infty} \theta(u+l) \cos lv \qquad (u \in [0,1],\ v \in \mathbf{R}). \tag{4}$$

Now our new-type summation method U_n^z ([1], p. 312) is defined by (1), where instead of the matrix $(\varphi(k/n))_{k=0,1,\ldots,n;\ n \in \mathbf{N}}$ we use the functional matrix with entries

$$\Phi\left(\frac{k}{n}, nx\right) := Z^{+}\theta\left(\frac{k}{n}, nx\right) \qquad (k = 0, 1, ..., n;\ n \in \mathbf{N},\ x \in \mathbf{R}), \tag{5}$$

explicitly,

$$U_n^z(f,x) := \frac{a_0}{2} + \sum_{k=1}^{n} \Phi\left(\frac{k}{n}, nx\right)(a_k \cos kx + b_k \sin kx). \tag{6}$$

First we remark that U_n^z is non-polynomial, i.e. in contrast with (1), $U_n^z f$ is not a trigonometric polynomial. Denote

$$A_0(f,x) := a_0/2, \quad A_k(f,x) := a_k \cos kx + b_k \sin kx \qquad (k \in \mathbf{N}). \tag{7}$$

Then, under the condition (3) we may write

$$U_n^z(f,x) = \sum_{l=0}^{\infty} \left(\sum_{k=0}^{n} \theta \left(l + \frac{k}{n} \right) A_k(f,x) \right) \cos lnx. \tag{8}$$

Now it is clear that U_n^z forms linear transformation on $C_{2\pi}$ into $C_{2\pi}$. We also see that the first summand of U_n^z in (8) is the ordinary summation method (1) defined by θ.

We must choose the function θ so that, like for φ in (1), $\Phi(0,v) = 1$ and $\Phi(1,v) = 0$ for all $v \in \mathbf{R}$. Moreover, under some additional conditions on θ we shall find the order of approximation by the method (6). It turns out that the approximation properties of U_n^z depend on the Fourier-cosine transform of θ on a discrete set of points. To understand the situation better, we recall an earlier result [5] which we call the subordination principle via the Rogosinski-type summation methods.

2. SUBORDINATION PRINCIPLE VIA ROGOSINSKI-TYPE SUMMATION METHODS

The Rogosinski-type summation methods (or means), denoted by $R_{n,j}$, were introduced by Rogosinski [6] in the form ($j \in \mathbf{N}$)

$$R_{n,j}(f,x) := \frac{a_0}{2} + \sum_{k=1}^{n} \cos \left(\left(j - \frac{1}{2} \right) \frac{k\pi}{n} \right) (a_k \cos kx + b_k \sin kx). \tag{9}$$

It is remarkable that here the generating functions $\varphi_j(t) := \cos(j - \frac{1}{2})\pi t$ ($j \in \mathbf{N}$) constitute an orthogonal system $\{\varphi_j\}$ on $[0,1]$ and the boundary conditions $\varphi_j(0) = 1$, $\varphi_j(1) = 0$ are valid for all $j \in \mathbf{N}$. This circumstance inspired us to represent the function (5) by a Fourier series using the system $\{\varphi_j\}$. To simplify the notations, let $m_j := (j - \frac{1}{2})\pi$. Let us evaluate the Fourier coefficients of Φ by the system $\{\varphi_j\}$. We have

$$d_j(v) := 2 \int_0^1 \Phi(u,v) \cos(m_j u) du \tag{10}$$

and formally

$$\Phi(u,v) = \sum_{j=1}^{\infty} d_j(v) \varphi_j(u). \tag{11}$$

Now our subordination principle (see [5], Theorem 1') reads as follows.

Theorem A. *Let the sequence of coefficients* (10) *satisfy*

$$\sum_{j=1}^{\infty} |d_j(v)| < \infty,$$

where the series is converging uniformly for $v \in \mathbf{R}$. *If the function* Φ *in* (5) *generates the summation method* U_n^z *by* (6), *then we have*

$$U_n^z(f,x) = \sum_{j=1}^{\infty} d_j(nx) R_{n,j}(f,x),$$

$$f(x) - U_n^z(f,x) = \sum_{j=1}^{\infty} d_j(nx)(f(x) - R_{n,j}(f,x)).$$

By Theorem A we see that the generalized Rogosinski means are very useful for the investigation of methods U_n^z. Therefore we continue with a theorem for the generalized Rogosinski means $R_{n,j}$ [5]. Let $\omega_k(f,\delta)$ be the kth modulus of continuity (see [7]) of the function $f \in C_{2\pi}$.

Theorem B. *For all* $j \in \mathbf{N}$

$$\sup_{n\in\mathbf{N}} \|R_{n,j}\| = \frac{4}{\pi^2} \log j + O(1)\,. \tag{12}$$

Moreover, there exist two positive absolute constants M_q ($q = 1$ *or* $q = 2$) *such that the generalized Rogosinski means have the order of approximation*

$$\|f - R_{n,j} f\|_{C_{2\pi}} \le M_q j^q \omega_q \left(f, \frac{1}{n}\right) \tag{13}$$

for $q = 1$ *or* $q = 2$, *respectively.*

As a straightforward consequence of Theorems A and B we formulate two theorems (cf. [5]) that we shall apply to the new summation method U_n^z introduced in [1].

Theorem 1. *Let for the sequence of coefficients in* (10) *there exist an absolute* (*independent of* $v \in \mathbf{R}$) *constant* m_0 *such that*

$$\sum_{j=1}^{\infty} |d_j(v)| \log j \le m_0 < \infty. \tag{14}$$

If the function Φ *in* (5) *generates the summation method* U_n^z *by* (6), *then* U_n^z *forms a uniformly bounded linear transformation on* $C_{2\pi}$ *into* $C_{2\pi}$.

Theorem 2. *Let for the sequence of coefficients in* (10) *there exist an absolute* (*independent of* $v \in \mathbf{R}$) *constant* m_q ($q = 1$ *or* $q = 2$) *such that*

$$\sum_{j=1}^{\infty} j^q |d_j(v)| \le m_q < \infty. \tag{15}$$

Then, for any $f \in C_{2\pi}$, *we have the order of approximation*

$$\|f - U_n^z f\|_{C_{2\pi}} \le M_q m_q \omega_q \Big(f, \frac{1}{n}\Big),$$

where $q = 1$ *or* $q = 2$, *respectively, and the constants* $M_q,\ m_q$ *are from Theorem B and from* (15), *respectively.*

3. APPLICATIONS TO SOME EXAMPLES OF APPROXIMATION METHODS

We are looking for some examples of the function θ in (5) so that the condition (15) would be valid.

Example 1. One of the simplest choices for θ is the sinc function defined by the equality

$$\operatorname{sinc} x := \begin{cases} \frac{\sin \pi x}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Since $\operatorname{sinc} l = 0$ for $l \in \mathbf{Z} \setminus \{0\}$, it is clear that for

$$\Phi_1(u,v) := Z^+ \operatorname{sinc}(u,v)$$

in (5) we have

$$\Phi_1(0,v) = 1, \qquad \Phi_1(1,v) = 0 \qquad (v \in \mathbf{R}). \tag{16}$$

Unfortunately, the sinc function decreases too slowly to quarantee that the condition (15) is satisfied. But here is another interesting feature.

First, we can find the Zak transform (2) explicitly, i.e.

$$Z \operatorname{sinc}(u,v) = \sum_{l \in \mathbf{Z}} \operatorname{sinc}(u+l) \cos lv = \sum_{l \in \mathbf{Z}} \operatorname{sinc}(u-l) \cos lv = \cos(uv)$$

for $u \in \mathbf{R},\ v \in (-\pi, \pi)$ (see [8], pp. 62–63). Here the last equality says that the Zak transform of the sinc function is the Fourier series of the function $\cos(uv)$. This case is interesting also because here the Zak transform does exist, but the condition (3) is not valid.

Second, in the case of the sinc function the first summand in (8) is the famous Lanczos' filter (see, e.g., [9], Ch. 22, Sec. 22.6). We shall consider the Lanczos' filter in our last section.

The function $\operatorname{sinc}^2 x$ has better decay than $\operatorname{sinc} x$ and preserves all nice properties of $\operatorname{sinc} x$. Therefore, in the following we shall use $\operatorname{sinc}^2 x$.

Example 2. Let the summation method $U_n^{z,2}$ be defined by (6), where instead of Φ we have

$$\Phi_2(u,v) := Z^+ \operatorname{sinc}^2(u,v).$$

Obviously, the conditions (16) are valid also for Φ_2. We need some technical lemmas to check (15). Since by (4)

$$\Phi_2(u,v) = \sum_{l=0}^{\infty} \operatorname{sinc}^2(u+l) \cos lv \qquad (u \in [0,1],\ v \in \mathbf{R}),$$

we first find estimations for shifted Fourier-cosine transform of $\operatorname{sinc}^2 x$, i.e. for

$$\delta_{l,j} := \int_0^1 \operatorname{sinc}^2(l+t)\cos(m_j t)dt \qquad (l=0,1,...; \quad j\in \mathbf{N}). \tag{17}$$

As before, $m_j := (j-\frac{1}{2})\pi$. Integration by parts in (17) gives

Lemma 1. *Let g be r-times differentiable on $[0,1]$. Then*

$$\begin{aligned}\int_0^1 g(t)\cos(at)dt &= (-1)^r \frac{1}{a^r}\int_0^1 g^{(r)}(t)\cos\left(at-\frac{\pi r}{2}\right)dt \\ &\quad + \sum_{k=0}^{r-1}(-1)^k \frac{1}{a^{k+1}} g^{(k)}(t)\cos\left(at-\frac{\pi(k+1)}{2}\right)\Bigg|_0^1 .\end{aligned}$$

To simplify notations, let

$$s_k(t) := \operatorname{sinc}^k t \qquad (k=1,2,3,4). \tag{18}$$

To use Lemma 1, we must evaluate several derivatives.

Lemma 2. *We have*

$$s_1'(0)=s_1'''(0)=0, \quad s_1''(0)=-\pi^2/3, \quad s_1^{(4)}(0)=\pi^4/5, \tag{19}$$

and for $l\in\mathbf{N}$

$$\begin{aligned} s_1(l)=0, \quad s_1'(l)=\frac{(-1)^l}{l}, \quad s_1''(l)=\frac{(-1)^{l+1}\,2}{l^2}, \\ s_1'''(l)=\frac{(-1)^{l+1}}{l}\left(\pi^2-\frac{6}{l^2}\right), \quad s_1^{(4)}(l)=\frac{(-1)^l\,4}{l^2}\left(\pi^2-\frac{6}{l^2}\right). \end{aligned} \tag{20}$$

Proof. We get the derivatives in (19) from the power series of the sinc function. For (20) we write

$$\pi t s_1(t) = \sin \pi t, \tag{21}$$

which by Leibniz's rule for the nth derivative yields

$$t s_1^{(n)}(t) + n s_1^{(n-1)}(t) = \pi^{n-1}\sin(\pi t + n\pi/2). \tag{22}$$

Putting $t=l\in\mathbf{N}$, we get step by step the equalities (20).

Lemma 3. *For all $l=0,1,...;\quad j\in\mathbf{N}$ the shifted cosine transform of $\operatorname{sinc}^2 t$ in* (17) *has the estimation*

$$\delta_{l,j} = O((l+1)^{-2}j^{-3}).$$

Proof. If in Lemma 1 we take $g(t) = s_2(l+t), r = 3, a = m_j := (j-1/2)\pi$, then we obtain

$$\delta_{l,j} = -\frac{s_2'(l)}{{m_j}^2} + (-1)^j \frac{s_2''(l+1)}{{m_j}^3} + \frac{1}{{m_j}^3} \int_0^1 s_2'''(l+t) \sin m_j t dt. \tag{23}$$

Since by (18) $s_2 = s_1^2$, we get

$$s_2' = 2s_1 s_1', \quad s_2'' = 2(s_1')^2 + 2s_1 s_1'', \quad s_2''' = 6s_1' s_1'' + 2s_1 s_1'''. \tag{24}$$

Now (20) yields for $l \in \mathbf{N}$

$$s_2(l) = s_2'(l) = 0, \quad s_2''(l) = \frac{2}{l^2}, \tag{25}$$

and for (23) we may write

$$\delta_{l,j} = \frac{(-1)^j \, 2}{(l+1)^2 {m_j}^3} + \frac{1}{{m_j}^3} \int_0^1 s_2'''(l+t) \sin m_j t dt. \tag{26}$$

Since by definition (18) $s_1(t) = O(t^{-1}) \quad (t > 0)$, it follows from (22) that $s_1'(t) = O(t^{-1}),\ s_1''(t) = O(t^{-1}),\ s_1'''(t) = O(t^{-1}) \quad (t \geq 1)$. Therefore, by (24) $s_2'''(t) = O(t^{-2}) \quad (t \geq 1)$ and for (26) we have for all $l \in \mathbf{N}$

$$|\delta_{l,j}| \leq \frac{2}{(l+1)^2 {m_j}^3} + \frac{M}{{m_j}^3} \int_0^1 \frac{dt}{(l+t)^2} = O((l+1)^{-2} j^{-3}),$$

as $m_j = (j-1/2)\pi$. Since by (24) and by Lemma 2 s_2''' is bounded on $[0, \infty)$, the equality (26) implies $\delta_{0,j} = O(j^{-3})$. The proof is complete.

Now it is easy to check [see (10), (5), (4), and (17)] that for

$$\begin{aligned} d_{j,2}(v) : &= 2 \int_0^1 Z^+ \mathrm{sinc}^2(u+v) \cos(m_j u) du \\ &= 2 \sum_{l=0}^{\infty} \cos lv \int_0^1 \mathrm{sinc}^2(u+l) \cos(m_j u) du \end{aligned}$$

Lemma 3 yields

$$d_{j,2}(v) = O(j^{-3}) \qquad (j \in \mathbf{N},\ v \in \mathbf{R}). \tag{27}$$

Therefore, the condition (15) is valid for $q = 1$ and by Theorem 2 we obtain as follows.

Theorem 3. *The summation method*

$$U_n^{z,2}(f,x) = \sum_{l=0}^{\infty} \left(\sum_{k=0}^{n} \operatorname{sinc}^2 \left(l + \frac{k}{n} \right) A_k(f,x) \right) \cos lnx, \qquad (28)$$

generated by the Zak transform of the sinc^2 *function, forms a uniformly bounded linear transformation on* $C_{2\pi}$ *into* $C_{2\pi}$*. Moreover, there exists an absolute constant* $M > 0$ *such that for all* $f \in C_{2\pi}$

$$\|f - U_n^{z,2} f\|_{C_{2\pi}} \le M \omega_1 \left(f, \frac{1}{n} \right).$$

Proceeding from Example 2, in the following we could attempt with $\operatorname{sinc}^3 t$. In this case we will have the same order of approximation as in Theorem 3 (see [10]). This observation fits well with the Fourier-cosine transform of the function φ in the definition of ordinary summation methods (1). Let us define the Fourier-cosine transform by

$$\varphi^{\wedge}(s) := \int_0^1 \varphi(u) \cos(\pi s u) du.$$

Then, for $\varphi_1(t) := 1$ (Fourier partial sums) we have $\varphi_1^{\wedge}(s) = \operatorname{sinc} s$, for $\varphi_2(t) := 1 - t$ (Fejér means) we have $\varphi_2^{\wedge}(s) = \frac{1}{2} \operatorname{sinc}^2(s/2)$, for $\varphi_3(t) := 1 - 6t^2 + 6t^3$ if $0 \le t \le \frac{1}{2}$ and $\varphi_3(t) := 2(1-t)^3$ if $\frac{1}{2} \le t \le 1$ (Jackson–de La Vallée Poussin means) we have $\varphi_3^{\wedge}(s) = \frac{3}{8} \operatorname{sinc}^4(s/4)$. It is known that in space $C_{2\pi}$ the Fourier partial sums may diverge, but the Fejér means and the Jackson–de La Vallée Poussin means have the order of approximation $\omega_2(f, 1/\sqrt{n})$ and $\omega_2(f, 1/n)$, respectively (see [11], pp. 77, 205). Let us remark that we can find this order of approximation by the Jackson–de La Vallée Poussin means also by using Theorem 2. Indeed, here $d_j = 2\varphi_3^{\wedge}(m_j) = O(j^{-4})$ and the condition (15) is valid with $q = 2$. After this discussion we continue with $\operatorname{sinc}^4 t$.

Example 3. Let the summation method $U_n^{z,4}$ be defined by (6), where instead of Φ we have

$$\Phi_4(u,v) := Z^{+}\operatorname{sinc}^4(u,v).$$

Obviously, the conditions (16) are valid also for Φ_4. This example is quite similar to Example 2. Therefore we only sketch the proofs. We deduce now from Lemma 1 that

$$\begin{aligned} \delta_{l,j} &= \frac{(-1)^{j+1}}{m_j} s_4(l+1) - \frac{s_4'(l)}{{m_j}^2} + (-1)^j \frac{s_4''(l+1)}{{m_j}^3} + \frac{s_4'''(l)}{{m_j}^4} \\ &\quad + \frac{1}{{m_j}^4} \int_0^1 s_4^{(4)}(l+t) \cos(m_j t) dt. \end{aligned}$$

If in (24) we replace s_1 by s_2 and s_2 by s_4, then we have by (25) for all $l \in \mathbf{N}$

$$s_4(l) = s_4'(l) = s_4''(l) = s_4'''(l) = 0.$$

This already yields

$$\delta_{l,j} = \frac{1}{{m_j}^4} \int_0^1 s_4^{(4)}(l+t)\cos(m_j t)dt.$$

Using

$$s_4^{(4)} = 6(s_2'')^2 + 8s_2's_2''' + 2s_2s_2^{(4)}$$

gives step by step the estimation $s_4^{(4)}(x) = O(x^{-4})$ for all $x \geq 1$. Hence,

$$\delta_{l,j} = O((l+1)^{-4}j^{-4})$$

and for

$$\begin{aligned} d_{j,4}(v) &:= 2\int_0^1 Z^+\mathrm{sinc}^4(u+v)\cos(m_j u)du \\ &= 2\sum_{l=0}^{\infty}\cos lv \int_0^1 \mathrm{sinc}^4(u+l)\cos(m_j u)du \end{aligned}$$

it follows that

$$d_{j,4}(v) = O(j^{-4}) \qquad (j \in \mathbf{N},\ v \in \mathbf{R}).$$

The condition (15) is fulfilled for $q = 2$ and by Theorem 2 we obtain as follows.

Theorem 4. *The summation method*

$$U_n^{z,4}(f,x) := \sum_{l=0}^{\infty}\left(\sum_{k=0}^{n}\mathrm{sinc}^4\left(l+\frac{k}{n}\right)A_k(f,x)\right)\cos lnx, \tag{29}$$

generated by the Zak transform of the sinc^4 *function, forms a uniformly bounded linear transformation on* $C_{2\pi}$ *into* $C_{2\pi}$*. Moreover, there exists an absolute constant* $M > 0$ *such that for all* $f \in C_{2\pi}$

$$\|f - U_n^{z,4}f\|_{C_{2\pi}} \leq M\omega_2\left(f, \frac{1}{n}\right).$$

4. LANCZOS' FILTER

Lanczos' filter is important for reducing the effect of the Gibbs phenomenon (see [9,12]). In [13], pp. 332–333, we find the order of approximation by Lanczos' method, defined by

$$L_n(f,x) := \frac{a_0}{2} + \sum_{k=1}^{n} \operatorname{sinc}\left(\frac{k}{n}\right)(a_k \cos kx + b_k \sin kx),$$

as follows:

$$\|f - L_n f\|_{C_{2\pi}} \le 3\rho\left(f, \frac{\pi}{n}\right).$$

Here

$$\rho(f,h) := \frac{1}{2h}\left\| \int_{-h}^{h} |f(\cdot + t) - f(\cdot)|dt \right\|_{C_{2\pi}} \qquad (h > 0). \tag{30}$$

So far we have not found any references to the order of approximation by L_n via the modulus of continuity. We present here a simple order of approximation by L_n. For the consideration we need our earlier result (here only a special case is given) as follows ([14]; cf. also [13], p. 312).

Theorem C. *Let the summation method U_n be defined by* (1), *where the function φ has for some $r \in \mathbf{N}$ the representation*

$$\varphi(u) = 1 - \sum_{j=r}^{\infty} c_j u^{2j}, \qquad \sum_{j=r}^{\infty} |c_j| < \infty. \tag{31}$$

Then there exists a constant $M(r) > 0$ such that

$$\|f - U_n f\|_{C_{2\pi}} \le \; M(r)\omega_{2r}\left(f, \frac{1}{n}\right)$$

for all $f \in C_{2\pi}$.

The condition (31) is valid for the sinc function due to the expansion

$$\operatorname{sinc}(u) = 1 - \sum_{j=1}^{\infty} (-1)^{j+1} \frac{\pi^{2j}}{(2j+1)!} u^{2j}.$$

Now by Theorem C we have the order of approximation of Lanczos' method.

Theorem 5. *For some $M > 0$ and for all $f \in C_{2\pi}$ we have the estimation*

$$\|f - L_n f\|_{C_{2\pi}} \le \; M\omega_2\left(f, \frac{1}{n}\right).$$

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ZAKI TEISENDUSEGA DEFINEERITUD TRIGONOMEETRILISTE FOURIER' RIDADE SUMMEERIMISMEETODID

On leitud Zaki teisendusega defineeritud trigonomeetriliste Fourier' ridade summeerimismeetodite koonduvuskiirused. Nimetatud meetodid on võetud kasutusele artiklis [1], kus neid rakendati teatud funktsioonialgebra baasi konstrueerimiseks.