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FAST SOLVERS OF GENERALIZED AIRFOIL EQUATION OF INDEX –1

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Abstract. We consider the generalized airfoil equation in the situation where the index of the problem is -1. We periodize the problem, then discretize it by a fully discrete version of the trigonometric collocation method and apply the conjugate gradient method to solve the discretized problem. The approximate solution appears to be of optimal accuracy in a scale of Sobolev norms, and the N parameters of the approximate solution can be determined by $\mathcal{O}(N \log N)$ arithmetical operations.

Key words: airfoil equations, fast solvers.

1. THE GENERALIZED AIRFOIL EQUATION AND ITS PERIODIZATION

Consider the generalized airfoil equation

$$(Bv)(x) := \int_{-1}^{1} \Big(\frac{1}{\pi} \frac{1}{x-y} + b_1(x,y) \log|x-y| + b_2(x,y] \Big) v(y)dy = g(x), \quad -1 < x < 1. \tag{1}$$

We assume that the kernel functions b_1 and b_2 are smooth. It is well known (see, e.g., [1−4]) that B represents a linear continuous Fredholm operator in different weighted spaces $L^2_\sigma(-1,1)$; the index of B depends on the weight. Particularly, $\operatorname{ind}(B) = 0$ if $\sigma(x) = \sqrt{(1+x)/(1-x)}$ or $\sigma(x) = \sqrt{(1-x)/(1+x)}$, and

$\operatorname{ind}(B) = 1$ if $\sigma(x) = \sqrt{1-x^2}$. Collocation solvers of Eq. (1) in these cases have been examined in [2] and [4], respectively. In the present paper, we put

$$\sigma(x) = \frac{1}{\sqrt{1-x^2}}\,, \qquad (u,v)_{L^2_\sigma} = \int_{-1}^{1} \sigma(x)u(x)\overline{v(x)}dx\,;$$

then the index of $B \in \mathcal{L}(L^2_\sigma(-1,1))$ is -1. We assume that the homogeneous equation $Bv = 0$ has in $L^2_\sigma(-1,1)$ only the trivial solution $v = 0$; then the range $\mathcal{R}(B) = BL^2_\sigma(-1,1)$ is of codimension 1. Let us fix a smooth function $\psi \in L^2_\sigma(-1,1)$ outside $\mathcal{R}(B)$. For any $g \in L^2_\sigma(-1,1)$ there exists a unique pair $(\omega, v) \in \mathbb{C} \times L^2_\sigma(-1,1)$ satisfying $\omega\psi + Bv = g$, and this pair can be treated as a generalized solution of (1). If $g \in \mathcal{R}(B)$, then $\omega = 0$, and the generalized solution $(0, v)$ can be identified with the usual solution $v \in L^2_\sigma(-1,1)$ of (1). In the sequel we design a numerical method yielding approximations (ω_N, v_N) such that $|\omega_N - \omega| \to 0$, $\|v_N - v\|_{L^2_\sigma} \to 0$ with a certain velocity. Thus, the convergence $\omega_N \to 0$ as $N \to \infty$ indicates that $\omega = 0$, $g \in \mathcal{R}(B)$, and (1) is solvable in $L^2_\sigma(-1,1)$ in the usual sense. An interpretation of the generalized solution (ω, v) with $\omega \neq 0$ can be given considering the flow ejection through a point of the airfoil (see [5]). In any case, the generalized solution (ω, v) is of interest also if $\omega \neq 0$, i.e. $g \notin \mathcal{R}(B)$. So we do not assume that $g \in \mathcal{R}(B)$.

With the cosine transformation

$$x = x(t) = -\cos(2\pi t)\left(0 \le t \le \frac{1}{2}\right), \quad y = x(s) = -\cos(2\pi s)\left(0 \le s \le \frac{1}{2}\right),$$

Eq. (1) can be reduced (see [3] for details) to the 1-periodic integral equation

$$\mathcal{A}u := A_0u + A_1u + A_2u = f, \tag{2}$$

where

$$(A_0u)(t) = \int_{-1/2}^{1/2} \cot \pi(t-s)u(s)ds \;\; \text{(the Hilbert transformation)},$$

$$(A_1u)(t) = \int_{-1/2}^{1/2} a_1(t,s)\log|\sin \pi(t-s)|u(s)ds\,,$$

$$(A_2u)(t) = \int_{-1/2}^{1/2} a_2(t,s)u(s)ds,$$

$$f(t) = g(x(t))\,, \qquad t \in \mathbb{R}\,,$$

$$a_1(t,s) = b_1(x(t),x(s))x'(s),$$

$$a_2(t,s) = \frac{1}{2}\big[b_2(x(t),x(s)) + (\log 2)b_1(x(t),x(s))\big]x'(s)\,, \quad t,s \in \mathbb{R}.$$

Clearly, f is 1-periodic and even, whereas a_1 and a_2 are 1-biperiodic, even in t and odd in s. The relation between solutions of (1) and (2) is somewhat more sophisticated: for $s \in \big(-\frac{1}{2}, \frac{1}{2}\,\big]$

$$u(s) = \begin{cases} v(x(s))\,, & 0 \le s \le \dfrac{1}{2}\,, \\ -v(x(-s))\,, & -\dfrac{1}{2} < s < 0\,, \end{cases}$$

and after that u is extended from $\big(-\frac{1}{2}, \frac{1}{2}\,\big]$ to $\mathbb{R}$ 1-periodically. Thus u is a 1-periodic odd function. To the generalized solution (ω, v) of (1) there corresponds the generalized solution (ω, u) of (2) satisfying $\omega\varphi + \mathcal{A}u = f$, where $\varphi(t) = \psi(x(t))$, $t \in \mathbb{R}$.

2. SOLVABILITY OF THE PROBLEM

Notice that $a_1, a_2 \in C^m(\mathbb{R})$, $f, \varphi \in C^m(\mathbb{R})$ if $b_1, b_2 \in C^m([-1,1]\times[-1,1])$, $g, \psi \in C^m[-1,1]$. Introduce the Sobolev space H^λ, $\lambda \ge 0$, of 1-periodic functions u having a finite norm

$$\|u\|_\lambda = \left(\sum_{k\in\mathbb{Z}} \underline{k}^{2\lambda}|\hat{u}(k)|^2\right)^{1/2}, \ \underline{k} = \max\{1,|k|\}\,,\ \hat{u}(k) = \int_{-1/2}^{1/2} u(s)e^{-ik2\pi s}ds\,.$$

We have $H^\lambda = H^\lambda_{\text{ev}} \oplus H^\lambda_{\text{od}}$, where H^λ_{ev} and H^λ_{od} are closed subspaces of H^λ consisting of even and odd functions, respectively. An orthogonal basis of H^λ_{ev} is given by $\{\cos(k2\pi t)\}_{k\ge 0}$, and an orthogonal basis of H^λ_{od} is given by $\{\sin(k2\pi t)\}_{k\ge 1}$. We also introduce the Sobolev space H^{λ_1,λ_2}, $\lambda_1 \ge 0$, $\lambda_2 \ge 0$, of 1-biperiodic functions a having a finite norm

$$\|a\|_{\lambda_1,\lambda_2} = \left(\sum_{(k_1,k_2)\in\mathbb{Z}^2} \underline{k}_1^{2\lambda_1}\underline{k}_1^{2\lambda_2}|\hat{a}(k_1,k_2]|^2\right)^{1/2},$$

$$\hat{a}(k_1,k_2) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} a(t,s)e^{-ik_1 2\pi t}e^{-ik_2 2\pi s}ds\,dt\,,$$

and the subspace $H^{\lambda_1,\lambda_2}_{\rm ev,od}$ of functions which are even in the first argument and odd in the second argument.

It is well known that

$$A_0 \sin(k2\pi t) = -\cos(k2\pi t)\,, \qquad k \geq 1\,,$$

$$A_0 1 = 0\,, \quad A_0 \cos(k2\pi t) = \sin(k2\pi t)\,, \qquad k \geq 1\,.$$

Thus $A_0 \in \mathcal{L}(H^\lambda_{\rm od}, H^\lambda_{\rm ev})$ is a Fredholm operator of index –1 for every $\lambda \geq 0$.

Lemma 2.1. *If $a_1 \in H^{\mu,\nu}_{\rm ev,od} \cap H^{\nu,\mu}_{\rm ev,od}$, $\frac{1}{2} < \nu \leq \mu$, then $A_1 \in \mathcal{L}(H^\lambda_{\rm od}, H^\lambda_{\rm ev})$ is compact for every $\lambda \in [0,\mu]$.*

Lemma 2.2. *If $a_2 \in H^{\mu,0}_{\rm ev,od}$, $\mu \geq 0$, then $A_2 \in \mathcal{L}(H^\lambda_{\rm od}, H^\lambda_{\rm ev})$ is compact for every $\lambda \in [0,\mu]$.*

Lemma 2.3. *Assume that $a_1 \in H^{\mu,\nu}_{\rm ev,od} \cap H^{\nu,\mu}_{\rm ev,od}$, $a_2 \in H^{\mu,0}_{\rm ev,od}$, $\frac{1}{2} < \nu \leq \mu$. Then $\mathcal{A} = A_0 + A_1 + A_2 \in \mathcal{L}(H^\lambda_{\rm od}, H^\lambda_{\rm ev})$ is a Fredholm operator of index -1 for every $\lambda \in [0,\mu]$.*

The proofs of Lemmas 2.1–2.3 can be constructed following the ideas of [4]. As a consequence of Lemma 2.3 we obtain the following result.

Theorem 2.1. *Assume the conditions of Lemma* 2.3*. Assume also that the homogeneous equation $\mathcal{A}u = 0$ has in $H^\mu_{\rm od}$ only the trivial solution. Then the range $\mathcal{A}H^\mu_{\rm od} \subset H^\mu_{\rm ev}$ is of codimension* 1*. Fixing a $\varphi \in H^\mu_{\rm ev} \backslash \mathcal{A}H^\mu_{\rm od}$, for every $f \in H^\mu_{\rm ev}$ we get a unique pair $(\omega, u) \in \mathbb{C} \times H^\mu_{\rm od}$ such that $\omega\varphi + \mathcal{A}u = f$, and this generalized solution of* (2) *is unique in $\mathbb{C} \times H^0_{\rm od}$.*

We have $H^\mu \subset C^m(\mathbb{R})$ for $m < \mu - \frac{1}{2}$, $\mu > \frac{1}{2}$, and under conditions of Theorem 2.1, $u \in C^m(\mathbb{R})$. For (ω, v), the generalized solution of (1), we have

$$v(x) = u\Big(\frac{1}{2\pi}\arccos(-x)\Big)\,, \qquad 1 \leq x \leq 1\,.$$

So v is continuous on $[-1,1]$, C^m-smooth in $(-1,1)$, satisfies $v(-1) = u(0) = 0$, $v(1) = u(1/2) = 0$, but the derivatives of v have certain singularities at the end points of the interval $(-1,1)$, e.g. $v \in C^1(-1,1)$ for $\mu > \frac{3}{2}$,

$$v'(x) - \frac{u'(0)}{2\pi\sqrt{1-x^2}} \to 0 \ \text{ as } \ x \to -1\,, \quad v'(x) - \frac{u'(\frac{1}{2})}{2\pi\sqrt{1-x^2}} \to 0 \ \text{ as } \ x \to 1\,.$$

3. A FULLY DISCRETE COLLOCATION METHOD

For $N \in \mathbb{N}$, introduce $m, M, n \in \mathbb{N}$ such that

$$\begin{gathered} 2m \le M \le n \le N\,, \quad m \sim N^{\varrho}, \quad M \sim N^{\sigma}, \quad n \sim N^{\tau}, \\ 0 < \varrho \le \sigma \le \tau < 1\,, \quad \sigma \le \frac{1}{2}\,, \quad \frac{\mu}{\mu+1} \le \tau < 1\,, \end{gathered} \tag{3}$$

where $n \sim N^{\tau}$ means that there are positive constants c_1 and c_2 such that $c_1 \le nN^{-\tau} \le c_2$ as $N \to \infty$. We approximate $\mathcal{A} = A_0 + A_1 + A_2 \in \mathcal{L}(H^0_{\rm od}, H^0_{\rm ev})$ by $\mathcal{A}_N \in \mathcal{L}(H^0_{\rm od}, H^0_{\rm ev})$ defined by

$$\mathcal{A}_N = A_0 + Q^{\rm ev}_M(A_1^{(M)} + A_2^{(M)})P^{\rm od}_m + Q^{\rm ev}_n A_1^{[d]}(P^{\rm od}_n - P^{\rm od}_m), \tag{4}$$

where $P^{\rm od}_n$ is the orthogonal projection operator in $H^0_{\rm od}$ to

$$\mathcal{T}^{\rm od}_n = \operatorname{span}\{\sin(k2\pi t)\,,\ \ k = 1, \ldots, n\}\,;$$

$Q^{\rm ev}_n$ is the interpolation projection operator defined by

$$\begin{gathered} Q^{\rm ev}_n u \in \mathcal{T}^{\rm ev}_n = \operatorname{span}\{\cos(k2\pi t)\,,\ \ k = 0, 1, \ldots, n\}\,, \\ (Q_n u)\Big(\frac{j}{2n+1}\Big) = u\Big(\frac{j}{2n+1}\Big)\,, \quad j = 0, 1, \ldots, n\,, \quad u \in H^{\mu}_{\rm ev}\,,\ \ \mu > \frac{1}{2}\,; \end{gathered}$$

the product integration approximations $A_1^{(M)}, A_2^{(M)} \in \mathcal{L}(H^{\mu}_{\rm od}, H^0_{\rm ev})$ are defined by

$$(A_1^{(M)}u)(t) = \int_{-1/2}^{1/2} \log|\sin\pi(t-s)|\, Q^{\rm ev}_{M,s}(a_1(t,s)u(s))ds\,,$$

$$(A_2^{(M)}u)(t) = \int_{-1/2}^{1/2} Q^{\rm ev}_{M,s}(a_2(t,s)u(s))ds\,, \quad u \in H^{\mu}_{\rm od}\,, \quad \mu > \frac{1}{2}\,,$$

where the index s in $Q^{\rm ev}_{M,s}$ indicates the interpolation with respect to the argument s; the asymptotic approximation $A_1^{[d]} \in \mathcal{L}(H^0_{\rm od}, H^0_{\rm ev})$ of A_1 is defined by

$$A_1^{[d]}\sin(k2\pi t) = \sum_{j=0}^{d-2} k^{-1-j} b_j(t) \left\{ \begin{array}{ll} \sin(k2\pi t), & j \text{ even} \\ \cos(k2\pi t), & j \text{ odd} \end{array} \right\}, \quad k = 1, 2, \ldots,$$

$$b_j(t) = -\left\{ \begin{array}{ll} (-1)^{j/2}, & j \text{ even} \\ (-1)^{(j-1)/2}, & j \text{ odd} \end{array} \right\} \frac{1}{2}\frac{1}{(2\pi)^j}\Big(\frac{\partial}{\partial s}\Big)^j a_1(t,s)\Big|_{s=t}\,, \qquad j = 0, \ldots, d-2\,,$$

$$\mathbb{N} \ni d \geq \frac{1-\varrho}{\varrho}\mu\,, \quad \mu > \frac{1}{2}\,; \qquad d = 1\,, \;\; A_1^{[d]} = 0 \text{ may be set if } \;\; \frac{1-\varrho}{\varrho}\mu \leq 1\,.$$

Lemma 3.1. *Let* (3) *be fulfilled with a* $\mu > \frac{1}{2}$*, and let* $d \geq \frac{1-\varrho}{\varrho}\mu$*. Further, assume that* $a_i = a_i(t,s)$*,* $i = 1,2$*, are even in* t*, odd in* s *and with a* $\nu > 1/2$*,*

$$a_1 \in H^{\nu,d+\nu} \cap H^{\mu+1,\nu} \cap H^{\nu+\mu(1-\sigma)/\sigma,\mu/\sigma} \cap H^{\mu/\sigma,\nu+\mu(1-\sigma)/\sigma},$$
$$a_2 \in H^{\nu,\mu/\sigma} \cap H^{\mu/\sigma,0} \cap H^{0,\mu(1-\varrho)/\varrho}.$$

Then

$$\|\mathcal{A} - \mathcal{A}_N\|_{\lambda,\mu} := \|\mathcal{A} - \mathcal{A}_N\|_{\mathcal{L}(H^\mu_{\text{ev}},H^\lambda_{\text{od}})} \leq cN^{\lambda-\mu} \qquad (0 \leq \lambda \leq \mu)\,,$$
$$\|\mathcal{A} - \mathcal{A}_N\|_{\lambda,\lambda} \to 0 \;\; \textit{as} \;\; N \to \infty \qquad (0 \leq \lambda \leq \mu)\,.$$

Theorem 3.1. *Assume the conditions of Lemma* 3.1*. Assume also that the homogeneous equation* $\mathcal{A}u = 0$ *has in* H^μ_{od} *only the trivial solution. Let* $\varphi \in H^\mu_{\text{ev}} \backslash \mathcal{A}H^\mu_{\text{od}}$*. Then there is a* $N_0 \in \mathbb{N}$ *such that for* $N \geq N_0$*, the approximate problem*

$$\omega Q_N^{\text{ev}}\varphi + \mathcal{A}_N u = Q_N^{\text{ev}} f \tag{5}$$

has for every $f \in H^\mu_{\text{ev}}$ *a solution* $(\omega_N, u_N) \in \mathbb{C} \times \mathcal{T}_N^{\text{od}}$ *which is unique in* $\mathbb{C} \times H^0_{\text{od}}$*, and*

$$|\omega_N - \omega| \leq cN^{-\mu}\|f\|_\mu\,, \quad \|u_N - u\|_\lambda \leq cN^{\lambda-\mu}\|f\|_\mu \qquad (0 \leq \lambda \leq \mu),$$

where $(\omega, u) \in \mathbb{C} \times H^\mu_{\text{ev}}$ *is the* (*unique*) *solution of the problem* $\omega\varphi + \mathcal{A}u = f$*.*

Notice that to (ω_N, u_N), $u_N = \sum_{j=1}^N c_j \sin(j2\pi t)$, there corresponds the approximate generalized solution (ω_N, v_N) of (1) with

$$v_N(x) = u_N\Big(\frac{1}{2\pi}\arccos(-x)\Big) = \sum_{j=1}^N c_j \sin(j\arccos(-x))$$
$$= \sqrt{1-x^2}\sum_{j=1}^N c_j U_{j-1}(-x),$$

where $U_j(x) = \sin((j+1)\arccos x)/\sqrt{1-x^2}$, $j = 0, 1, \ldots$, are the Chebyshev polynomials of the second kind. Moreover, by Theorem 3.1

$$\|v_N - v\|_{L^2_\sigma} = \|u_N - u\|_0 \leq cN^{-\mu}\|f\|_\mu,$$

where (ω, v), $v(x) = u(\frac{1}{2\pi}\arccos(-x))$, is the generalized solution of problem (1). Also estimates of $v_N - v$ in weighted Sobolev norms follow from Theorem 3.1.

4. MATRIX FORM OF THE METHOD AND CONJUGATE GRADIENTS

The dimension of the problem (5) can be reduced from N to n. Namely, if (ω_N, u_N) with $u_N = \sum_{j=1}^{N} c_j \sin(j2\pi t)$ is the solution of (5), then $\omega_n = \omega_N$, $u_n = P_n^{\mathrm{od}} u_N = \sum_{j=1}^{n} c_j \sin(j2\pi t)$ is the solution of the problem

$$\omega\varphi_n + \mathcal{A}_N u = f_n \tag{6}$$

with $\varphi_n = P_n^{\mathrm{ev}} Q_N^{\mathrm{ev}} \varphi$, $f_n = P_n^{\mathrm{ev}} Q_N^{\mathrm{ev}} f$, and u_N can be reconstructed by the formula $u_N = u_n + \sum_{j=n+1}^{N} (\omega_n \alpha_j - d_j) \sin(j2\pi t)$, where α_j and d_j are the Fourier coefficients of $Q_N^{\mathrm{ev}}\varphi$ and $Q_N^{\mathrm{ev}} f$, respectively,

$$Q_N^{\mathrm{ev}}\varphi = \sum_{j=0}^{N} \alpha_j \cos(j2\pi t)\,, \qquad Q_N^{\mathrm{ev}} f = \sum_{j=0}^{N} d_j \cos(j2\pi t)\,.$$

Denoting $\underline{c}_n = (c_1, \ldots, c_n)^\top$, $\underline{d}_n = (d_0, d_1, \ldots, d_n)^\top$, $\underline{\alpha}_n = (\alpha_0, \alpha_1, \ldots, \alpha_n)^\top$, we have problem (6) in the matrix form

$$\omega\underline{\alpha}_n + \mathbb{M}_n \underline{c}_n = \underline{d}_n \tag{7}$$

with the $(n+1) \times n$ matrix $\mathbb{M}_n$ defined by

$$\mathbb{M}_n = \mathbb{A}_0 + \mathbb{I}_{n,M} \tilde{\mathcal{C}}_M (\mathbb{A}_1^{(M)} + \mathbb{A}_2^{(M)}) \mathcal{S}_M \mathbb{P}_{M,m,n} + \tilde{\mathcal{C}}_n \sum_{j=0}^{d-2} \mathbb{B}_n^{(j)} \left\{ \begin{array}{ll} \mathcal{C}_n \mathbb{J}_n, & j \text{ even} \\ \mathbb{J}_n \mathcal{S}_n, & j \text{ odd} \end{array} \right\} \mathbb{G}_N^{(j)},$$

where

$\mathbb{A}_0 = -\mathbb{J}_n$, $\mathbb{J}_n = \begin{pmatrix} 0 \\ \mathbb{I}_n \end{pmatrix}$ are $(n+1) \times n$ matrices,

$\mathbb{I}_n$ is an $n \times n$ identity matrix,

$\mathbb{I}_{n,M} = \begin{pmatrix} \mathbb{I}_{M+1} \\ 0 \end{pmatrix}$ is an $(n+1) \times (M+1)$ matrix,

$\mathbb{P}_{M,m,n} = \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & 0 \end{pmatrix}$ is an $M \times n$ matrix;

$\mathcal{C}_n = \left(\cos\left(kj \dfrac{2\pi}{2n+1} \right) \right)_{j,k=0}^{n}$, $\tilde{\mathcal{C}}_n = \dfrac{4}{2n+1} \mathbb{D}_n \mathcal{C}_n \mathbb{D}_n$,

$\mathbb{D}_n = \mathrm{diag}\left\{ \dfrac{1}{2}, 1, \ldots, 1 \right\}$, $\mathcal{S}_n = \left(\sin\left(kj \dfrac{2\pi}{2n+1} \right) \right)_{j,k=1}^{n}$;

$\mathbb{A}_1^{(M)} = \left(a_{kj}^{(1)} \right)$, $\mathbb{A}_2^{(M)} = \left(a_{kj}^{(2)} \right)$ are $(M+1) \times M$ matrices with the entries

$$
\begin{aligned}
a_{kj}^{(1)} &= -\frac{1}{2M+1} a_1\Big(\frac{k}{2M+1}, \frac{j}{2M+1}\Big)\big(\gamma_{|k-j|} + \gamma_{k+j}\big),\\
a_{kj}^{(2)} &= \frac{2}{2M+1} a_2\Big(\frac{k}{2M+1}, \frac{j}{2M+1}\Big), \quad k = 0, 1, \ldots, M, \ \ j = 1, \ldots, M,\\
\gamma_k &= \log 2 + \sum_{l=1}^{M} \frac{1}{l} \cos\Big(kl \frac{2\pi}{2M+1}\Big), \quad k = 0, 1, \ldots, M,\\
\gamma_{M+k} &= \gamma_{M+1-k}\,, \qquad 1 \le k \le M;
\end{aligned}
$$

$\mathbb{G}_n^{(j)} = \operatorname{diag}\big\{0, \ldots, 0, (m+1)^{-1-j}, \ldots, n^{-1-j}\big\}$ is an $n \times n$ matrix,

$\mathbb{B}_n^{(j)} = \operatorname{diag}\Big\{b_j(0), b_j\Big(\frac{1}{2n+1}\Big), \ldots, b_j\Big(\frac{n}{2n+1}\Big)\Big\}$ is an $(n+1) \times (n+1)$ matrix.

The application of $\mathbb{M}_n$ to an n-vector, as well as the application of $\mathbb{M}_n'$, the Hermite adjoint matrix of $\mathbb{M}_n$, to an $(n+1)$-vector costs $\mathcal{O}(n \log n) + \mathcal{O}(M^2) = \mathcal{O}(N^\tau \log N) + \mathcal{O}(N^{2\sigma})$ arithmetical operations, provided that the fast Fourier technique is used for the cosine and sine transformations $\mathcal{C}_n$ and $\mathcal{S}_n$. The computation of the entries of $\mathbb{M}_n$ costs $\mathcal{O}(M^2) + \mathcal{O}(N) = \mathcal{O}(N)$ arithmetical operations. This enables us to design fast solvers of problem (2) on the basis of iteration methods. We specify a classical conjugate gradient iteration algorithm (see [6,7]) to solve (7).

Denote by $\underline{x}_n = (\omega, c_1, \ldots, c_n)$ the $(n+1)$-vector of unknowns and rewrite the system (7) in the form

$$\mathbb{A}_n \underline{x}_n = \underline{d}_n,$$

where

$$\mathbb{A}_n = \begin{pmatrix} \underline{\alpha}_n & \mathbb{M}_n \end{pmatrix} \text{ is an } (n+1) \times (n+1) \text{ matrix}\,.$$

Algorithm 1.

Step 0: $\underline{x}_n^0 = 0, \ \underline{y}_n^0 = -\underline{d}_n, \ \underline{r}_n^0 = -\mathbb{A}_n' \underline{d}_n$.

For $k = 0, 1, 2 \ldots$:

(i) if $\|\underline{y}_n^k\| \le \|\underline{d}_n\| \delta N^{-\mu}$, then terminate;

(ii) if $\|\underline{y}_n^k\| > \|\underline{d}_n\| \delta N^{-\mu}$, then go to step $k+1$, and compute

$$
\begin{aligned}
\underline{z}_n^k &= \begin{cases} -\underline{r}_n^0\,, & k = 0\,,\\ -\underline{r}_n^k + \big(\|\underline{r}_n^k\| / \|\underline{r}_n^{k-1}\|\big)^2 \underline{z}_n^{k-1}\,, & k \ge 1\,, \end{cases}\\
\underline{x}_n^{k+1} &= \underline{x}_n^k + \gamma_k \underline{z}_n^k\,, \qquad \gamma_k = \big(\|\underline{r}_n^k\| / \|\mathbb{A}_n \underline{z}_n^k\|\big)^2\,,\\
\underline{y}_n^{k+1} &= \underline{y}_n^k + \gamma_k \mathbb{A}_n \underline{z}_n^k\,,\\
\underline{r}_n^{k+1} &= \underline{r}_n^k + \gamma_k \mathbb{A}_n' \mathbb{A}_n \underline{z}_n^k\,.
\end{aligned}
$$

In this algorithm the usual norm $\|\underline{d}_n\| = \left(\sum_{k=0}^{n} |d_k|^2\right)^{1/2}$ is used for $(n+1)$-vectors. We have incorporated the residual termination rule into the algorithm: the iterations stop on the first k such that $\|\mathbb{A}_n \underline{x}_n^k - \underline{d}_n\| \leq \|\underline{d}_n\| \delta N^{-\mu}$. Here $\delta > 0$ is a parameter.

Theorem 4.1. *Under conditions of Theorem* 3.1, *for* $N \geq N_0$, *Algorithm* 1 *terminates at an iteration number* k *of order* $o(\log N)$ *as* $N \to \infty$. *The corresponding iteration approximation* $\underline{x}_n^k = (\omega^k, c_1^k, \ldots, c_n^k)$ *defines an iteration solution* (ω_N^k, u_N^k) *to* (5) *with*

$$\omega_N^k = \omega^k\,, \quad u_N^k = \sum_{j=1}^{n} c_j^k \sin(j2\pi t) + \sum_{j=n+1}^{N} (\omega^k \alpha_j - d_j) \sin(j2\pi t)$$

for which there hold the optimal order estimates

$$|\omega_N^k - \omega| \leq cN^{-\mu} \|f\|_\mu\,, \qquad \|u_N^k - u\|_\lambda \leq cN^{\lambda-\mu} \|f\|_\mu\,, \qquad 0 \leq \lambda \leq \mu\,,$$

where $(\omega, u) \in \mathbb{C} \times H_{\mathrm{od}}^\mu$ *is the unique generalized solution of integral equation* (2).

The computation of $\underline{d}_N = \tilde{\mathcal{C}}_N \underline{f}_N$ and $\underline{\alpha}_N = \tilde{\mathcal{C}}_N \underline{\varphi}_N$ from the vectors of grid values $\underline{f}_N = \left(f(0), f\left(\frac{1}{2N+1}\right), \ldots, f\left(\frac{N}{2N+1}\right)\right)$ and $\underline{\varphi}_N = \left(\varphi(0),\ \varphi\left(\frac{1}{2N+1}\right), \ldots, \varphi\left(\frac{N}{2N+1}\right)\right)$ by the fast algorithm costs $\mathcal{O}(N \log N)$ arithmetical operations. All other computations are cheaper, costing asymptotically $o(\log N)(\mathcal{O}(N^\tau \log N) + \mathcal{O}(N^{2\sigma}))$ arithmetical operations, which is $o(N)$ for $\sigma < \frac{1}{2}$; notice that an iteration step by Algorithm 1 contains one application of $\mathbb{A}_n$ and one application of $\mathbb{A}_n'$. If the Fourier coefficients of f and φ with respect to $\cos(k2\pi t)$ $(k = 0, 1, 2, \ldots)$ are known, we can use $P_N^{\mathrm{ev}} f$ and $P_N^{\mathrm{ev}} \varphi$ instead of $Q_N^{\mathrm{ev}} f$ and $Q_N^{\mathrm{ev}} \varphi$. In this case the full number of arithmetical and logical operations reduces to $N + o(N)$.

If functions a_1, a_2, and φ are real, then $\mathbb{A}_n$ is real.

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ÜLDISTATUD TIIVAVÕRRANDI KIIRED LAHENDUSMEETODID INDEKSI –1 KORRAL

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Üldistatud tiivavõrrandit on käsitletud situatsioonis, kui vastava integraal-operaatori Fredholmi indeks on -1. On esitatud vastava laiendatud ülesande lahendusmeetod, mis põhineb trigonomeetrilisele kollokatsioonimeetodile, on aga täielikult diskreetne ning võimaldab teatud mõttes optimaalse täpsusastmega lähislahendi N parameetrit määrata $\mathcal{O}(N \log N)$ aritmeetilise tehtega.