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MATHEMATICS

Induced 3-Lie algebras, superalgebras and induced representations

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Abstract. We construct 3-Lie superalgebras on a commutative superalgebra by means of involution and even degree derivation. We construct a representation of induced 3-Lie algebras and superalgebras by means of a representation of initial (binary) Lie algebra, trace and supertrace. We show that the induced representation of 3-Lie algebra, that we constructed, is a representation by traceless matrices, that is, lies in the Lie algebra $\mathfrak{sl}(V)$, where V is a representation space. In the case of 2-dimensional representation we find conditions under which the induced representation of induced 3-Lie algebra is irreducible. We give the example of irreducible representation of induced 3-Lie algebra of 2nd order complex matrices.

Key words: 3-Lie algebra, 3-Lie superalgebra, Filippov–Jacobi identity, commutative algebras with involution and derivation, representations of 3-Lie algebras and superalgebras.

1. INTRODUCTION

A generalization of such an important concept as Lie algebra and superalgebra occurs in several directions. One of such directions is the development of the theory of n-ary Lie algebras and superalgebras, which was originated by Filippov in [9]. A n-ary Lie algebra $\mathfrak{g}$ is a vector space endowed with n-ary Lie bracket $(x_1,x_2,\ldots,x_n)\in\mathfrak{g}\times\mathfrak{g}\times\ldots\times\mathfrak{g}\to[x_1,x_2,\ldots,x_n]\in\mathfrak{g}$, which is a multilinear totally skew-symmetric mapping and satisfies the Filippov-Jacobi identity

$$\begin{aligned}[x_1,x_2,\ldots,x_{n-1},[y_1,y_2,\ldots,y_n]] = [[x_1,x_2,\ldots,x_{n-1},y_1],y_2,\ldots,y_n]+[y_1,[x_1,x_2,\ldots,x_{n-1},y_2],y_3,\ldots,y_n]\\ +\ldots+[y_1,y_2,\ldots,y_{n-1},[x_1,x_2,\ldots,x_{n-1},y_n]],\end{aligned}$$

where $x_1,x_2,\ldots,x_{n-1},y_1,y_2,\ldots,y_n\in\mathfrak{g}$. The extension of this definition to Lie superalgebras is based on the signs matching rule in $\mathbb{Z}_2$-graded structures. A n-ary Lie superalgebra is a super vector space $\mathfrak{h}=\mathfrak{h}_0\oplus\mathfrak{h}_1$ endowed with a multilinear graded skew-symmetric mapping $(x_1,x_2,\ldots,x_n)\in\mathfrak{h}\times\mathfrak{h}\times\ldots\times\mathfrak{h}\to[x_1,x_2,\ldots,x_n]\in\mathfrak{h}$ such that it satisfies the graded Filippov–Jacobi identity

$$\begin{aligned}[x_1,x_2,\ldots,x_{n-1},[y_1,y_2,\ldots,y_n]] = {}&[[x_1,x_2,\ldots,x_{n-1},y_1],y_2,\ldots,y_n]\\ &+(-1)^{\mu_1}[y_1,[x_1,x_2,\ldots,x_{n-1},y_2],y_3,\ldots,y_n]\\ &+\ldots+(-1)^{\mu_{n-1}}[y_1,y_2,\ldots,y_{n-1},[x_1,x_2,\ldots,x_{n-1},y_n]],\end{aligned}$$

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where

$$\mu_1 = \hat{y}_1(\hat{x}_1 + \ldots + \hat{x}_{n-1}), \ldots, \mu_{n-1} = (\hat{y}_1 + \hat{y}_2 + \ldots + \hat{y}_{n-1})(\hat{x}_1 + \ldots + \hat{x}_{n-1}),$$

and $\hat{x}$ denotes the $\mathbb{Z}_2$-grading of an element $x \in \mathfrak{h}$. In the particular case $n = 3$ a Lie (super)algebra with ternary (graded) Lie bracket is usually called 3-Lie (super)algebra.

A well known example of an n-ary Lie algebra can be constructed by means of the analogue of a vector product of n vectors in an $(n+1)$-dimensional vector space. In this case, the n-ary Lie bracket is constructed with the help of determinant of order $n+1$. Another important example of an n-ary Lie bracket is the analogue of the Poisson bracket on an algebra of smooth functions proposed by Nambu as part of his approach to generalization of Hamiltonian mechanics [13]. In this case, the n-ary Lie bracket for n smooth functions, where n is a positive odd integer, is constructed with the help of Jacobian. Another natural way to construct new n-ary Lie brackets is the method of constructing such a bracket making use of an existing $(n-1)$-ary Lie bracket. Thus, the resulting n-ary Lie bracket is usually called the induced n-ary Lie bracket and the corresponding n-ary Lie algebra is called the induced n-ary Lie algebra. In this paper, we consider induced 3-Lie algebras and superalgebras.

At first glance, it seems that the induced ternary (graded) Lie brackets do not give anything new, since they are based on a more fundamental structure, that is, a binary (graded) Lie bracket. But that is not true. It is well known that Lie groups and their Lie algebras are used in gauge field theories, where the Lie group is a gauge group and its Lie algebra (Lie bracket) is used in infinitesimal gauge transformations. Now, suppose we were able to construct a ternary Lie bracket using a binary Lie bracket of some gauge group. Then we can trace a relation of our ternary bracket (through a binary Lie bracket) with the infinitesimal gauge transformations and we can generalize them introducing two (or more) gauge transformation parameters. This idea and a method of constructing induced ternary Lie brackets were proposed in [6]. This method in a more general form can be described as follows: Given a Lie algebra $\mathfrak{g}$ and a generalized trace $\tau : \mathfrak{g} \to \mathbb{C}$ on it, one can define the induced ternary Lie bracket by the formula

$$[x,y,z] = \tau(x)\,[y,z] + \tau(y)\,[z,x] + \tau(z)\,[x,y], \tag{1}$$

where $x,y,z \in \mathfrak{g}$ and $[\cdot,\cdot]$ is a Lie bracket of $\mathfrak{g}$. The structure of induced n-Lie algebras with n-ary Lie brackets constructed by means of a generalized trace, their cohomologies and Hom-generalizations were studied in the papers [4,5,12]. An extension of the method of constructing induced 3-Lie algebras to 3-Lie superalgebras by means of a generalized supertrace and possible application of this method in BRST-formalism of quantum field theory was proposed in the papers [1,2]. Later the method of constructing 3-Lie superalgebras by means of supertrace proposed in [1,2] was extended to ternary Hom–Lie superalgebras in [10].

In this paper, we construct induced 3-Lie superalgebras, whose ternary graded Lie brackets have the structure similar to (1). First of all, we propose two identities, which give sufficient and necessary conditions for graded skew-symmetric ternary bracket to satisfy the graded Filippov–Jacobi identity, in other words, to determine a 3-Lie superalgebra. This result can be considered as an extension of result obtained in [7] for 3-Lie algebras to 3-Lie superalgebras. Next we construct binary graded Lie brackets and then ternary graded Lie brackets on a commutative superalgebra with involution, where the structure of a ternary graded Lie bracket is similar to (1) and it is constructed by means of even degree derivation and involution. The starting point for our constructions are the results obtained in [8], in which the authors construct ternary Lie brackets on a commutative algebra with involution. Then we study representations of induced 3-Lie algebras and superalgebras. Given a representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ of Lie algebra $\mathfrak{g}$, where V is a representation space, we construct the mapping $\rho : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{gl}(V)$ as follows:

$$\rho(x,y) = \mathrm{Tr}(\pi(x))\,\pi(y) - \mathrm{Tr}(\pi(y))\,\pi(x), \tag{2}$$

where $x,y \in \mathfrak{g}$, and prove that ρ is a representation of induced 3-Lie algebra. We call this representation of induced 3-Lie algebra induced representation. Note that the induced representation (2) of induced 3-Lie algebra was introduced in the PhD dissertation [11], where the author also proposed an example of such a representation constructed by means of the Lie algebra of 2nd order complex matrices, which is similar to example given in the present paper. In this paper, we consider the question of irreducibility of the induced representation of 3-Lie algebra. We show that for any $x, y \in \mathfrak{g}$ the matrix $\rho(x,y) \in \mathfrak{gl}(V)$ is traceless, i.e. $\rho : \otimes\mathfrak{g} \to \mathfrak{sl}(V)$, where $\mathfrak{sl}(V)$ is the Lie algebra of special linear group of endomorphisms of a vector space V. Then we study the irreducibility of induced representation ρ and propose conditions, when the induced representation of induced 3-Lie algebra is irreducible. As an example, we consider the basic representation of $\mathfrak{gl}_2(\mathbb{C})$ by 2nd order complex matrices and the corresponding induced 3-Lie algebra of 2nd order matrices and, making use of previously mentioned conditions, show that this is an irreducible representation of 3-Lie algebra. Next we propose the extension of induced representation (2) to induced 3-Lie superalgebras by means of supertrace and prove that this is a representation of induced 3-Lie superalgebra.

2. 3-LIE SUPERALGEBRAS INDUCED BY MEANS OF DERIVATION AND INVOLUTION

In this section we consider a commutative superalgebra $\mathscr{A} = \mathscr{A}_0 \oplus \mathscr{A}_1$ endowed with an involution $x \in \mathscr{A} \mapsto x^\star \in \mathscr{A}$ and an even degree derivation $\delta : \mathscr{A} \to \mathscr{A}$. By involution we mean an even degree linear mapping (even degree means that it preserves grading of any homogeneous element), which satisfies $(x^\star)^\star = x$, $x \in \mathscr{A}$. By derivation of degree m, where m is an integer either 0 or 1, we mean linear mapping $\delta : \mathscr{A} \to \mathscr{A}$, which satisfies the graded Leibniz rule $\delta(xy) = \delta(x)y + (-1)^{m\hat{x}\,\hat{y}}\, x\delta(y)$, where $\hat{x}$, $\hat{y}$ are gradings of homogeneous elements x, y, respectively. Using an involution and derivation we construct three graded Lie brackets on superalgebra $\mathscr{A}$. Furthermore, by considering a generalized supertrace we apply methods described in [3] and yield induced 3-Lie superalgebras whose bracket is defined using involution, derivation and both of them, together with generalized supertrace.

First of all, we start by proposing equivalent form to the graded Filippov–Jacobi identity. To simplify the equations, we will use the notation $\widehat{xy} = \hat{x} + \hat{y}$.

Proposition 1. *Assume $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a super vector space and let*

$$[\cdot,\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \tag{3}$$

be a skew-symmetric multilinear map, such that $\widehat{[x,y,z]} = \widehat{xyz}$. Then $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ is a 3-Lie superalgebra if and only if the equalities

$$[[x,y,z],u,v] = (-1)^{\hat{u}\,\widehat{xyz} + \hat{x}\,\widehat{yz}}\,[[u,y,z],x,v] + (-1)^{\hat{u}\,\widehat{yz} + \hat{y}\,\hat{z}}\,[[x,u,z],y,v] + (-1)^{\hat{u}\,\hat{z}}\,[[x,y,u],z,v] \tag{4}$$

and

$$\begin{aligned} &[[x,y,z],u,v] + (-1)^{\widehat{xy}\,\widehat{zuv} + \hat{z}\,\widehat{uv}}\,[[u,v,z],x,y] - (-1)^{\hat{y}\,\hat{u} + \hat{y}\,\hat{z} + \hat{z}\,\hat{u}}\,[[x,u,z],y,v] \\ &-(-1)^{\hat{v}\,\widehat{yzu} + \hat{x}\,\widehat{yzu} + \hat{x}\,\hat{v}}\,[[v,y,z],u,x] - (-1)^{\widehat{yv}\,\widehat{zu} + \hat{v}\,\hat{y}}\,[[x,v,z],u,y] - (-1)^{\widehat{xu}\,\widehat{yz} + \hat{x}\,\hat{u}}\,[[u,y,z],x,v] = 0 \end{aligned} \tag{5}$$

hold for bracket $[\cdot,\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Proof. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a super vector space and assume that $(\mathfrak{g}, [\cdot,\cdot,\cdot])$ is a 3-Lie superalgebra. If this is the case, then the Filippov–Jacobi identity must hold:

$$[x,y,[z,u,v]] = [[x,y,z],u,v] + (-1)^{\widehat{xy}\,\hat{z}}\,[z,[x,y,u],v] + (-1)^{\widehat{xy}\,\widehat{zu}}\,[z,u,[x,y,v]]. \tag{6}$$

To show that identity (4) holds for bracket (3), apply Filippov–Jacobi identity (6) recursively to itself on the right-most bracket. This yields us the following result:

$$
\begin{aligned}
\cancel{[x,y,[z,u,v]]} &= [[x,y,z],u,v] + (-1)^{\widehat{xy}\,\hat{z}}\,[z,[x,y,u],v] \\
&\quad + (-1)^{\widehat{xy}\,\widehat{zu}}\Big([[z,u,x],y,v] + (-1)^{\widehat{zu}\,\hat{x}}\,[x,[z,u,y],v] + (-1)^{\widehat{zu}\,\widehat{xy}}\,[x,y,[z,u,v]]\Big) \\
&= [[x,y,z],u,v] + (-1)^{\widehat{xy}\,\hat{z}}\,[z,[x,y,u],v] + (-1)^{\widehat{xy}\,\widehat{zu}}\,[[z,u,x],y,v] \\
&\quad + (-1)^{\widehat{xy}\,\widehat{zu}+\widehat{zu}\,\hat{x}}\,[x,[z,u,y],v] + (-1)^{\widehat{xy}\,\widehat{zu}+\widehat{zu}\,\widehat{xy}}\,\cancel{[x,y,[z,u,v]]},
\end{aligned}
$$

which gives

$$
[[x,y,z],u,v] = -(-1)^{\widehat{xy}\,\hat{z}}\,[z,[x,y,u],v] - (-1)^{\widehat{xy}\,\widehat{zu}}\,[[z,u,x],y,v] - (-1)^{\widehat{zu}\,\hat{y}}\,[x,[z,u,y],v]. \tag{7}
$$

We need to show that the right hand sides of (4) and (7) coincide. It is indeed the case:

$$
-(-1)^{\widehat{zu}\,\hat{y}}\,[x,[z,u,y],v] = -(-1)^3(-1)^{\widehat{zu}\,\hat{y}+\hat{x}\,\widehat{zuy}+\hat{u}\,\hat{z}+\hat{z}\,\hat{y}}\,[[u,y,z],x,v] = (-1)^{\hat{u}\,\widehat{xyz}+\hat{x}\,\widehat{yz}}\,[[u,y,z],x,v],
$$

$$
-(-1)^{\widehat{xy}\,\widehat{zu}}\,[[z,u,x],y,v] = -(-1)^3(-1)^{\widehat{xy}\,\widehat{zu}+\hat{z}\,\hat{u}+\hat{z}\,\hat{x}+\hat{x}\,\hat{u}}\,[[x,u,z],y,v] = (-1)^{\hat{u}\,\widehat{yz}+\hat{y}\,\hat{z}}\,[[x,u,z],y,v],
$$

$$
-(-1)^{\widehat{xy}\,\hat{z}}\,[z,[x,y,u],v] = -(-1)^1(-1)^{\widehat{xy}\,\hat{z}+\hat{z}\,\widehat{xyu}}\,[[x,y,u],z,v] = (-1)^{\hat{u}\,\hat{z}}\,[[x,y,u],z,v].
$$

From those equalities we can deduce that if $(\mathfrak{g},[\cdot,\cdot,\cdot])$ is indeed a 3-Lie superalgebra, then identity (4) holds for bracket $[\cdot,\cdot,\cdot]$.

In order to prove the identity (5), we can apply Filippov–Jacobi identity recursively to itself once again, but this time to both the right-most and middle brackets on the right hand side of identity (6).

$$
\begin{aligned}
\cancel{[x,y,[z,u,v]]} &= [[x,y,z],u,v] - (-1)^{\widehat{xy}\,\hat{z}+\hat{v}\,\widehat{xyu}}\,[z,v,[x,y,u]] + (-1)^{\widehat{xy}\,\widehat{zu}}\,[z,u,[x,y,v]] \\
&= [[x,y,z],u,v] \\
&\quad - (-1)^{\widehat{xy}\,\hat{z}+\hat{v}\,\widehat{xyu}}\Big([[z,v,x],y,u] + (-1)^{\widehat{zv}\,\hat{x}}\,[x,[z,v,y],u] + (-1)^{\widehat{zv}\,\widehat{xy}}\,[x,y,[z,v,u]]\Big) \\
&\quad + (-1)^{\widehat{xy}\,\widehat{zu}}\Big([[z,u,x],y,v] + (-1)^{\widehat{zu}\,\hat{x}}\,[x,[z,u,y],v] + (-1)^{\widehat{zu}\,\widehat{xy}}\,[x,y,[z,u,v]]\Big) \\
&= [[x,y,z],u,v] - (-1)^{\widehat{xy}\,\hat{z}+\hat{v}\,\widehat{xyu}}\,[[z,v,x],y,u] - (-1)^{\widehat{xy}\,\hat{z}+\hat{v}\,\widehat{xyu}+\widehat{zv}\,\hat{x}}\,[x,[z,v,y],u] \\
&\quad - (-1)^{\widehat{xy}\,\hat{z}+\hat{v}\,\widehat{xyu}+\widehat{zv}\,\widehat{xy}}\,[x,y,[z,v,u]] + (-1)^{\widehat{xy}\,\widehat{zu}}\,[[z,u,x],y,v] \\
&\quad + (-1)^{\widehat{xy}\,\widehat{zu}+\widehat{zu}\,\hat{x}}\,[x,[z,u,y],v] + (-1)^{\widehat{xy}\,\widehat{zu}+\widehat{zu}\,\widehat{xy}}\,\cancel{[x,y,[z,u,v]]}.
\end{aligned}
$$

Reordering the summands in the last equation results in

$$
\begin{aligned}
&[[x,y,z],u,v] - (-1)^{\widehat{xy}\,\hat{z}+\hat{v}\,\widehat{xyu}+\widehat{zv}\,\widehat{xy}}\,[x,y,[z,v,u]] \\
&+(-1)^{\widehat{xy}\,\widehat{zu}}\,[[z,u,x],y,v] - (-1)^{\widehat{xy}\,\hat{z}+\hat{v}\,\widehat{xyu}+\widehat{zv}\,\hat{x}}\,[x,[z,v,y],u] \\
&-(-1)^{\widehat{xy}\,\hat{z}+\hat{v}\,\widehat{xyu}}\,[[z,v,x],y,u] + (-1)^{\widehat{xy}\,\widehat{zu}+\widehat{zu}\,\hat{x}}\,[x,[z,u,y],v] = 0,
\end{aligned}
$$

which is exactly (5) once the elements are ordered accordingly within the brackets:

$$-(-1)^{\widehat{xy}\,\hat{z}+\hat{v}\,\widehat{xyu}+\widehat{zv}\,\widehat{xy}}\,[x,y,[z,v,u]] = +\,(-1)^{\widehat{xy}\,\widehat{zuv}+\hat{z}\,\widehat{uv}}\,[[u,v,z],x,y]$$

$$+(-1)^{\widehat{xy}\,\widehat{zu}}\,[[z,u,x],y,v] = -\,(-1)^{\hat{y}\,\hat{u}+\hat{y}\,\hat{z}+\hat{z}\,\hat{u}}\,[[x,u,z],y,v]$$

$$-(-1)^{\widehat{xy}\,\hat{z}+\hat{v}\,\widehat{xyu}+\widehat{zv}\,\hat{x}}\,[x,[z,v,y],u] = -\,(-1)^{\hat{v}\,\widehat{yzu}+\hat{x}\,\widehat{yzu}+\hat{x}\,\hat{v}}\,[[v,y,z],u,x]$$

$$-(-1)^{\widehat{xy}\,\hat{z}+\hat{v}\,\widehat{xyu}}\,[[z,v,x],y,u] = -\,(-1)^{\widehat{yv}\,\widehat{zu}+\hat{v}\,\hat{y}}\,[[x,v,z],u,y]$$

$$+(-1)^{\widehat{xy}\,\widehat{zu}+\widehat{zu}\,\hat{x}}\,[x,[z,u,y],v] = -\,(-1)^{\widehat{xu}\,\widehat{yz}+\hat{x}\,\hat{u}}\,[[u,y,z],x,v].$$

This completes the proof of necessity. Sufficiency can be shown analogously. □

Let $\mathscr{A}=\mathscr{A}_0\oplus\mathscr{A}_1$ be a superalgebra. We say that superalgebra $\mathscr{A}$ is commutative superalgebra if for any two homogeneous elements $u,v\in\mathscr{A}$ it holds that $uv=(-1)^{\hat{u}\,\hat{v}}\,vu$. Let $m\in \mathrm{Z}_2$. Linear mapping $\delta:\,\mathscr{A}\to\mathscr{A}$ is said to be a degree m derivation of a superalgebra $\mathscr{A}$ if $\widehat{\delta(u)}\;=\hat{u}\;+m$, for all $u\in\mathscr{A}$, and it satisfies the graded Leibniz rule

$$\delta(uv)=\delta(u)v+(-1)^{m\hat{u}\,\hat{v}}\,u\delta(v).$$

In case $m=0$, we call it even degree derivation, and otherwise, for $m=1$, we call it odd degree derivation. We denote the degree of δ as $\hat{\delta}$. Consequently, if δ is an even degree derivation of superalgebra $\mathscr{A}$, then $\widehat{\delta(u)}\;=\hat{u}$. Or, in other words, derivation δ does not change the degree of homogeneous element $u\in\mathscr{A}$. Furthermore, in case of even δ the Leibniz rule for any $u,v\in\mathscr{A}$ simplifies to

$$\delta(uv)=\delta(u)v+u\delta(v).$$

Mapping $(\cdot)^\star\colon\,\mathscr{A}\to\mathscr{A},\quad u\mapsto u^\star$ is said to be an involution of a superalgebra $\mathscr{A}$ if it satisfies the following conditions:
(1) involution is an even degree mapping of a superalgebra $\mathscr{A}$, $\mathscr{A}_i^\star\subset\mathscr{A}_i,\;\; i\in\mathrm{Z}_2,\;\widehat{u^\star}\;=\hat{u}$,
(2) it is linear, $(\lambda u+\mu v)^\star=\lambda u^\star+\mu v^\star$,
(3) $(\cdot)^\star\colon\,\mathscr{A}\to\mathscr{A}$ is its own inverse, $(u^\star)^\star=u$,
(4) $(uv)^\star=(-1)^{\hat{u}\,\hat{v}}\,v^\star u^\star$.
In the case of commutative superalgebra the condition 4 takes on the form $(uv)^\star=u^\star v^\star$.

Making use of involution and even degree derivation we can construct graded Lie brackets on a superalgebra $\mathscr{A}$. To achieve that, let us define

$$[u,v]_\star=u^\star v-(-1)^{\hat{u}\,\hat{v}}\,v^\star u, \tag{8}$$

$$[u,v]_\delta=u\,\delta(v)-(-1)^{\hat{u}\,\hat{v}}\,v\,\delta(u), \tag{9}$$

$$[u,v]_{\star,\delta}=(u-u^\star)\,\delta(v)-(-1)^{\hat{u}\,\hat{v}}\,(v-v^\star)\,\delta(u). \tag{10}$$

Proposition 2. *Brackets (8) and (9) are graded Lie brackets. If involution $(\cdot)^\star:\,\mathscr{A}\to\mathscr{A}$ and an even degree derivation $\delta:\,\mathscr{A}\to\mathscr{A}$ satisfy the condition*

$$(\delta(u))^\star=-\delta\left(u^\star\right),$$

then bracket (10) is also a graded Lie bracket.

Proof. Proving the proposition is similar for all three brackets (8), (9) and (10). Let us only observe (10), which is the most involved. To assure linearity, pick coefficients $\lambda,\mu\in$ K and homogeneous elements $u,v,w\in\mathscr{A}$ such that $\hat{v}=\widehat{w}$. Note that $\widehat{\omega}=\widehat{\lambda v+\mu w}=\hat{v}=\widehat{w}$ as both v and w are homogeneous, and calculate

$$\begin{aligned}
[u,\lambda v+\mu w]_{\star,\delta} &= (u-u^\star)\,\delta(\lambda v+\mu w)-(-1)^{\widehat{u}\,\widehat{\omega}}\,((\lambda v+\mu w)-(\lambda v+\mu w)^\star)\,\delta(u)\\
&= \lambda\,(u-u^\star)\,\delta(v)+\mu\,(u-u^\star)\,\delta(w)-(-1)^{\widehat{u}\,\widehat{\omega}}\,(\lambda(v-v^\star)+\mu(w-w^\star))\,\delta(u)\\
&= \lambda\,(u-u^\star)\,\delta(v)-\lambda(-1)^{\widehat{u}\,\hat{v}}\,(v-v^\star)\delta(u)+\mu\,(u-u^\star)\,\delta(w)-\mu(-1)^{\widehat{u}\,\widehat{w}}\,(w-w^\star)\delta(u)\\
&= \lambda[u,v]_{\star,\delta}+\mu[u,w]_{\star,\delta}.
\end{aligned}$$

Antisymmetry is a result of direct computation

$$\begin{aligned}
[u,v]_{\star,\delta} &= (u-u^\star)\,\delta(v)-(-1)^{\widehat{u}\,\hat{v}}\,(v-v^\star)\delta(u)\\
&= -(-1)^{\widehat{u}\,\hat{v}}\,\Big((v-v^\star)\delta(u)-(-1)^{\widehat{u}\,\hat{v}}\,(u-u^\star)\,\delta(v)\Big)=-(-1)^{\widehat{u}\,\hat{v}}\,[v,u]_{\star,\delta}\,.
\end{aligned}$$

In order to complete the proof, we still need to show that bracket defined by (10) satisfies the Jacobi identity. To achieve that, first observe that due to the commutativity of $\mathscr{A}$ we can write bracket $[\cdot,\cdot]_{\star,\delta}$ as

$$[u,v]_{\star,\delta}=u\delta(v)-u^\star\delta(v)-\delta(u)v+\delta(u)v^\star.$$

Furthermore, if δ is even and $(\delta(u))^\star=-\delta(u^\star)$, then we can write

$$\begin{aligned}
[u,v\delta(w)]_{\star,\delta} = {}& u\delta(v)\delta(w)+uv\delta^2(w)-u^\star\delta(v)\delta(w)\\
&-u^\star v\delta^2(w)-\delta(u)v\delta(w)-\delta(u)v^\star\delta(w^\star),\\
[u,\delta(v)w]_{\star,\delta} = {}& u\delta^2(v)w+u\delta(v)\delta(w)-u^\star\delta^2(v)w\\
&-u^\star\delta(v)\delta(w)-\delta(u)\delta(v)w-\delta(u)\delta(v^\star)w^\star.
\end{aligned}$$

Using the results above we can now calculate $[u,[v,w]_{\star,\delta}]_{\star,\delta}$:

$$\begin{aligned}
[u,[v,w]_{\star,\delta}]_{\star,\delta} = {}& [u,v\delta(w)]_{\star,\delta}-[u,v^\star\delta(w)]_{\star,\delta}-[u,\delta(v)w]_{\star,\delta}+[u,\delta(v)w^\star]_{\star,\delta}\\
= {}& u\delta(v)\delta(w)+uv\delta^2(w)-u^\star\delta(v)\delta(w)-u^\star v\delta^2(w)-\delta(u)v\delta(w)-\delta(u)v^\star\delta(w^\star)\\
&-u\delta(v^\star)\delta(w)-uv^\star\delta^2(w)+u^\star\delta(v^\star)\delta(w)+u^\star v^\star\delta^2(w)+\delta(u)v^\star\delta(w)+\delta(u)v\delta(w^\star)\\
&-u\delta^2(v)w-u\delta(v)\delta(w)+u^\star\delta^2(v)w+u^\star\delta(v)\delta(w)+\delta(u)\delta(v)w+\delta(u)\delta(v^\star)w^\star\\
&+u\delta^2(v)w^\star+u\delta(v)\delta(w^\star)-u^\star\delta^2(v)w^\star-u^\star\delta(v)\delta(w^\star)-\delta(u)\delta(v)w^\star-\delta(u)\delta(v^\star)w. \quad (11)
\end{aligned}$$

As a next step we can apply (11) also to $[v,[w,u]_{\star,\delta}]_{\star,\delta}$ and $[w,[u,v]_{\star,\delta}]_{\star,\delta}$, yielding all elements on the left hand side of Jacobi identity.

$$[u,[v,w]_{\star,\delta}]_{\star,\delta}+(-1)^{\widehat{u}\,\widehat{v}+\widehat{u}\,\widehat{w}}\,[v,[w,u]_{\star,\delta}]_{\star,\delta}+(-1)^{\widehat{w}\,\widehat{u}+\widehat{w}\,\widehat{v}}\,[w,[u,v]_{\star,\delta}]_{\star,\delta}=$$

$$\left.\begin{aligned}
&+\cancel{u\delta(v)\delta(w)}^{(1)}+\cancel{uv\delta^2(w)}^{(2)}-\cancel{u^\star\delta(v)\delta(w)}^{(3)}-\cancel{u^\star v\delta^2(w)}^{(4)}\\
&-\cancel{\delta(u)v\delta(w)}^{(5)}-\cancel{\delta(u)v^\star\delta(w^\star)}^{(6)}-\cancel{u\delta(v^\star)\delta(w)}^{(7)}-\cancel{uv^\star\delta^2(w)}^{(8)}\\
&+\cancel{u^\star\delta(v^\star)\delta(w)}^{(9)}+\cancel{u^\star v^\star\delta^2(w)}^{(10)}+\cancel{\delta(u)v^\star\delta(w)}^{(11)}+\cancel{\delta(u)v\delta(w^\star)}^{(12)}\\
&-\cancel{u\delta^2(v)w}^{(1)}-\cancel{u\delta(v)\delta(w)}^{(1)}+\cancel{u^\star\delta^2(v)w}^{(2)}+\cancel{u^\star\delta(v)\delta(w)}^{(3)}\\
&+\cancel{\delta(u)\delta(v)w}^{(3)}+\cancel{\delta(u)\delta(v^\star)w^\star}^{(4)}+\cancel{u\delta^2(v)w^\star}^{(5)}+\cancel{u\delta(v)\delta(w^\star)}^{(6)}\\
&-\cancel{u^\star\delta^2(v)w^\star}^{(7)}-\cancel{u^\star\delta(v)\delta(w^\star)}^{(8)}-\cancel{\delta(u)\delta(v)w^\star}^{(9)}-\cancel{\delta(u)\delta(v^\star)w}^{(10)}
\end{aligned}\right\}[u,[v,w]_{\star,\delta}]_{\star,\delta}$$

$$\left.\begin{aligned}
&+\cancel{\delta(u)v\delta(w)}^{(5)}+\cancel{\delta^2(u)vw}^{(11)}-\cancel{\delta(u)v^\star\delta(w)}^{(11)}-\cancel{\delta^2(u)v^\star w}^{(12)}\\
&-\cancel{\delta(u)\delta(v)w}^{(3)}-\cancel{\delta(u^\star)\delta(v)w^\star}^{(1)}-\cancel{\delta(u)v\delta(w^\star)}^{(12)}-\cancel{\delta^2(u)vw^\star}^{(2)}\\
&+\cancel{\delta(u)v^\star\delta(w^\star)}^{(6)}+\cancel{\delta^2(u)v^\star w^\star}^{(3)}+\cancel{\delta(u)\delta(v)w^\star}^{(9)}+\cancel{\delta(u^\star)\delta(v)w}^{(4)}\\
&-\cancel{uv\delta^2(w)}^{(2)}-\cancel{\delta(u)v\delta(w)}^{(5)}+\cancel{uv^\star\delta^2(w)}^{(8)}+\cancel{\delta(u)v^\star\delta(w)}^{(6)}\\
&+\cancel{u\delta(v)\delta(w)}^{(7)}+\cancel{u^\star\delta(v)\delta(w^\star)}^{(8)}+\cancel{u^\star v\delta^2(w)}^{(4)}+\cancel{\delta(u^\star)v\delta(w)}^{(8)}\\
&-\cancel{u^\star v^\star\delta^2(w)}^{(10)}-\cancel{\delta(u^\star)v^\star\delta(w)}^{(9)}-\cancel{u^\star\delta(v)\delta(w)}^{(10)}-\cancel{u\delta(v)\delta(w^\star)}^{(6)}
\end{aligned}\right\}(-1)^{\widehat{u}\,\widehat{vw}}\,[v,[w,u]_{\star,\delta}]_{\star,\delta}$$

$$\left.\begin{aligned}
&+\cancel{\delta(u)\delta(v)w}^{(11)}+\cancel{u\delta^2(v)w}^{(1)}-\cancel{\delta(u)\delta(v)w^\star}^{(12)}-\cancel{u\delta^2(v)w^\star}^{(5)}\\
&-\cancel{u\delta(v)\delta(w)}^{(7)}-\cancel{u^\star\delta(v^\star)\delta(w)}^{(9)}-\cancel{\delta(u^\star)\delta(v)w}^{(4)}-\cancel{u^\star\delta^2(v)w}^{(2)}\\
&+\cancel{\delta(u^\star)\delta(v)w^\star}^{(1)}+\cancel{u^\star\delta^2(v)w^\star}^{(7)}+\cancel{u^\star\delta(v)\delta(w)}^{(10)}+\cancel{u\delta(v^\star)\delta(w)}^{(7)}\\
&-\cancel{\delta^2(u)vw}^{(11)}-\cancel{\delta(u)\delta(v)w}^{(11)}+\cancel{\delta^2(u)vw^\star}^{(2)}+\cancel{\delta(u)\delta(v)w^\star}^{(12)}\\
&+\cancel{\delta(u)v\delta(w)}^{(5)}+\cancel{\delta(u^\star)v^\star\delta(w)}^{(9)}+\cancel{\delta^2(u)v^\star w}^{(12)}+\cancel{\delta(u)\delta(v^\star)w}^{(10)}\\
&-\cancel{\delta^2(u)v^\star w^\star}^{(3)}-\cancel{\delta(u)\delta(v^\star)w^\star}^{(4)}-\cancel{\delta(u)v^\star\delta(w)}^{(6)}-\cancel{\delta(u^\star)v\delta(w)}^{(8)}
\end{aligned}\right\}(-1)^{\widehat{w}\,\widehat{uv}}\,[w,[u,v]_{\star,\delta}]_{\star,\delta}$$

$$=0.$$

This means that Jacobi identity indeed holds, and $[\cdot,\cdot]_{\star,\delta}$ is a graded Lie bracket. □

What the proposition tells us is that graded Lie brackets (8), (9) naturally define Lie superalgebra structures $(\mathscr{A},[\cdot,\cdot]_\delta)$ and $(\mathscr{A},[\cdot,\cdot]_\star)$ on commutative superalgebra $\mathscr{A}$. In case we further assume that the derivation δ is even, and together with involution it satisfies the condition $(\delta(u))^\star=-\delta(u^\star)$, then graded Lie bracket (10) defines another Lie superalgebra structure $\left(\mathscr{A},[\cdot,\cdot]_{\star,\delta}\right)$ on $\mathscr{A}$.

Next let us consider the generalized supertraces on those Lie superalgebras:

$$\xi:\ (\mathscr{A},[\cdot,\cdot]_\star)\to K, \tag{12}$$

$$\eta:\ (\mathscr{A},[\cdot,\cdot]_\delta)\to K, \tag{13}$$

$$\chi:\ \left(\mathscr{A},[\cdot,\cdot]_{\star,\delta}\right)\to K. \tag{14}$$

With the help of those generalized supertraces we can induce ternary Lie superalgebras out of the binary Lie superalgebras, by following the construction described in [3].

Theorem 1. *Let $\mathscr{A} = \mathscr{A}_0 \oplus \mathscr{A}_1$ be a commutative superalgebra, and let $(\cdot)^\star \colon \mathscr{A} \to \mathscr{A}$ and $\delta \colon \mathscr{A} \to \mathscr{A}$ be involution and derivation of $\mathscr{A}$, respectively. If (12) and (13) are supertraces, then graded ternary brackets $[\cdot,\cdot,\cdot]_\star \colon \mathscr{A}^{\oplus 3} \to A$ and $[\cdot,\cdot,\cdot]_\delta \colon \mathscr{A}^{\oplus 3} \to A$, defined by*

$$[u,v,w]_\star = \xi(u)[v,w]_\star + (-1)^{\widehat{u}\,\widehat{vw}}\,\xi(v)[w,u]_\star + (-1)^{\widehat{w}\,\widehat{uv}}\,\xi(w)[u,v]_\star, \tag{15}$$

$$[u,v,w]_\delta = \eta(u)[v,w]_\delta + (-1)^{\widehat{u}\,\widehat{vw}}\,\eta(v)[w,u]_\delta + (-1)^{\widehat{w}\,\widehat{uv}}\,\eta(w)[u,v]_\delta \tag{16}$$

are graded ternary Lie brackets. If δ is even, $(\delta(u))^\star = -\delta(u^\star)$ and (14) is a supertrace, then graded ternary bracket $[\cdot,\cdot,\cdot]_{\star,\delta} \colon \mathscr{A}^{\oplus 3} \to A$, defined as

$$[u,v,w]_{\star,\delta} = \chi(u)[v,w]_\delta + (-1)^{\widehat{u}\,\widehat{vw}}\,\chi(v)[w,u]_{\star,\delta} + (-1)^{\widehat{w}\,\widehat{uv}}\,\chi(w)[u,v]_{\star,\delta} \tag{17}$$

is ternary Lie bracket.

Consequences of the theorem are that each and every one of $(\mathscr{A},[\cdot,\cdot,\cdot]_\delta)$, $(\mathscr{A},[\cdot,\cdot,\cdot]_\star)$ and $\left(\mathscr{A},[\cdot,\cdot,\cdot]_{\star,\delta}\right)$ are all ternary Lie superalgebras. Of course granted that for the latter the derivation and involution satisfy the required condition.

3. 3-LIE ALGEBRAS INDUCED BY MEANS OF TRACE AND THEIR REPRESENTATIONS

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. By a generalized trace on $\mathfrak{g}$ we will mean a linear function $\tau : \mathfrak{g} \to C$ such that $\tau([x,y]) = 0$ for any $x,y \in \mathfrak{g}$. If a Lie algebra $\mathfrak{g}$ is equipped with a generalized trace τ, then one can construct a ternary bracket using the binary Lie bracket of $\mathfrak{g}$ and a generalized trace τ as follows:

$$[x,y,z] = \tau(x)\,[y,z] + \tau(y)\,[z,x] + \tau(z)\,[x,y], \tag{18}$$

and then prove that this ternary bracket satisfies the Filippov–Jacobi identity. For a matrix Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ of complex matrices of nth order, this was proved in the paper [6] in which the authors proposed an approach to quantum theory of Nambu generalization of Hamilton mechanics based on the ternary Lie bracket (18). Thus, an initial binary Lie algebra $\mathfrak{g}$ endowed with the ternary Lie bracket (18) becomes a 3-Lie algebra, which is usually called an induced 3-Lie algebra. We will denote this induced 3-Lie algebra by $\mathfrak{tg}_\tau$. This method of constructing the induced 3-Lie algebra is applicable, for example, in the case when we have a representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ of a Lie algebra $\mathfrak{g}$ in a vector space V. Let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a Lie algebra $\mathfrak{g}$ in a vector space V. Then one can construct the induced 3-Lie algebra with the help of ternary bracket

$$[x,y,z] = \mathrm{Tr}(\pi(x))\,[y,z] + \mathrm{Tr}(\pi(y))\,[z,x] + \mathrm{Tr}(\pi(z))\,[x,y]. \tag{19}$$

In what follows we will denote the 3-Lie algebra induced with the help of (19) by $\mathfrak{tg}_\pi$.

Let $\mathfrak{h}$ be a 3-Lie algebra and V be a finite-dimensional vector space.

Definition 3.1. *A bilinear skew-symmetric mapping $\rho : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{gl}(V)$ is said to be a representation of a 3-Lie algebra $\mathfrak{h}$ if the following conditions are satisfied*:
(1) $[\rho(x,y),\rho(u,v)] = \rho([x,y,u],v) + \rho(u,[x,y,v])$,
(2) $\rho([x,y,z],u) = \rho(x,y)\rho(z,u) + \rho(y,z)\rho(x,u) + \rho(z,x)\rho(y,u)$,
where $x,y,z,u,v \in \mathfrak{h}$. We will denote this representation of 3-Lie algebra $\mathfrak{h}$ in a vector space V by $(\mathfrak{h},\rho,V)$.

Let $\mathfrak{g}$ be a Lie algebra, $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ and $\mathfrak{tg}_\pi$ be the induced 3-Lie algebra. We show that it is possible to construct a representation of the 3-Lie algebra $\mathfrak{tg}_\pi$ using a representation π of $\mathfrak{g}$.

Theorem 2. *The mapping $\rho : \mathfrak{g}\times\mathfrak{g}\to gl(V)$ defined by*

$$\rho(x,y) = Tr(\pi(x))\,\pi(y) - Tr(\pi(y))\,\pi(x), \tag{20}$$

where $x,y\in\mathfrak{g}$, is a representation of induced 3-Lie algebra $\mathfrak{tg}_\pi$ in a vector space V.

Proof. First of all, we obtain the formula for terms of the form $\rho([x,y,u],v)$. Making use of the formula for ternary Lie bracket (19), we get

$$\begin{aligned}\rho([x,y,u],v) &= \rho\big(\mathrm{Tr}(\pi(x))[y,u]+\mathrm{Tr}(\pi(y))[u,x]+\mathrm{Tr}(\pi(u))[x,y],v\big)\\ &= \mathrm{Tr}(\pi(x))\rho([y,u],v)+\mathrm{Tr}(\pi(y))\rho([u,x],v)+\mathrm{Tr}(\pi(u))\rho([x,y],v).\end{aligned}$$

Now applying the formula for a representation (20) and taking into account that $\mathrm{Tr}([\pi(x),\pi(y)])=0$ for any $x,y\in\mathfrak{g}$, we get

$$\rho([y,u],v) = -\mathrm{Tr}(\pi(v))[\pi(y),\pi(u)]. \tag{21}$$

Hence,

$$\begin{aligned}\rho([x,y,u],v) = &-\mathrm{Tr}(\pi(v))\mathrm{Tr}(\pi(x))[\pi(y),\pi(u)]-\mathrm{Tr}(\pi(v))\mathrm{Tr}(\pi(y))[\pi(u),\pi(x)]\\ &-\mathrm{Tr}(\pi(v))\mathrm{Tr}(\pi(u))[\pi(x),\pi(y)].\end{aligned} \tag{22}$$

Analogously,

$$\begin{aligned}\rho(u,[x,y,v]) = &\;\mathrm{Tr}(\pi(u))\mathrm{Tr}(\pi(x))[\pi(y),\pi(v)]+\mathrm{Tr}(\pi(u))\mathrm{Tr}(\pi(y))[\pi(v),\pi(x)]\\ &+\mathrm{Tr}(\pi(u))\mathrm{Tr}(\pi(v))[\pi(x),\pi(y)].\end{aligned} \tag{23}$$

Summing up the obtained expressions (22),(23), we see that the last terms cancel each other and the remaining four terms can be combined into the binary commutator

$$\big[\mathrm{Tr}(\pi(x))\pi(y)-\mathrm{Tr}(\pi(y))\pi(x),\mathrm{Tr}(\pi(u))\pi(v)-\mathrm{Tr}(\pi(v))\pi(u)\big],$$

which is equal to $[\rho(x,y),\rho(u,v)]$. Thus we get

$$[\rho(x,y),\rho(u,v)] = \rho([x,y,u],v)+\rho(u,[x,y,v])$$

and the first condition for representation of 3-Lie algebra is proved. In order to prove the second condition of Definition 3.1 we calculate the right hand side of this condition. We get

$$\begin{aligned}\rho(x,y)\,\rho(z,u) &= \big(\mathrm{Tr}(\pi(x))\pi(y)-\mathrm{Tr}(\pi(y))\pi(x)\big)\big(\mathrm{Tr}(\pi(z))\pi(u)-\mathrm{Tr}(\pi(u))\pi(z)\big)\\ &= \cancel{\mathrm{Tr}(\pi(x))\mathrm{Tr}(\pi(z))\pi(y)\pi(u)} - \underbrace{\mathrm{Tr}(\pi(x))\mathrm{Tr}(\pi(u))\pi(y)\pi(z)}_{1}\\ &\quad -\cancel{\mathrm{Tr}(\pi(y))\mathrm{Tr}(\pi(z))\pi(x)\pi(u)}+\underbrace{\mathrm{Tr}(\pi(y))\mathrm{Tr}(\pi(u))\pi(x)\pi(z)}_{2}.\end{aligned}$$

Analogously,

$$\begin{aligned}\rho(y,z)\,\rho(x,u) &= \cancel{\mathrm{Tr}(\pi(x))\mathrm{Tr}(\pi(y))\pi(z)\pi(u)} - \underbrace{\mathrm{Tr}(\pi(y))\mathrm{Tr}(\pi(u))\pi(z)\pi(x)}_{2}\\ &\quad -\cancel{\mathrm{Tr}(\pi(x))\mathrm{Tr}(\pi(z))\pi(y)\pi(u)}+\underbrace{\mathrm{Tr}(\pi(z))\mathrm{Tr}(\pi(u))\pi(y)\pi(x)}_{3},\end{aligned}$$

$$\begin{aligned}\rho(z,x)\,\rho(y,u) \;=\; & \cancel{\mathrm{Tr}(\pi(y))\mathrm{Tr}(\pi(z))\pi(x)\pi(u)} - \underbrace{\mathrm{Tr}(\pi(z))\mathrm{Tr}(\pi(u))\pi(x)\pi(y)}_{3} \\ & - \cancel{\mathrm{Tr}(\pi(x))\mathrm{Tr}(\pi(y))\pi(z)\pi(u)} + \underbrace{\mathrm{Tr}(\pi(x))\mathrm{Tr}(\pi(u))\pi(z)\pi(y)}_{1}.\end{aligned}$$

Taking the sum of these expressions, we see that strike through terms cancel each other, while terms marked with 1,2,3 give the sum of following terms, each containing the corresponding binary commutator,

$$-\mathrm{Tr}(\pi(u))\mathrm{Tr}(\pi(x))[\pi(y),\pi(z)] - \mathrm{Tr}(\pi(u))\mathrm{Tr}(\pi(y))[\pi(z),\pi(x)] - \mathrm{Tr}(\pi(u))\mathrm{Tr}(\pi(z))[\pi(x),\pi(y)]. \quad (24)$$

Now the formula (22) shows that the expression (24) is equal to $\rho([x,y,z],u)$ and this ends the proof. □

Proposition 3. *Let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a Lie algebra $\mathfrak{g}$. Then for induced representation ρ of induced 3-Lie algebra $\mathfrak{tg}_\pi$ we have $\rho : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{sl}(V)$, i.e. for any $x,y \in \mathfrak{g}$ the matrix $\rho(x,y)$ is traceless.*

Proof. We have

$$\mathrm{Tr}\,\rho(x,y) = \mathrm{Tr}(\pi(x))\,\mathrm{Tr}(\pi(y)) - \mathrm{Tr}(\pi(y))\,\mathrm{Tr}(\pi(x)) = 0.$$

□

An important question related with the induced representation ρ of induced 3-Lie algebra $\mathfrak{tg}_\pi$ is the question of how the irreducibility (reducibility) of an initial representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is related to the irreducibility (reducibility) of the induced representation ρ of $\mathfrak{tg}_\pi$. It is quite easy to show that if an initial representation π of a Lie algebra $\mathfrak{g}$ is reducible, then the induced representation ρ of induced 3-Lie algebra $\mathfrak{tg}_\pi$ is reducible as well. Indeed, assume that $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is reducible, i.e. there is a non-trivial subspace $W \subset V$, which is invariant under a representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. By other words, for any $x \in \mathfrak{g}, v \in W$ we have $\pi(x)\cdot v \in W$. Now it is easy to show that a subspace W is invariant under the induced representation ρ of induced 3-Lie algebra $\mathfrak{tg}_\pi$. Indeed, for any $x,y \in \mathfrak{g}$ and $v \in W$ we have

$$\rho(x,y)\cdot v = \mathrm{Tr}(\pi(x))\pi(y)\cdot v - \mathrm{Tr}(\pi(y))\pi(x)\cdot v \in W.$$

The question of whether the induced representation ρ of the induced 3-Lie algebra $\mathfrak{tg}_\pi$ will be irreducible if an initial representation π is irreducible is more subtle. The following theorem gives an answer to this question in the case when the representation space is two-dimensional. Let $e_1,e_2,\ldots,e_r$ be a basis for a Lie algebra $\mathfrak{g}$ and $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$, where V is a 2-dimensional complex vector space. We denote by μ_i the trace of the 2nd order matrix $\pi(e_i)$. We assume that generators $e_1,e_2,\ldots,e_r$ of the Lie algebra $\mathfrak{g}$ are ordered in such a way that the first k have non-zero trace, i.e. $\mu_1 \neq 0,\ldots,\mu_k \neq 0$, and the next $r-k$ have the trace equal to zero, i.e. $\mu_{k+1} = \ldots = \mu_r = 0$.

Theorem 3. *Let $\mathfrak{g}$ be a Lie algebra, $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$, where V is a 2-dimensional complex vector space, $e_1,e_2,\ldots,e_r$ be a basis for a Lie algebra $\mathfrak{g}$, $\mu_1,\mu_2,\ldots,\mu_r$ be traces of matrices $\pi(e_1),\pi(e_2),\ldots,\pi(e_r)$, respectively, and $\mu_1 \neq 0,\ldots,\mu_k \neq 0, \mu_{k+1} = \ldots = \mu_r = 0$. If the matrices*

$$\pi(\mu_i\, e_j - \mu_j\, e_i), \pi(e_{k+1}),\ldots,\pi(e_r), \quad (25)$$

where $1 \leq i < j \leq k$, have no common eigenvector, then a representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ and the induced representation $\rho : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{sl}(V)$ are both irreducible representations.

Proof. First of all, we prove that if the assumption of theorem holds then π is an irreducible representation of $\mathfrak{g}$. We will prove this by contradiction. Thus we assume that π is reducible and our aim to show that then the matrices $\pi(\mu_i\, e_j - \mu_j\, e_i), \pi(e_{k+1}),\ldots,\pi(e_r)$ have common eigenvector and this will contradict our assumption. From reducibility of π it follows that the matrices $\pi(e_1),\pi(e_2),\ldots,\pi(e_r)$ have a common eigenvector, which we denote by v, i.e. $\pi(e_i)\cdot v = \lambda_i\, v$. Particularly, the matrices $\pi(e_{k+1}),\ldots,\pi(e_r)$ have common eigenvector v. Hence, we only need to show that v is a common eigenvector for the matrices $\pi(\mu_i e_j - \mu_j e_i)$, where $1 \leq i < j \leq k$. We have

$$\pi(\mu_i e_j - \mu_j e_i)\cdot v = \mu_i \pi(e_j)\cdot v - \mu_j \pi(e_i)\cdot v = (\mu_i \lambda_j - \mu_j \lambda_i)\, v.$$

Hence v is a common eigenvector for the matrices $\pi(\mu_i e_j - \mu_j e_i), \pi(e_{k+1}), \ldots, \pi(e_r)$ and this is contradiction.

The second part of the proof, that is, that ρ is an irreducible representation similar to the first, that is, we prove by contradiction. Hence, we assume that $\rho : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{sl}(V)$ is a reducible representation of induced 3-Lie algebra $\mathfrak{tg}_\pi$. Since we are considering a two-dimensional vector space V, this means that there is a one-dimensional invariant subspace in V for all matrices $\rho(x,y)$, in other words, the matrices $\rho(e_i,e_j)$, where $1 \leq i < j \leq r$ have a common eigenvector. Let us denote this common eigenvector by v. Then $\rho(e_i,e_j)\cdot v = \lambda_{ij}\, v$ or

$$\big(\mu_i \pi(e_j) - \mu_j \pi(e_i)\big)\cdot v = \lambda_{ij}\, v, \;\; 1 \leq i < j \leq r. \tag{26}$$

As a vector e_j in this formula, we take an arbitrary vector from the set $\{e_{k+1},\ldots,e_r\}$, i.e. $k+1 \leq j \leq r$, and as a vector e_i we take an arbitrary vector from the set $\{e_1,\ldots,e_k\}$, i.e. $1 \leq i \leq k$. Then $\mu_i \neq 0, \mu_j = 0$ and the relation (26) takes on the form

$$\pi(e_j)\cdot v = \frac{\lambda_{ij}}{\mu_i}\, v,$$

and this relation shows that a vector v is a common eigenvector for the matrices $\pi(e_{k+1}),\ldots,\pi(e_r)$. The fact that this vector v is a common eigenvector for the matrices $\pi(\mu_i e_j - \mu_j e_i)$, where $1 \leq i < j \leq k$ follows directly from the relation (26). □

As an illustration of the application of this theorem, we consider the Lie algebra of 2nd order square matrices $\mathfrak{gl}_2(\mathbb{C})$. This means that we consider these matrices as a basic (irreducible) representation of Lie algebra $\mathfrak{gl}_2(\mathbb{C})$ in a complex plane $\mathbb{C}^2$. We choose the standard basis for this Lie algebra

$$E^1_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\; E^2_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\; E^2_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\; E^1_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{27}$$

Notice that we order the generators of $\mathfrak{gl}_2(\mathbb{C})$ as in Theorem 3, i.e. the generators E^1_1, E^2_2 have non-zero trace and the generators E^2_1, E^1_2 have zero trace. Then the Lie algebra $\mathfrak{gl}_2(\mathbb{C})$ has the following commutation relations

$$\begin{aligned}
&[E^1_1,E^2_2] = 0,\; [E^1_1,E^2_1] = E^2_1,\; [E^1_1,E^1_2] = -E^1_2,\\
&[E^2_2,E^2_1] = -E^2_1,\; [E^2_2,E^1_2] = E^1_2,\; [E^2_1,E^1_2] = E^1_1 - E^2_2.
\end{aligned}$$

The ternary commutation relations of the induced 3-Lie algebra we find by means of (19)

$$[E^1_1,E^2_2,E^2_1] = -2E^2_1, \tag{28}$$
$$[E^1_1,E^2_2,E^1_2] = \;\; 2E^1_2, \tag{29}$$
$$[E^1_1,E^2_1,E^1_2] = \;\; E^1_1 - E^2_2, \tag{30}$$
$$[E^2_2,E^2_1,E^1_2] = \;\; E^1_1 - E^2_2. \tag{31}$$

The induced representation of induced 3-Lie algebra we compute by means of the formula given in Theorem 2

$$\begin{aligned}
&\rho(E^1_1,E^2_2) = E^2_2 - E^1_1,\; \rho(E^1_1,E^2_1) = E^2_1,\; \rho(E^1_1,E^1_2) = E^1_2,\\
&\rho(E^2_2,E^2_1) = E^2_1, \qquad \rho(E^2_2,E^1_2) = E^1_2,\; \rho(E^2_1,E^1_2) = 0.
\end{aligned}$$

It is easy to see that in accordance with Proposition 3, the induced representation of 3-Lie algebra (28) – (31) gives the generators of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of the special linear group $\mathrm{SL}_2(\mathbb{C})$

$$E_1^1 - E_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ E_1^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_2^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that these matrices are exactly the matrices (25) in Theorem 3. The first matrix $E_1^1 - E_2^2$ has two eigenvalues $1,-1$ and two eigenvectors $(1,0),(0,1)$, respectively, the second matrix E_1^2 has one eigenvalue 0 and the corresponding eigenvector is $(1,0)$, third matrix E_2^1 has also one eigenvalue 0 and the corresponding eigenvector is $(0,1)$. Thus, these matrices have no common eigenvector and the induced representation of 3-Lie algebra (28)–(31) is irreducible.

4. INDUCED 3-LIE SUPERALGEBRAS AND THEIR INDUCED REPRESENTATIONS

In this section we consider 3-Lie superalgebras. It was shown [1,2] that the method of constructing an induced 3-Lie algebra with the help of a generalized trace can be extended to the case of Lie superalgebras by means of a concept of a generalized supertrace. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra. Then by generalized supertrace we mean a linear function $S\tau : \mathfrak{g} \to \mathbb{C}$ such that it vanishes on graded Lie bracket of $\mathfrak{g}$, i.e. $S\tau([x,y]) = 0$, and it also vanishes when restricted to $\mathfrak{g}_1$, i.e. $S\tau|_{\mathfrak{g}_1} \equiv 0$.

Let $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ be a 3-Lie superalgebra, $V = V_0 \oplus V_1$ be a super vector space and $\mathrm{End}(V)$ be the super vector space of endomorphisms of V. The graded commutator of two endomorphisms $A,B \in \mathrm{End}(V)$ of a super vector space V, defined by formula $[A,B] = AB - (-1)^{\widehat{A}\,\widehat{B}}\,BA$, where A,B are homogeneous endomorphisms and $\widehat{A}$, $\widehat{B}$ are their gradings, determines the structure of the Lie superalgebra on $\mathrm{End}(V)$ and we denote this Lie superalgebra by $\mathfrak{sgl}(V)$. There is a canonical structure of a super vector space on the tensor product $\mathfrak{h} \otimes \mathfrak{h}$, which is defined as follows:

$$\mathfrak{h} \otimes \mathfrak{h} = (\mathfrak{h} \otimes \mathfrak{h})_0 \oplus (\mathfrak{h} \otimes \mathfrak{h})_1,$$

where $(\mathfrak{h} \otimes \mathfrak{h})_0 = (\mathfrak{h}_0 \otimes \mathfrak{h}_0) \oplus (\mathfrak{h}_1 \otimes \mathfrak{h}_1)$ and $(\mathfrak{h} \otimes \mathfrak{h})_1 = (\mathfrak{h}_0 \otimes \mathfrak{h}_1) \oplus (\mathfrak{h}_1 \otimes \mathfrak{h}_0)$.

Definition 4.1. *A mapping $\rho : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{sgl}(V)$ is said to be a representation of a 3-Lie superalgebra $\mathfrak{h}$ if the following conditions are satisfied*:

(1) ρ, *as a mapping between two super vector spaces, has grading zero, i.e.* $\rho : (\mathfrak{h} \otimes \mathfrak{h})_0 \to V_0$ *and* ρ : $(\mathfrak{h} \otimes \mathfrak{h})_1 \to V_1$, *or, equivalently,* $\widehat{\rho}\ (x,y) = \hat{x} + \hat{y}$,

(2) $\rho(x,y) = -(-1)^{\hat{x}\,\hat{y}}\,\rho(y,x)$,

(3) $[\rho(x,y),\rho(u,v)] = \rho([x,y,u],v) + (-1)^{\hat{u}\,\widehat{xy}}\,\rho(u,[x,y,v])$,

(4) $\rho([x,y,z],u) = \rho(x,y)\rho(z,u) + (-1)^{\hat{x}\,\widehat{yz}}\,\rho(y,z)\rho(x,u) + (-1)^{\hat{z}\,\widehat{xy}}\,\rho(z,x)\rho(y,u)$,
where $x,y,z,u,v \in \mathfrak{h}$. We will denote this representation of 3-Lie superalgebra $\mathfrak{h}$ in a super vector space V by $(\mathfrak{h},\rho,V)$.

An evident example of a representation of 3-Lie superalgebra is an adjoint representation. Fix two elements x,y of a 3-Lie superalgebra $\mathfrak{h}$ and for any $u \in \mathfrak{h}$ define $\mathrm{ad}_{(x,y)}u = [x,y,u]$. Hence, $\mathrm{ad} : \mathfrak{h} \otimes \mathfrak{h} \to \mathrm{End}(\mathfrak{h})$. Conditions 1 and 2 of Definition 4.1 immediately follow from the properties of graded ternary Lie bracket. In order to prove condition 3 of Definition 4.1, we calculate the graded commutator of two linear operators $\mathrm{ad}_{(x,y)}$ and $\mathrm{ad}_{(u,v)}$ by means of graded Filippov–Jacobi identity. Then we have

$$\begin{aligned}
[\mathrm{ad}_{(x,y)},\mathrm{ad}_{(u,v)}]\,z &= (\mathrm{ad}_{(x,y)}\mathrm{ad}_{(u,v)} - (-1)^{\widehat{xy}\,\widehat{uv}}\,\mathrm{ad}_{(u,v)}\mathrm{ad}_{(x,y)})z = [x,y,[u,v,z]] - (-1)^{\widehat{xy}\,\widehat{uv}}\,[u,v,[x,y,z]] \\
&= [[x,y,u],v,z] + (-1)^{\hat{u}\,\widehat{xy}}\,[u,[x,y,v],z] + \cancel{(-1)^{\widehat{xy}\,\widehat{uv}}\,[u,v,[x,y,z]]} - \cancel{(-1)^{\widehat{xy}\,\widehat{uv}}\,[u,v,[x,y,z]]} \\
&= [[x,y,u],v,z] + (-1)^{\hat{u}\,\widehat{xy}}\,[u,[x,y,v],z] = \mathrm{ad}_{([x,y,u],v)}z + (-1)^{\hat{u}\,\widehat{xy}}\,\mathrm{ad}_{(u,[x,y,v])}z.
\end{aligned}$$

The last property of Definition 4.1 can be checked as follows;

$$\mathrm{ad}_{([x,y,z],u)}v = [[x,y,z],u,v] = (-1)^{\widehat{uv}\,\widehat{xyz}}\,[u,v,[x,y,z]]. \tag{32}$$

Now making use of graded Filippov–Jacobi identity we obtain

$$[u,v,[x,y,z]] = [[u,v,x],y,z] + (-1)^{\hat{x}\,\widehat{uv}}\,[x,[u,v,y],z] + (-1)^{\widehat{uv}\,\widehat{xy}}\,[x,y,[u,v,z]].$$

Now we should substitute this expression into the right hand side of formula (32), but first we will calculate the sign of each term in resulting expression. In the term $[x,y,[u,v,z]]$, we will do the following permutation of the arguments $[x,y,[z,u,v]]$, which will entail multiplication by $(-1)^{\hat{z}\,\widehat{uv}}$. Therefore, the coefficient of this term will be (-1) in power

$$\widehat{uv}\,\widehat{xyz} + \widehat{uv}\,\widehat{xy} + \hat{z}\,\widehat{uv} = \widehat{uv}\,\widehat{xyz} + \widehat{uv}\,\widehat{xyz} = 0.$$

Analogously, we permute the arguments of the double bracket $[[u,v,x],y,z]$ as follows $[y,z,[x,u,v]]$ and this entails the appearance of the factor (-1) to power $\widehat{yz}\,\widehat{uvx} + \hat{x}\,\widehat{uv}$. All together it gives the following sign

$$\widehat{uv}\,\widehat{xyz} + \widehat{yz}\,\widehat{uvx} + \hat{x}\,\widehat{uv} = \widehat{uv}\,\widehat{yz} + \widehat{yz}\,\widehat{uvx} = \widehat{yz}\,(\widehat{uv} + \widehat{uvx}) = \widehat{xy}\,\hat{x}.$$

Similarly, we permute the arguments of the double bracket $[x,[u,v,y],z]$ to cast it into the form $[z,x,[y,u,v]]$, then calculate the sign, which turns out to be $\hat{z}\,\widehat{xy}$. Hence, we get

$$\mathrm{ad}_{([x,y,z],u)} = \mathrm{ad}_{(x,y)}\mathrm{ad}_{(z,u)} + (-1)^{\hat{x}\,\widehat{yz}}\,\mathrm{ad}_{(y,z)}\mathrm{ad}_{(x,u)} + (-1)^{\hat{z}\,\widehat{xy}}\,\mathrm{ad}_{(z,x)}\mathrm{ad}_{(y,u)}.$$

Now we assume that $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ is a 3-Lie superalgebra, $V = V_0 \oplus V_1$ is a super vector space and $\rho : \mathfrak{h} \otimes \mathfrak{h} \to \mathrm{End}(V)$ is a graded skew-symmetric mapping. Consider the direct sum $\mathfrak{h} \oplus V$. We equip it with a structure of super vector space if we associate grade 0 to elements $x+v$ (elements of even grade), where $x \in \mathfrak{h}_0, v \in V_0$, and grade 1 to elements $x+v$ (elements of odd grade), where $x \in \mathfrak{h}_1, v \in V_1$. Then $\mathfrak{h} \oplus V = (\mathfrak{h} \oplus V)_0 \oplus (\mathfrak{h} \oplus V)_1$, where $(\mathfrak{h} \oplus V)_0 = \mathfrak{h}_0 \oplus V_0$ and $(\mathfrak{h} \oplus V)_1 = \mathfrak{h}_1 \oplus V_1$. In analogy with representations of 3-Lie algebras [7], we define the ternary bracket on the super vector space $\mathfrak{h} \oplus V$ as follows:

$$\begin{aligned}[x_1+v_1, x_2+v_2, x_3+v_3] &= [x_1,x_2,x_3] + \rho(x_1,x_2)\,v_3 \\ &\quad + (-1)^{\widehat{x_1}\,\widehat{x_2x_3}}\,\rho(x_2,x_3)\,v_1 + (-1)^{\widehat{x_3}\,\widehat{x_1x_2}}\,\rho(x_3,x_1)\,v_2. \end{aligned} \tag{33}$$

It is easy to show that this ternary bracket is a graded ternary bracket. Indeed, if we assume that all arguments of this ternary bracket are homogenous elements of $\mathfrak{h} \oplus V$, then the grading of $x_i + v_i$ is equal to the grading of x_i (or v_i). Thus it is sufficient to show that the grading of ternary bracket (33) is $\widehat{x_1} + \widehat{x_2} + \widehat{x_3}$. But this is true, because the grading of the first term $[x_1,x_2,x_3]$ is $\widehat{x_1} + \widehat{x_2} + \widehat{x_3}$ and the grading of each term of the form $\rho(x_i,x_j)\,v_k$, where i,j,k is a cyclic permutation of 1,2,3, is the same integer, because

$$\widehat{x_i} + \widehat{x_j} + \widehat{v_k} = \widehat{x_i} + \widehat{x_j} + \widehat{x_k} = \widehat{x_1} + \widehat{x_2} + \widehat{x_3}.$$

The fact that this ternary bracket has the correct graded symmetries is checked on the permutation of the first two arguments x_1+v_1, x_2+v_2. Making use of the graded symmetry properties of a graded ternary Lie bracket in $\mathfrak{h}$ and the property 2 of Definition 4.1, we get

$$\begin{aligned}[x_2+v_2, x_1+v_1, x_3+v_3] &= [x_2,x_1,x_3] + \rho(x_2,x_1)\,v_3 + (-1)^{\widehat{x_2}\,\widehat{x_1x_3}}\,\rho(x_1,x_3)\,v_1 + (-1)^{\widehat{x_3}\,\widehat{x_1x_2}}\,\rho(x_3,x_2)\,v_1 \\ &= -(-1)^{\widehat{x_1}\,\widehat{x_2}}\,[x_1,x_2,x_3] - (-1)^{\widehat{x_1}\,\widehat{x_2}}\,\rho(x_1,x_2)\,v_3 - (-1)^{\widehat{x_1}\,\widehat{x_3}}\,\rho(x_2,x_3)\,v_1 \\ &\qquad - (-1)^{\widehat{x_1}\,\widehat{x_2} + \widehat{x_3}\,\widehat{x_1x_2}}\,\rho(x_3,x_1)\,v_2 \\ &= -(-1)^{\widehat{x_1}\,\widehat{x_2}}\,[x_1+v_1, x_2+v_2, x_3+v_3]. \end{aligned}$$

The following theorem extends the result, obtained in the paper [7] for 3-Lie algebras, to 3-Lie superalgebras. Actually, this extension is not complicated and consists of checking the rule for the consistency of signs that arise in the case of graded structures. Since we did not find formulation and proof of this theorem in the literature, we decided to give its proof here. Note that we will need this theorem for the induced representation, which will be discussed later in this paper.

Theorem 4. *Let $\mathfrak{h}=\mathfrak{h}_0\oplus\mathfrak{h}_1$ be a 3-Lie superalgebra, $V=V_0\oplus V_1$ be a super vector space, $\rho:\mathfrak{h}\otimes\mathfrak{h}\to\mathfrak{sgl}(V)$ be a graded skew-symmetric bilinear mapping. Then $(\mathfrak{h},\rho,V)$ is a representation of 3-Lie superalgebra $\mathfrak{h}$ in a super vector space V if and only if the direct sum of super vector spaces $\mathfrak{h}\oplus V$ equipped with the graded ternary bracket (33) is a 3-Lie superalgebra, or, in other words, the graded ternary bracket (33) satisfies the graded Filippov–Jacobi identity.*

Proof. First of all, we prove that if $(\mathfrak{h},\rho,V)$ is a representation of a 3-Lie superalgebra $\mathfrak{h}$, then the graded ternary bracket (33) defines the structure of 3-Lie superalgebra on the direct sum of super vector spaces $\mathfrak{h}\oplus V$. Since we have already proved that the ternary bracket (33) is a graded ternary bracket, the only thing we need to prove is that this bracket satisfies the graded Filippov-Jacobi identity. To this end, we introduce the following notations:

$$Y=y+v,\ Z=z+w,\ X_i=x_i+u_i,$$

where $i=1,2,3,\ y,z,x_i\in\mathfrak{h}$ and $v,w,u_i\in V$. We assume that all elements Y,Z,X_i are homogeneous with respect to super vector space structure of $\mathfrak{h}+V$. Evidently the grading of Y is equal to $\hat{y}$, grading of Z is $\hat{z}$ and the grading of X_i is $\widehat{x_i}$. Now our aim is to prove the graded Filippov–Jacobi identity for graded ternary bracket (33), that is, we need to show that the following expression

$$[Y,Z,[X_1,X_2,X_3]]-[[Y,Z,X_1],X_2,X_3]-(-1)^{\widehat{x_1}\,\widehat{yz}}[X_1,[Y,Z,X_2],X_3]-(-1)^{\widehat{x_1x_2}\,\widehat{yz}}[X_1,X_2,[Y,Z,X_3]] \quad (34)$$

is equal to zero. If we expand each double ternary bracket in this expression by means of (33), then, the $\mathfrak{h}$-component of resulting expression is

$$[y,z,[x_1,x_2,x_3]]-[[y,z,x_1],x_2,x_3]-(-1)^{\widehat{x_1}\,\widehat{yz}}[x_1,[y,z,x_2],x_3]-(-1)^{\widehat{x_1x_2}\,\widehat{yz}}[x_1,x_2,[y,z,x_3]] \quad (35)$$

and this is zero by virtue of the graded Filippov–Jacobi identity in a 3-Lie superalgebra $\mathfrak{h}$. The V-component of the resulting expression can be written in the form

$$\Psi_1(u_1)+\Psi_2(u_2)+\Psi_3(u_3)+\Psi_v(v)+\Psi_w(w), \quad (36)$$

where $\Psi_1,\Psi_2,\Psi_3,\Psi_v,\Psi_w\in\mathfrak{gl}(V)$ and

$$\begin{aligned}
\Psi_1 &= (-1)^{\widehat{x_1}\,\widehat{x_2x_3}}\Big([\rho(y,z),\rho(x_2,x_3)]-\rho([y,z,x_2],x_3)-(-1)^{\widehat{x_2}\,\widehat{yz}}\rho(x_2,[y,z,x_3])\Big),\\
\Psi_2 &= (-1)^{\widehat{x_3}\,\widehat{x_1x_2}}\Big([\rho(y,z),\rho(x_3,x_1)]-\rho([y,z,x_3],x_1)-(-1)^{\widehat{x_3}\,\widehat{yz}}\rho(x_3,[y,z,x_1])\Big),\\
\Psi_3 &= [\rho(y,z),\rho(x_1,x_2)]-\rho([y,z,x_1],x_2)-(-1)^{\widehat{x_1}\,\widehat{yz}}\rho(x_1,[y,z,x_2]),\\
\Psi_v &= (-1)^{\alpha}\Big(\rho([x_1,x_2,x_3],z)-\rho(x_1,x_2)\rho(x_3,z)\\
&\qquad -(-1)^{\widehat{x_1}\,\widehat{x_2x_3}}\rho(x_2,x_3)\rho(x_1,z)-(-1)^{\widehat{x_3}\,\widehat{x_1x_2}}\rho(x_3,x_1)\rho(x_2,z)\Big),\\
\Psi_w &= (-1)^{\beta}\Big(\rho([x_1,x_2,x_3],y)-\rho(x_1,x_2)\rho(x_3,y)\\
&\qquad -(-1)^{\widehat{x_1}\,\widehat{x_2x_3}}\rho(x_2,x_3)\rho(x_1,y)-(-1)^{\widehat{x_3}\,\widehat{x_1x_2}}\rho(x_3,x_1)\rho(x_2,y)\Big),
\end{aligned}$$

where $\alpha=\widehat{yz}\ \widehat{[x_1,x_2,x_3]}+\hat{y}\ \hat{z}+1,\beta=\widehat{yz}\ \widehat{[x_1,x_2,x_3]}$. Expressions Ψ_1,Ψ_2,Ψ_3 vanish by virtue of condition 3 of Definition 4.1 and expressions Ψ_v,Ψ_w by virtue of condition 4. Hence, the V-component of expression

(34) also vanishes and this means that the graded ternary bracket (33) is a graded ternary Lie bracket, i.e. it satisfies the graded Filippov-Jacobi identity.

Now we prove that if the graded ternary bracket (33) satisfies the graded Filippov–Jacobi identity, then $(\mathfrak{h},\rho,V)$ is a representation of 3-Lie superalgebra $\mathfrak{h}$. By other words, we assume that the expression (34) vanishes. Vanishing of the $\mathfrak{h}$-component of this expression gives us nothing, because it reduces to the graded Filippov-Jacobi identity in $\mathfrak{h}$, which already holds according to our assumption that $\mathfrak{h}$ is a 3-Lie superalgebra. From the equality to zero of the V-component, it immediately follows that the expression (36), where u_1,u_2,u_3,v,w are arbitrary vectors of V, is equal to zero. Taking $u_2=u_3=v=w=0$, we get $\Psi_1(u_1)=0$ for any u_1, which means that $\Psi_1=0$. Hence, condition 3 of Definition 4.1 is satisfied. Analogously we can prove that condition 4 is also satisfied and $(\mathfrak{g},\rho,V)$ is a representation of 3-Lie superalgebra. □

Recall that if a Lie algebra is equipped with a generalized trace, then one can construct the induced ternary Lie algebra (Section 3). This method of constructing the induced ternary Lie algebras can be extended by means of a generalized supertrace to Lie superalgebras, as was shown in [1,2]. Let $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ be a Lie superalgebra and $S\tau$ be a generalized supertrace of this Lie superalgebra. It can be proved then [2] that the graded ternary bracket

$$[x,y,z]=S\tau(x)\,[y,z]+(-1)^{\hat{x}\,\widehat{yz}}\,S\tau(y)\,[z,x]+(-1)^{\hat{z}\,\widehat{xy}}\,S\tau(z)\,[x,y],\ \ x,y,z\in\mathfrak{g} \tag{37}$$

determines the 3-Lie superalgebra on the super vector space of a Lie superalgebra $\mathfrak{g}$. We will call this 3-Lie superalgebra constructed by means of a generalized supertrace induced 3-Lie superalgebra. Particularly, if we have a representation $\pi:\mathfrak{g}\to\mathfrak{sgl}(V)$ of a Lie superalgebra $\mathfrak{g}$, then we construct the induced 3-Lie superalgebra (37) by simply using the supertrace of matrices in $\mathfrak{sgl}(V)$, i.e. we define the ternary bracket as follows:

$$[x,y,z]=\mathrm{Str}(\pi(x))\,[y,z]+(-1)^{\hat{x}\,\widehat{yz}}\,\mathrm{Str}(\pi(y))\,[z,x]+(-1)^{\hat{z}\,\widehat{xy}}\,\mathrm{Str}(\pi(z))\,[x,y],\ \ x,y,z\in\mathfrak{g}. \tag{38}$$

We will denote the induced 3-Lie superalgebra with graded ternary bracket (38) by $\mathfrak{tg}_\pi$. We can also extend the method of constructing induced representations of induced 3-Lie algebras to induced 3-Lie superalgebras.

Theorem 5. *Let $\mathfrak{g}$ be a Lie superalgebra and $\pi:\mathfrak{g}\to\mathfrak{sgl}(V)$ be a representation of $\mathfrak{g}$. Then mapping $\rho:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{sgl}(V)$, defined by the formula*

$$\rho(x,y)=Str(\pi(x))\,\pi(y)-(-1)^{\hat{x}\,\hat{y}}\,Str(\pi(y))\,\pi(x), \tag{39}$$

where $x,y\in\mathfrak{g}$, is a representation of induced 3-Lie superalgebra $\mathfrak{tg}_\pi$.

We will prove this theorem by means of Theorem 4 and the following lemma.

Lemma 6. *Let $\mathfrak{g}$ be a Lie superalgebra, $\pi:\mathfrak{g}\to\mathfrak{sgl}(V)$ be a representation of this Lie superalgebra. If we equip the super vector space $\mathfrak{g}\oplus V$ with the graded skew-symmetric bracket*

$$[\![x+v,y+w]\!]=[x,y]+\pi(x)\cdot w-(-1)^{\hat{x}\,\hat{y}}\,\pi(y)\cdot v, \tag{40}$$

where $x,y\in\mathfrak{g}$, $v,w\in V$ and $[x,y]$ is a Lie bracket in $\mathfrak{g}$, then the direct sum of two super vector spaces $\mathfrak{g}\oplus V$ becomes a Lie superalgebra, i.e. the graded skew-symmetric bracket (40) satisfies the graded Jacobi identity.

Proof. The proof of this lemma is simply to verify the graded Jacobi identity for the bracket (40). In order to simplify notations, we will denote $\mu_1=\widehat{x_1}\,\widehat{x_2x_3}$, $\mu_2=\widehat{x_2}\,\widehat{x_1x_3}$, $\mu_3=\widehat{x_3}\,\widehat{x_1x_2}$ and $\nu=\widehat{x_1}\,\widehat{x_2}+\widehat{x_2}\,\widehat{x_3}+\widehat{x_1}\,\widehat{x_3}$. Then the first term of the graded Jacobi identity can be expanded as follows:

$$[\![[\![x_1+v_1,x_2+v_2]\!],x_3+v_3]\!]=\underline{[[x_1,x_2],x_3]}+\pi([x_1,x_2])\cdot v_3-(-1)^{\mu_3}\pi(x_3)\,\pi(x_1)\cdot v_2+(-1)^{\nu}\pi(x_3)\pi(x_2)\cdot v_1.$$

The second term of the identity gives

$$(-1)^{\mu_1}[\![[\![x_2+v_2,x_3+v_3]\!],x_1+v_1]\!] = \underline{(-1)^{\mu_1}[[x_2,x_3],x_1]} + (-1)^{\mu_1}\pi([x_2,x_3])\cdot v_1 - \pi(x_1)\,\pi(x_2)\cdot v_3 \\ +(-1)^{\widehat{x_2}\,\widehat{x_3}}\,\pi(x_1)\pi(x_3)\cdot v_2$$

and the third one yields the expression

$$(-1)^{\mu_3}[\![[\![x_3+v_3,x_1+v_1]\!],x_2+v_2]\!] = \underline{(-1)^{\mu_3}[[x_3,x_1],x_2]} + (-1)^{\mu_3}\pi([x_3,x_1])\cdot v_2 - (-1)^{\mu_1}\pi(x_2)\,\pi(x_3)\cdot v_1 \\ +(-1)^{\widehat{x_1}\,\widehat{x_2}}\,\pi(x_2)\pi(x_1)\cdot v_3.$$

If we now take the sum of the left hand sides of these relations, we get the left hand side of the graded Jacobi identity for bracket (40). The sum of the right-hand sides of these relations gives zero. Indeed, terms underlined by a solid line add up to zero, because the graded Jacobi identity holds in the Lie superalgebra $\mathfrak{g}$. The terms underlined with dashed lines or not underlined at all also add up to zero due to the fact that the terms in each group simply cancel each other. □

Proof. Now we prove Theorem 5. According to Theorem 4, if we show that the graded ternary bracket (33), where the first term at the right hand side of (33) is the graded ternary bracket (38) and ρ is (39), determines 3-Lie superalgebra on the direct sum $\mathfrak{g}\oplus V$, then we prove that (39) is a representation of induced 3-Lie superalgebra $\mathfrak{tg}_\pi$. Substituting (38) and (39) into (33), we find

$$\begin{aligned}[x_1+v_1,x_2+v_2,x_3+v_3] \;=\; & \mathrm{Str}(\pi(x_1))[\![x_2+v_2,x_3+v_3]\!] + \mathrm{Str}(\pi(x_2))[\![x_3+v_3,x_1+v_1]\!] \\ & +\mathrm{Str}(\pi(x_3))[\![x_1+v_1,x_2+v_2]\!]. \end{aligned} \tag{41}$$

According to Lemma 6, the bracket $[\![x,y]\!]$ determines the structure of Lie superalgebra on $\mathfrak{g}\oplus V$. Thus, the graded ternary bracket (41) has the form of a graded ternary bracket for an induced 3-Lie superalgebra constructed with the help of a graded Lie bracket and the super trace. Hence, the graded ternary bracket (41) determines the induced 3-Lie superalgebra on $\mathfrak{g}\oplus V$ and therefore, (39) is a representation of 3-Lie superalgebra. □

5. CONCLUSIONS

The goal of the present paper was to construct a 3-Lie superalgebra on the basis of a given binary Lie superalgebra and study representations of constructed 3-Lie superalgebra induced by the representations of an initial binary Lie superalgebra. The 3-Lie superalgebra constructed in this way is called induced 3-Lie superalgebra. In the present paper the ternary graded Lie bracket of induced 3-Lie superalgebra was constructed with the help of a generalized trace and binary graded bracket of an initial Lie superalgebra. This method is applied to a commutative superalgebra with involution and even degree derivation. Furthermore, we proposed a method for constructing a representation of a 3-Lie algebra or superalgebra if a representation of an initial binary Lie algebra or superalgebra is given. We call this representation of induced 3-Lie algebra or superalgebra induced representation. It is shown that if a representation of an initial Lie algebra is reducible, then the induced representation of the induced 3-Lie algebra is reducible as well. In the case of the induced representation of induced 3-Lie superalgebra, we proposed conditions under which the induced representation is irreducible. Our hypothesis is that for the irreducibility of the induced representation of induced 3-Lie algebra, the irreducibility of an initial representation of a binary Lie algebra (without any additional conditions) is sufficient and we plan to consider this question in subsequent publications.

In the present paper we show that the induced representation of induced 3-Lie algebra maps the tensor square of a Lie algebra to the Lie subalgebra of traceless matrices $\mathfrak{sl}_N(\mathbb{C}) \subset \mathfrak{gl}_N(\mathbb{C})$, where N is the dimension of a representation space V. Next, assume $\pi : \mathfrak{g} \to \mathfrak{u}(V)$ is unitary representation of a Lie algebra $\mathfrak{g}$ in complex vector space V. Then $\pi(x)$ for any $x \in \mathfrak{g}$ is a skew-Hermitian matrix, i.e. $(\pi(x))^\dagger = -\pi(x)$. Then for any $x, y \in \mathfrak{g}$

$$(\rho(x,y))^\dagger = \mathrm{Tr}(\pi(x)^\dagger)\pi(y)^\dagger - \mathrm{Tr}(\pi(y)^\dagger)\pi(x)^\dagger = \rho(x,y),$$

where ρ is the induced representation of induced 3-Lie algebra. Hence, $\rho(x,y)$ is a Hermitian matrix and $i\rho(x,y)$ is a skew-Hermitian matrix, i.e. $i\rho(x,y) \in \mathfrak{su}(N)$. In the paper [6] the authors propose an analog of infinitesimal gauge transformation defined by means of ternary commutator

$$\delta A = i\,[X,Y,A], \tag{42}$$

where X,Y,A are Nth order complex matrices and the ternary bracket is the ternary bracket (1) of induced 3-Lie algebra, where τ is the usual trace of a matrix. In (42) a matrix A plays the role of gauge field and matrices X,Y can be considered as parameters of infinitesimal gauge transformation. Here we consider Nth order complex matrices as fundamental representations of Lie algebras $\mathfrak{sl}(V), \mathfrak{u}(V), \mathfrak{su}(V)$. It is important that if we apply (1) to the right hand side of gauge transformation (42), then there appears the induced representation ρ of induced 3-Lie algebra, which we introduced in the present paper, as follows:

$$\delta A = [i\rho(X,Y),A] + i\,\mathrm{Tr}(A)\,[X,Y]. \tag{43}$$

As $i\rho(X,Y) \in \mathfrak{su}(N)$, the first term at the right hand side of the above equation is the usual infinitesimal $\mathfrak{su}(N)$-gauge transformation and the parameter of this transformation is $\rho(X,Y)$. Thus, we see that, if we construct analogues of gauge transformations using the ternary commutator of the induced 3-Lie algebra, the representation of this induced 3-algebra makes it possible to establish a connection between these transformations and usual $\mathfrak{su}(N)$-gauge transformations. We plan to develop further this connection of induced representations of 3-Lie algebras with gauge theories.

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Indutseeritud 3-Lie algebrad, superalgebrad ja nende esitused

Viktor Abramov ja Priit Lätt

On näidatud, kuidas on kommutatiivse superalgebra $\mathscr{A}$ korral võimalik konstrueerida 3-Lie superalgebraid, kasutades selleks esialgse algebra involutsiooni ja paarisgradueeringuga dervivatsiooni või neid mõlemaid üheskoos. Artiklis on esitatud skeem Lie (super)algebra esituse ja (super)jälje põhjal vastava indutseeritud 3-Lie (super)algebra esituse konstrueerimiseks. Näitame, et 3-Lie algebra indutseeritud esitus on sisestatav Lie algebrasse $\mathfrak{sl}(V)$, kus V on esituse ruum. Juhul kui esituse dimensioon on 2, on leitud tingimused, mille korral vastava 3-Lie algebra indutseeritud esitus on taandumatu. On antud ka näide 3-Lie algebra taandumatust esitusest teist järku kompleksarvuliste maatriksite näol.