Proceedings of the Estonian Academy of Sciences,
2019, **68**, 1, 89–99
https://doi.org/10.3176/proc.2019.1.09
Available online at www.eap.ee/proceedings

TOPOLOGICAL ALGEBRAS

Products and coproducts in the category $S(B)$ of Segal topological algebras

Mart Abel

School of Digital Technologies, Tallinn University, Narva 29, 10120 Tallinn, Estonia
Institute of Mathematics and Statistics, University of Tartu, J. Liivi 2, 50409 Tartu, Estonia; mart.abel@tlu.ee, mart.abel@ut.ee

Received 12 September 2018, accepted 10 December 2018, available online 29 January 2019

Abstract. Let B be a topological algebra and $S(B)$ the category of Segal topological algebras. In the present paper we show that all coproducts of two objects of the category $S(B)$ always exist. We also find necesssary and sufficient conditions under which the products of two objects of the category $S(B)$ exist.

Key words: Segal topological algebra, category, product, coproduct.

1. INTRODUCTION

The study of Segal topological algebras started in [1]. It was followed by [2], where the category $S(B)$ of Segal topological algebras was represented as triples (A, f, B) where B was fixed. Further study of the category $S(B)$ was carried out in [3].

The present paper deals with the question of the existence of products and coproducts of objects in the category $S(B)$. While the coproducts exist always and have a form similar to the form of coproducts in the category of algebras, the products might or might not exist and have a bit different description, similar to the description of a Whitney sum known in the theory of fibre spaces.

Let us start by recalling the necessary definitions from [1] and [2].

A *topological algebra* is a topological linear space over the field $\mathbb{K}$ (where $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$), in which there is defined a separately continuous associative multiplication.

A topological algebra (A, τ_A) is a left (right or two-sided) *Segal topological algebra* in a topological algebra (B, τ_B) via an algebra homomorphism $f : A \to B$ if
(1) $\mathrm{cl}_B(f(A)) = B$;
(2) f is continuous;
(3) $f(A)$ is a left (respectively, right or two-sided) ideal of B.

Notice that condition (2) is equivalent to the following condition:

$$(2')\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\},$$

used in [1]. In what follows, we will denote a Segal topological algebra shortly by a triple (A, f, B).

From now on, we will fix a topological algebra (B, τ_B), which we will not change for this paper. The set $\mathrm{Ob}(S(B))$ of objects of the category $S(B)$ will consist of all Segal topological algebras in the same topological algebra B, i.e., all Segal algebras in the form of triples $(A, f, B), (C, g, B), \ldots.$

The set $\mathrm{Mor}((A, f, B), (C, g, B))$ of morphisms between Segal topological algebras (A, f, B) and (C, g, B) will consist of all continuous algebra homomorphisms $\alpha : A \to C$ with the property $g(\alpha(a)) = f(a)$ for every $a \in A$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle 1_B} \\ C & \xrightarrow{g} & B \end{array}.$$

In [2], it was shown that $S(B)$ is really a category, but not an additive category. In what follows, we will denote by $1_A : A \to A$ the identity map on A for every algebra A, i.e., $1_A(a) = a$ for every $a \in A$. It is easy to see that if B_0 is a dense left (right or two-sided) ideal of B, then $(B_0, 1_{B_0}, B) \in \mathrm{Ob}(S(B))$.

For conciseness of the text, we will write everywhere just “ideal” instead of “left (right or two-sided) ideal”. In what follows, every claim about “ideal” holds in all three cases. One just has to fix the sideness of all ideals and then to continue with the same sideness throughout the paper.

2. FREE PRODUCT OF TWO OBJECTS OF THE CATEGORY $S(B)$

In algebra it is known that, for any finite collection $V_1, \ldots, V_n$ of linear spaces, their tensor product $V_1 \otimes \cdots \otimes V_n$ is a linear space and consists of all finite sums of the form

$$\sum_{j=1}^{k} v_{j,i} \otimes \cdots \otimes v_{j,n},$$

where $k \in \mathbb{N}$ is finite and $v_{j,i} \in V_i$ for every $i \in \{1, \ldots, n\}$.

It is also known in algebra that, for any collection $(A_i)_{i\in\mathbb{N}}$ of linear spaces, their direct sum

$$\bigoplus_{n\in\mathbb{N}} A_n$$

consists of all tuples $(a_i)_{i\in\mathbb{N}}$ with $a_i \in A_i$ for every $i \in \mathbb{N}$ and $a_i = \theta_{A_i}$ for all but finitely many $i \in \mathbb{N}$. Hence, we can write a general element $(A_i)_{i\in\mathbb{N}} \neq (\theta_{A_i})_{i\in\mathbb{N}}$ of the direct sum of algebras $(A_i)_{i\in\mathbb{N}}$ in the form

$$(a_i)_{i\in\mathbb{N}} = \bigoplus_{l=1}^{k} b_l$$

for some $k \in \mathbb{Z}^+ = \{1, 2, \ldots\}$, where there exist $j_1, \ldots, j_k \in \mathbb{N}$ with $1 \leqslant j_1 < j_2 < j_k$ such that

$$a_i = \begin{cases} b_l & \text{if } i = j_l \text{ for some } l \in \{1, \ldots, k\} \\ \theta_{A_i}, & \text{otherways} \end{cases}.$$

For simplicity, let us denote the element $(\theta_{A_i})_{i\in\mathbb{N}}$ by

$$\bigoplus_{l=1}^{1} b_l,$$

where $b_1 = \theta_{A_1}$. By doing it, we can write every element of the direct sum in the form

$$\bigoplus_{l=1}^{k} b_l$$

for some $k \in \mathbb{Z}^+$, some $j_1, \ldots, j_k \in \mathbb{N}$ with $1 \leqslant j_1 < j_2 < j_k$, and some $b_1 \in A_{j_1}, b_2 \in A_{j_2} \ldots, b_k \in A_{j_k}$.

Next, we follow the ideas of [5], p. 9, about the free product of modules over a commutative unital ring. In our case, we will apply them to algebras and give the formulas for algebraic operations for the general element of the free product of two algebras over the field $\mathbb{K}$.

Let A and C be algebras, which are made disjoint by setting $a = (a, 1)$ and $c = (c, 2)$ for every $a \in A$ and $c \in C$ if $A \cap C \neq \emptyset$ originally. Consider the set

$$T = A \oplus C \oplus A \otimes A \oplus A \otimes C \oplus C \otimes A \oplus C \otimes C \oplus A \otimes A \otimes A \oplus \ldots.$$

By the aforementioned formulas, we can write every element of T in the form

$$t = \bigoplus_{i=1}^{n} \sum_{j=1}^{k_i} t_{i,j,1} \otimes \cdots \otimes t_{i,j,N_i}$$

for some $n, k_1, \ldots, k_n, N_1, \ldots, N_n \in \mathbb{Z}^+$ and for some $t_{i,j,l} \in A \cup C$.

For every

$$t = \bigoplus_{i=1}^{n} \sum_{j=1}^{k_i} t_{i,j,1} \otimes \cdots \otimes t_{i,j,N_i}, \quad s = \bigoplus_{i=1}^{m} \sum_{j=1}^{l_i} s_{i,j,1} \otimes \cdots \otimes s_{i,j,M_i}$$

and $\lambda \in \mathbb{K}$, define the operations on T as follows:

$$t + s = \bigoplus_{p=1}^{n+m} \sum_{j=1}^{q_p} r_{p,j,1} \otimes \cdots \otimes r_{p,j,K_p},$$

where

$$K_p = \begin{cases} N_p & \text{if } p \leqslant n \\ M_p & \text{if } n < p \end{cases}, \quad q_p = \begin{cases} k_p & \text{if } p \leqslant n \\ l_p & \text{if } n < p \end{cases}, \quad r_{p,j,i} = \begin{cases} t_{p,j,i} & \text{if } p \leqslant n \\ s_{p-n,j,i} & \text{if } n < p \end{cases},$$

$$\lambda t = \bigoplus_{i=1}^{n} \sum_{j=1}^{k_i} \lambda t_{i,j,1} \otimes \cdots \otimes t_{i,j,N_i},$$

and

$$ts = \bigoplus_{p=1}^{nm} \sum_{j=1}^{k_{\lceil \frac{p}{m} \rceil} l_{p-m\lfloor \frac{p-1}{m} \rfloor}} q_{p,j,1} \otimes \cdots \otimes q_{p,j,N_{\lceil \frac{p}{m} \rceil} + M_{p-m\lfloor \frac{p-1}{m} \rfloor}},$$

where

$$q_{p,j,i} = \begin{cases} t_{\lceil \frac{p}{m} \rceil, \left\lceil \frac{j}{k_{\lceil \frac{p}{m} \rceil}} \right\rceil, i} & \text{if } i \leqslant N_{\lceil \frac{p}{m} \rceil} \\ s_{p-m\lfloor \frac{p-1}{m} \rfloor, j-k_{\lceil \frac{p}{m} \rceil} \left\lfloor \frac{j-1}{k_{\lceil \frac{p}{m} \rceil}} \right\rfloor, i-N_{\lceil \frac{p}{m} \rceil}} & \text{if } N_{\lceil \frac{p}{m} \rceil} < i \end{cases}.$$

Then T becomes an algebra with respect to those operations. This algebra is called *the tensor algebra of A and C*.

Let

$$t_r = \sum_{j=1}^{k_r} t_{r,j,1} \otimes \cdots \otimes t_{r,j,N_r} = \bigoplus_{i=r}^{r} \sum_{j=1}^{k_i} t_{i,j,1} \otimes \cdots \otimes t_{i,j,N_i}$$

for every $r \in \{1, \ldots, n\}$. As the addition of tensor products in T is defined through direct sum,

$$\bigoplus_{i=1}^{n} \sum_{j=1}^{k_i} t_{i,j,1} \otimes \cdots \otimes t_{i,j,N_i} = \sum_{r=1}^{n} t_r = \sum_{i=1}^{n} \sum_{j=1}^{k_i} t_{i,j,1} \otimes \cdots \otimes t_{i,j,N_i}. \tag{1}$$

Similarly, as the multiplication of tensor products in T is defined through tensor multiplication, we have that

$$t_{i,j,1} \otimes \cdots \otimes t_{i,j,N_i} = \prod_{l=1}^{N_i} t_{i,j,l}, \tag{2}$$

when the elements $t_{i,j,l}$ are considered as elements of the direct summand A of T or of the direct summand C of T.

Suppose that $(A, f, B), (C, g, B) \in \mathrm{Ob}(S(B))$, let T be the tensor algebra of A and C and define a map $h_T : T \to B$ as follows:

$$h_T(t) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \prod_{l=1}^{N_i} \tilde{h_T}(t_{i,j,l}) \tag{3}$$

for every element

$$t = \bigoplus_{i=1}^{n} \sum_{j=1}^{k_i} t_{i,j,1} \otimes \cdots \otimes t_{i,j,N_i}$$

of T, where

$$\tilde{h_T}(t_{i,j}) = \begin{cases} f(t_{i,j,l}) & \text{if } t_{i,j,l} \in A \\ g(t_{i,j,l}) & \text{if } t_{i,j,l} \in C \end{cases}.$$

Let τ_{h_T} be the topology, induced on T by the map h_T, i.e., $\tau_{h_T} = \{h_T^{-1}(U) : U \in \tau_B\}$, where τ_B denotes the topology of B. Then h_T is a continuous map in the topology τ_{h_T}.

Notice that $h_T(s+t) = h_T(s) + h_T(t), h_T(st) = h_T(s)h_T(t)$, and $h_T(\lambda t) = \lambda h_T(t)$ for every $s, t \in T$ and $\lambda \in K$. Thus, h_T is an algebra homomorphism and $h_T(T)$ is closed with respect to the algebraic operations.

Next, let us show that the addition and scalar multiplication are continuous and multiplication is separately continuous in the topology τ_{h_T}. For this, let O be an arbitrary neighbourhood of zero in T, t be an arbitrary element of T, and λ an arbitrary scalar from $\mathbb{K}$. Then $h_T(t) \in B$ and there exists a neighbourhood O_B of zero in B such that $h_T^{-1}(O_B) \subseteq O$. Since the addition and scalar multiplication are continuous in B and the multiplication is separately continuous in B, there exist neighbourhoods U, V, and W of zero in B such that $U + U, \lambda V, h_T(t)W, Wh_T(t) \subseteq O_B$. Now, $h_T^{-1}(U), h_T^{-1}(V)$ and $h_T^{-1}(W)$ are neighbourhoods of zero in T such that

$$h_T^{-1}(U) + h_T^{-1}(U) \subseteq h_T^{-1}(U+U) \subseteq h_T^{-1}(O_B) \subseteq O,$$

$$\lambda h_T^{-1}(V) = h_T^{-1}(\lambda V) \subseteq h_T^{-1}(O_B) \subseteq O,$$

$$t h_T^{-1}(W) \subseteq h_T^{-1}(h_T(t)W) \subseteq h_T^{-1}(O_B) \subseteq O,$$

and

$$h_T^{-1}(W)t \subseteq h_T^{-1}(Wh_T(t)) \subseteq h_T^{-1}(O_B) \subseteq O.$$

Thus, T is a topological algebra.

Notice that $h_T(T)$ is an ideal of B. Its "sideness" is the same as it is for the dense ideals $f(A)$ and $g(C)$. If $f(A)$ and $g(C)$ are left ideals of B, then $bf(a) \in f(A)$ and $bg(c) \in g(C)$ for every $a \in A, b \in B$, and $c \in C$. Let

$$t = \bigoplus_{i=1}^{n} \sum_{j=1}^{k_i} t_{i,j,1} \otimes \cdots \otimes t_{i,j,N_i} \in T.$$

Suppose that $t_{i,j,1} \in A$ and $b \in B$. Then $\tilde{h_T}(t_{i,j,1}) \in f(A)$ and there exists $\hat{t}_{i,j,1} \in A$ such that

$$b\tilde{h_T}(t_{i,j,1}) = f(\hat{t}_{i,j,1}) = \tilde{h_T}(\hat{t}_{i,j,1}).$$

Similarly, if $t_{i,j,1} \in C$, then there exists $\hat{t}_{i,j,1} \in C$ such that $b\tilde{h_T}(t_{i,j,1}) = \tilde{h_T}(\hat{t}_{i,j,1})$. Therefore,

$$bh_T(t) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \Big(\tilde{h_T}(\hat{t}_{i,j,1}) \prod_{l=2}^{N_i} \tilde{h_T}(t_{i,j,l})\Big) \in h_T(T).$$

Similarly, if $f(A)$ and $g(C)$ are right ideals of B, we find elements $\hat{t}_{i,j,N_i}$ in A or C such that

$$h_T(t)b = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \Big(\Big(\prod_{l=1}^{N_i-1} \tilde{h_T}(t_{i,j,l})\Big) \tilde{h_T}(\hat{t}_{i,j,N_i})\Big) \in h_T(T).$$

Thus, $h_T(T)$ is an ideal of B, which has the same sideness as the ideals $f(A)$ and $g(C)$ had.

Moreover, since $f(A) \subset h_T(T)$ and $f(A)$ was dense in B, $h_T(T)$ is a dense left (right or two-sided) ideal of B. With this, we have proved the following result.

Lemma 2.1. *Let $(A, f, B), (C, g, B) \in \mathrm{Ob}(S(B))$ and let T be the tensor algebra of A and C. Define the map $h_T : T \to B$ as in (3) and equip T with the topology τ_{h_T}. Then $(T, h_T, B) \in \mathrm{Ob}(S(B))$.*

Let I be the two-sided ideal of T, generated by the set

$$\{a_1 \otimes a_2 - a_1a_2, c_1 \otimes c_2 - c_1c_2 : a_1, a_2 \in A, c_1, c_2 \in C\}.$$

Then $h_T(I) = \{\theta_B\}$, because

$$h_T(a_1 \otimes a_2 - a_1a_2) = f(a_1)f(a_2) - f(a_1a_2) = \theta_B = g(c_1)g(c_2) - g(c_1c_2) = h_T(c_1 \otimes c_2 - c_1c_2)$$

for every $a_1, a_2 \in A$ and $c_1, c_2 \in C$. Let $A \sqcup C = T/I$, equipped with the quotient topology $\tau_{A\sqcup C}$ (induced by the topology τ_{h_T}). Then $(A \sqcup C, \tau_{A\sqcup C})$ is a topological algebra.

In algebra, the set $A \sqcup C$ is called the *free product of A and C*.

Let $\kappa_I : T \to T/I = A \sqcup C$ be the quotient map and define a map $h : A \sqcup C \to B$ by $h(\kappa_I(t)) = h_T(t)$. Then h is correctly defined, because $h_T(I) = \{\theta_B\}$. Moreover, h is a continuous algebra homomorphism, taking $A \sqcup C$ to a dense ideal $h(A \sqcup C) = h_T(T)$ of B. With that, we have proved another result.

Lemma 2.2. *Let $(A, f, B), (C, g, B) \in \mathrm{Ob}(S(B))$ and let T be the tensor algebra of A and C. Define the map $h_T : T \to B$ as in (3) and equip T with the topology τ_{h_T}. Let I be the two-sided ideal of T, generated by the set*

$$\{a_1 \otimes a_2 - a_1a_2, c_1 \otimes c_2 - c_1c_2 : a_1, a_2 \in A, c_1, c_2 \in C\}$$

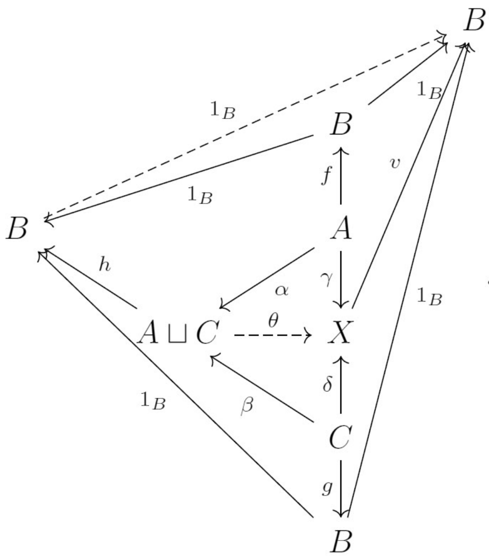
and $A \sqcup C = T/I$ be equipped with the quotient topology. Let $\kappa_I : T \to T/I$ be the quotient map. Then the triple $(A \sqcup C, h, B)$, where $h(\kappa_I(t)) = h_T(t)$ for every $t \in T$ and every $\kappa_I(t) \in A \sqcup C$, is an object of the category $S(B)$.

3. COPRODUCTS IN THE CATEGORY $S(B)$

Let us recall from [6] (see Definition in Chapter 5.1, p. 214) that the *coproduct* of the objects A and B of a category $\mathscr{C}$ is a triple $(A \sqcup B, \alpha, \beta)$, where $A \sqcup B$ is an object in $\mathscr{C}$ and $\alpha : A \to A \sqcup B, \beta : B \to A \sqcup B$ are morphisms of the category $\mathscr{C}$ such that for every object X in $\mathscr{C}$ and every pair of morphisms $f : A \to X, g : B \to X$ of $\mathscr{C}$ there exists a unique morphism $\theta : A \sqcup B \to X$ of $\mathscr{C}$ such that $\theta \circ \alpha = f$ and $\theta \circ \beta = g$.

Now we will formulate this definition for the category $S(B)$.

Definition 3.1. *The coproduct of* $(A, f, B), (C, g, B) \in \mathrm{Ob}(S(B))$ *is a triple* $((A \sqcup C, h, B), \alpha, \beta)$*, where* $(A \sqcup C, h, B) \in \mathrm{Ob}(S(B))$*,* $\alpha \in \mathrm{Mor}((A, f, B), (A \sqcup C, h, B)), \beta \in \mathrm{Mor}((C, g, B), (A \sqcup C, h, B))$ *such that for every* $(X, j, B) \in \mathrm{Ob}(S(B))$ *and every pair of morphisms* $\gamma \in \mathrm{Mor}((A, f, B), (X, j, B))$ *and* $\delta \in \mathrm{Mor}((C, g, B), (X, j, B))$ *there exists a unique morphism* $\theta \in \mathrm{Mor}((A \sqcup C, h, B), (X, j, B))$ *such that* $\theta \circ \alpha = \gamma$ *and* $\theta \circ \beta = \delta$


.

With this, we are ready to describe the coproducts in the category $S(B)$.

Proposition 3.2. *For any* $(A, f, B), (C, g, B) \in \mathrm{Ob}(S(B))$*, their coproduct in* $S(B)$ *exists and is the triple* $((A \sqcup C, h, B), \alpha, \beta)$*, where* $(A \sqcup C, h, B)$ *is the object of* $S(B)$ *described in Lemma* 2.2*,* $\alpha : A \to A \sqcup C$ *and* $\beta : C \to A \sqcup C$ *are morphisms defined by* $\alpha(a) = \kappa_I(a), \beta(c) = \kappa_I(c)$ *for all* $a \in A$*, and* $c \in C$*, where* κ_I *is the quotient map defined in Lemma* 2.2*.*

Proof. Let T be the tensor algebra of algebras A and C. Let $i_A : A \to T$ and $i_C : C \to T$ be the inclusion maps sending elements of A and B into the direct summands A and C of T, respectively, i.e., $i_A(a) = a \in A \subset T$ and $i_C(c) = c \in C \subset T$ for every $a \in A$ and $c \in C$. Then the maps i_A and i_C are continuous algebra homomorphisms. Moreover, the quotient map $\kappa_I : T \to A \sqcup C$ is a continuous algebra homomorphism. Hence, the maps $\alpha = \kappa_I \circ i_A$ and $\beta = \kappa_I \circ i_C$ are also continuous algebra homomorphisms.

Notice that $f(a) = h_T(a) = h_T(i_A(a))$ and $g(c) = h_T(c) = h_T(i_C(c))$ for all $a \in A$ and $c \in C$. Thus, $f = h_T \circ i_A$ and $g = h_T \circ i_C$. By Lemma 2.2, $h \circ \kappa_I = h_T$. Take any $a \in A$ and $c \in C$. Then

$$(h \circ \alpha)(a) = (h \circ (\kappa_I \circ i_A))(a) = ((h \circ \kappa_I) \circ i_A)(a) = (h_T \circ i_A)(a) = f(a)$$

and

$$(h \circ \beta)(c) = (h \circ (\kappa_I \circ i_C))(c) = ((h \circ \kappa_I) \circ i_C)(c) = (h_T \circ i_C)(c) = g(c).$$

Thus, we have demonstrated that $\alpha \in \mathrm{Mor}((A,f,B),(A\sqcup C,h,B))$ and $\beta \in \mathrm{Mor}((C,g,B),(A\sqcup C,h,B))$.

Take any $(X,v,B) \in \mathrm{Ob}(S(B))$, any $\gamma \in \mathrm{Mor}((A,f,B),(X,v,B))$, and any $\delta \in \mathrm{Mor}((C,g,B),(X,v,B))$. Then $v\circ\gamma = f$ and $v\circ\delta = g$.

Define a map $\omega : T \to X$ by

$$\omega(t) = \omega\Big(\bigoplus_{i=1}^{n}\sum_{j=1}^{k_i} t_{i,j,1}\otimes\cdots\otimes t_{i,j,N_i}\Big) := \sum_{i=1}^{n}\sum_{j=1}^{k_i}\prod_{l=1}^{N_i}\Omega(t_{i,j,l})$$

for every element

$$t = \bigoplus_{i=1}^{n}\sum_{j=1}^{k_i} t_{i,j,1}\otimes\cdots\otimes t_{i,j,N_i}$$

of T, where

$$\Omega(t_{i,j,l}) = \begin{cases}\gamma(t_{i,j,l}) & \text{if } t_{i,j,l}\in A\\ \delta(t_{i,j,l}) & \text{if } t_{i,j,l}\in C\end{cases}.$$

Then ω is an algebra homomorphism, because, by the definition of ω, $\omega(s+t) = \omega(s)+\omega(t)$, $\omega(st) = \omega(s)\omega(t), \omega(\lambda t) = \lambda\omega(t)$ for every $s,t\in T$ and $\lambda\in K$. Moreover, ω is continuous, because it is defined using continuous maps γ and δ and arithmetic operations, which are continuous.

Let $\theta : A\sqcup C \to X$ be defined by $\theta(\kappa_I(t)) = \omega(t)$. Then θ is also a continuous algebra homomorphism, because ω was a continuous algebra homomorphism and κ_I was an open algebra homomorphism.

Take any element y of $A\sqcup C$. Then there exists an element

$$t = \bigoplus_{i=1}^{n}\sum_{j=1}^{k_i} t_{i,j,1}\otimes\cdots\otimes t_{i,j,N_i}$$

of T such that $y = \kappa_I(t)$. Now, as v is an algebra homomorphism,

$$(v\circ\theta)(y) = v(\theta(\kappa_I(t))) = v(\omega(t)) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\prod_{l=1}^{N_i}(v\circ\Omega)(t_{i,j,l}).$$

Notice that

$$(v\circ\Omega)(t_{i,j,l}) = \begin{cases}(v\circ\gamma)(t_{i,j,l}) & \text{if } t_{i,j,l}\in A\\ (v\circ\delta)(t_{i,j,l}) & \text{if } t_{i,j,l}\in C\end{cases} = \begin{cases}f(t_{i,j,l}) & \text{if } t_{i,j,l}\in A\\ g(t_{i,j,l}) & \text{if } t_{i,j,l}\in C\end{cases} = \tilde{h_T}(t_{i,j,l})$$

for every $i\in\{1,\dots,n\}, j\in\{1,\dots,k_i\}$, and $l\in\{1,\dots,N_i\}$. Therefore,

$$(v\circ\theta)(y) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\prod_{l=1}^{N_i}\tilde{h_T}(t_{i,j,l}) = h_T(t) = h(\kappa_I(t)) = h(y).$$

Thus, $v\circ\theta = h$. Hence, $\theta\in\mathrm{Mor}((A\sqcup C,h,B),(X,v,B))$.

It is also easy to check that $(\theta\circ\alpha)(a) = \gamma(a)$ for every $a\in A$ and that $(\theta\circ\beta)(c) = \delta(c)$ for every $c\in C$. Thus, $\theta\circ\alpha = \gamma$ and $\theta\circ\beta = \delta$.

Take any $\psi\in\mathrm{Mor}((A\sqcup C,h,B),(X,j,B))$ such that $\psi\circ\alpha = \gamma$ and $\psi\circ\beta = \delta$. As ψ is an algebra homomorphism,

$$\psi(I) = \psi(\theta_{A\sqcup C}) = \{\theta_X\}.$$

Take any $y\in A\sqcup C$. Then there exists an element

$$t = \bigoplus_{i=1}^{n}\sum_{j=1}^{k_i} t_{i,j,1}\otimes\cdots\otimes t_{i,j,N_i}$$

of T such that $y = \kappa_I(t)$. By using first (2) and then (1), we obtain that

$$\psi(y) = \psi(\kappa_I(t)) = \psi\Big(\kappa_I\Big(\bigoplus_{i=1}^{n}\sum_{j=1}^{k_i} t_{i,j,1}\otimes\cdots\otimes t_{i,j,N_i}\Big)\Big) = \psi\Big(\kappa_I\Big(\bigoplus_{i=1}^{n}\sum_{j=1}^{k_i}\prod_{l=1}^{N_i} t_{i,j,l}\Big)\Big) = \psi\Big(\kappa_I\Big(\sum_{i=1}^{n}\sum_{j=1}^{k_i}\prod_{l=1}^{N_i} t_{i,j,l}\Big)\Big).$$

Using the facts that κ_I and ψ are algebra homomorphisms and that $t_{i,j,l} = \sigma(t_{i,j,l})$, where

$$\sigma(t_{i,j,l}) = \begin{cases} \alpha(t_{i,j,l}) = t_{i,j,l} & \text{if } t_{i,j,l}\in A \\ \beta(t_{i,j,l}) = t_{i,j,l} & \text{if } t_{i,j,l}\in C \end{cases},$$

we obtain that

$$\begin{aligned} \psi(y) &= \sum_{i=1}^{n}\sum_{j=1}^{k_i}\prod_{l=1}^{N_i}\psi(\kappa_I(t_{i,j,l})) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\prod_{l=1}^{N_i}\psi(t_{i,j,l}+I) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\prod_{l=1}^{N_i}(\psi(t_{i,j,l})+\psi(I)) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{k_i}\prod_{l=1}^{N_i}\psi(\sigma(t_{i,j,l})) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\prod_{l=1}^{N_i}\Omega(t_{i,j,l}) = \omega(t) = \theta(\kappa_I(t)) = \theta(y). \end{aligned}$$

Therefore, $\psi = \theta$ and $\theta \in \mathrm{Mor}((A\sqcup C,h,B),(X,v,B))$ is a unique morphism with the properties $\theta\circ\alpha = \gamma$ and $\theta\circ\beta = \delta$.

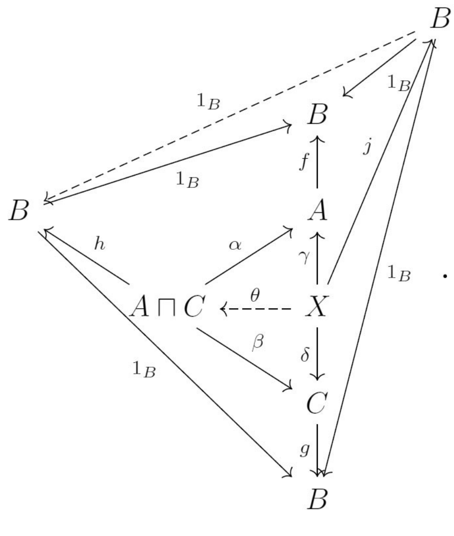
Consequently, $(A\sqcup C,h,B)$ is the coproduct of (A,f,B) and (C,g,B). □

4. PRODUCTS IN $S(B)$

Let us recall from [6] (see Definition in Chapter 5.1, p. 217) that the *product* of the objects A and B of a category $\mathscr{C}$ is a triple $(A\sqcap B,p,q)$, where $A\sqcap B$ is an object in $\mathscr{C}$ and $p: A\sqcap B\to A, q: A\sqcap B\to B$ are morphisms of the category $\mathscr{C}$ such that for every object X in $\mathscr{C}$ and every pair of morphisms $f: X\to A, g: X\to C$ of $\mathscr{C}$ there exists a unique morphism $\theta: X\to A\sqcap B$ of $\mathscr{C}$ such that $p\circ\theta = f$ and $q\circ\theta = g$.

Now we will formulate this definition for the category $S(B)$.

Definition 4.1. *The product of* $(A,f,B),(C,g,B)\in\mathrm{Ob}(S(B))$ *is a triple* $((A\sqcap C,h,B),\alpha,\beta)$*, where* $(A\sqcap C,h,B)\in\mathrm{Ob}(S(B))$, $\alpha\in\mathrm{Mor}((A\sqcap C,h,B),(A,f,B)), \beta\in\mathrm{Mor}((A\sqcap C,h,B),(C,g,B))$ *such that for every* $(X,j,B)\in\mathrm{Ob}(S(B))$ *and every pair of morphisms* $\gamma\in\mathrm{Mor}((X,j,B),(A,f,B))$, $\delta\in\mathrm{Mor}((X,j,B),(C,g,B))$ *there exists a unique morphism* $\theta\in\mathrm{Mor}((X,j,B),(A\sqcap C,h,B))$ *such that* $\alpha\circ\theta = \gamma$ *and* $\beta\circ\theta = \delta$



.

Let us remind that, when A and C are topological algebras, then $A \times C = \{(a,c) : a \in A, c \in C\}$, equipped with the product topology, is also a topological algebra with respect to the algebraic operations defined by

$$(a_1,c_1)+(a_2,c_2) = (a_1+a_2,c_1+c_2), \;\; \lambda(a_1,c_1) = (\lambda a_1, \lambda c_1), \;\text{ and }\; (a_1,c_1)(a_2,c_2) = (a_1a_2,c_1c_2)$$

for all $(a_1,c_1),(a_2,c_2) \in A \times C$ and every $\lambda \in \mathbb{K}$.

In the case of the category of modules over a fixed ring, the product of objects A and C was defined to be $A \sqcap C = A \times C$ and the maps $\alpha = \mathrm{pr}_A : A \times C \to A$ and $\beta = \mathrm{pr}_C : A \times C \to C$ were chosen as projections.

In the category $S(B)$, the conditions $\alpha \in \mathrm{Mor}((A \sqcap C,h,B),(A,f,B))$ and $\beta \in \mathrm{Mor}((A \sqcap C,h,B),(C,g,B))$ induce the condition $f \circ \alpha = h = g \circ \beta$. Hence, choosing $A \sqcap C = A \times C$, $\alpha = \mathrm{pr}_A$, and $\beta = \mathrm{pr}_C$, we would have the condition $f(a) = h((a,c)) = g(c)$, which is not true for all $(a,c) \in A \times C$, in general. Thus, we have to limit ourselves to some subset $D := \{(a,c) \in A \times C : f(a) = g(c)\}$ of $A \times C$.

The construction of D is similar to the construction of the Whitney sum, known for fibre bundles. The difference in our case is that, unlike the case of the Whitney sums of fibre bundles, not all elements $b = f(a)$ of the image $f(A)$ have to have such $c \in C$ that $g(c) = b$, because we do not demand that $f(A) = g(C)$.

Fortunately, D is still an algebra and, choosing the subspace topology on D, induced by the product topology of $A \times C$, we still obtain a topological algebra and are able to define $h : D \to B$ by $h((a,c)) = f(a) = g(c)$. But now we can not guarantee that $h(D)$ is dense in B. We faced a similar situation (with a bit more difficult obstacles) in [2], while we were describing the equalizers in the category $S(B)$.

Let us continue with a result similar to Lemma 2 from [2].

Lemma 4.2. *Let* $(A,f,B),(C,g,B) \in \mathrm{Ob}(S(B))$,

$$D = \{(a,c) \in A \times C : f(a) = g(c)\},$$

and $h : D \to B$ *be defined by* $h((a,c)) = f(a) = g(c)$ *for every* $(a,c) \in D$. *Consider on* $A \times C$ *the product topology induced by the topologies of* A *and* C *and consider on* D *the subspace topology* τ_D *induced by the product topology on* $A \times C$. *If* $\tilde{D}$ *is a subalgebra of* D, *equipped with the subspace topology, such that* $h(\tilde{D})$ *is a dense ideal of* B, *then* $(\tilde{D},h|_{\tilde{D}},B) \in \mathrm{Ob}(S(B))$, $\mathrm{pr}_A|_{\tilde{D}} \in \mathrm{Mor}((\tilde{D},h|_{\tilde{D}},B),(A,f,B))$, *and* $\mathrm{pr}_C|_{\tilde{D}} \in \mathrm{Mor}((\tilde{D},h|_{\tilde{D}},B),(C,g,B))$.

Proof. By the definition of $\tilde{D}$, conditions (1) and (3) of the Segal topological algebra are fulfilled. It is easy to see, by the definition of h, that

$$h|_{\tilde{D}} = f \circ \mathrm{pr}_A|_{\tilde{D}} = g \circ \mathrm{pr}_C|_{\tilde{D}}\,.$$

As f, g, pr_A, pr_C are all continuous algebra homomorphisms, $\mathrm{pr}_A|_{\tilde{D}}$, $\mathrm{pr}_C|_{\tilde{D}}$, and $h|_{\tilde{D}}$ are also continuous algebra homomorphisms. Thus, condition (2) of the Segal topological algebra is fulfilled. Hence,

$$(\tilde{D},h|_{\tilde{D}},B) \in \mathrm{Ob}(S(B)).$$

From the first part of the proof, we also conclude that

$$\mathrm{pr}_A|_{\tilde{D}} \in \mathrm{Mor}((\tilde{D},h|_{\tilde{D}},B),(A,f,B))$$

and

$$\mathrm{pr}_C|_{\tilde{D}} \in \mathrm{Mor}((\tilde{D},h|_{\tilde{D}},B),(C,g,B)).$$ □

Remark 4.3. When the present paper had been submitted and was waiting for the opinion of the referees, another paper ([4]) was written, where the situation of products in the category $S(B)$ was studied for an arbitrary collection of objects in $S(B)$ instead of the product of just two objects. Therefore, several results of the present paper become as a special case of more general results and will be given here without proofs.

Now we are ready to give a sufficient condition in order to have a product in the category $S(B)$.

Lemma 4.4. *Let $(A,f,B),(C,g,B) \in \mathrm{Ob}(S(B))$ and take D and h as in Lemma 4.2. Then $h(D)$ is an ideal of B. If $B_0 = f(A) \cap g(C)$ is dense in B, then there exists a product $((D,h,B), \mathrm{pr}_A|_{\tilde{D}}, \mathrm{pr}_C|_{\tilde{D}})$ of (A,f,B) and (C,g,B).*

Proof. See the proof of Proposition 1 in [4], pp. 29–31 and take there $I = \{1,2\}$, $(A_1, f_1, B) = (A,f,B)$, and $(A_2, f_2, B) = (C,g,B)$. □

Next, we will show that the denseness condition is not only sufficient for the existence of a product but also necessary.

Lemma 4.5. *Let $(A,f,B),(C,g,B) \in \mathrm{Ob}(S(B))$. If the product of (A,f,B) and (C,g,B) exists in the category $S(B)$, then $f(A) \cap g(C)$ is dense in B.*

Proof. See the proof of Proposition 2 in [4], pp. 31–32 and take there $I = \{1,2\}$, $(A_1, f_1, B) = (A,f,B)$, and $(A_2, f_2, B) = (C,g,B)$. □

From Lemma 4.4 and Lemma 4.5, we will obtain the following result.

Theorem 4.6. *Let $(A,f,B),(C,g,B) \in \mathrm{Ob}(S(B))$. The product of (A,f,B) and (C,g,B) exists in the category $S(B)$ if and only if $f(A) \cap g(C)$ is dense in B. If the product of (A,f,B) and (C,g,B) exists in the category $S(B)$, then it is isomorphic to the triple $((D,h,B), \mathrm{pr}_A|_D, \mathrm{pr}_C|_D)$, where $D = \{(a,c) \in A \times C : f(a) = g(c)\}$, $h : D \to B$ is defined by $h((a,c)) = f(a) = g(c)$ for every $(a,c) \in D$ and $\mathrm{pr}_A : A \times C \to A$ and $\mathrm{pr}_C : A \times C \to C$ are the projections of $A \times C$ to A and C, respectively.*

Proof. Suppose that $f(A) \cap g(C)$ is dense in B. Then the product of (A,f,B) and (C,g,B) exists in the category $S(B)$, by Lemma 4.4. Moreover, Lemma 4.4 also tells us that the product of (A,f,B) and (C,g,B) is of the form $((D,h,B), \mathrm{pr}_A|_D, \mathrm{pr}_C|_D)$, described in the text of Theorem 4.6.

Suppose that the product $((A \sqcap C, h, B), \alpha, \beta)$ of (A,f,B) and (C,g,B) exists in the category $S(B)$. Then $f(A) \cap g(C)$ is dense in B, by Lemma 4.5. Now, by Lemma 4.4, we know that the triple $((D,h,B), \mathrm{pr}_A|_D, \mathrm{pr}_C|_D)$, described above, is also the product of (A,f,B) and (C,g,B). By Proposition 5.7 in [6], p. 218, we know that any two products of two fixed objects of a category are isomorphic. Hence, the product $((A \sqcap C, h, B), \alpha, \beta)$ of (A,f,B) and (C,g,B) is isomorphic to $((D,h,B), \mathrm{pr}_A|_D, \mathrm{pr}_C|_D)$. □

With Theorem 4.6, we have transferred the problem of the existence of products in the category $S(B)$ to the problem of denseness of the intersection of two dense ideals (of the same sideness).

Since $(B_0, 1_{B_0}, B) \in S(B)$ for every dense ideal B_0 of B, in order to ensure that all products in the category $S(B)$ exist, we have to ensure that the intersection $B_0 \cap B_1$ is dense in B for every pair of dense ideals B_0 and B_1 of B. Although we are at the moment unable to describe the class of all topological algebras where the intersection of any two dense ideals (of the same sideness) is dense in the whole algebra, we can have at least a criterion for the existence of products in the category $S(B)$.

Corollary 4.7. *Let B be a topological algebra. Then all coproducts in the category $S(B)$ always exist. Moreover, all products in the category $S(B)$ exist if and only if B has the property*

(*) *the intersection of any two dense ideals (of the same sideness) of B is dense in B.*

5. OPEN QUESTIONS

(1) Is the intersection of two dense ideals (of the same sideness) of a topological algebra always dense?

(2) Describe all topological algebras, where the intersection of any two dense ideals (of the same sideness) is again a dense ideal (of the same sideness) of the same topological algebra.

6. CONCLUSIONS

In the present paper, we showed that all coproducts of elements of the category $S(B)$ of Segal topological algebras exist for every topological algebra B. We also found a necessary and sufficient condition for a topological algebra B under which all products of elements of the category $S(B)$ exist.

ACKNOWLEDGEMENTS

The research was supported by the institutional research funding IUT20-57 of the Estonian Ministry of Education and Research. The publication costs of this article were covered by the Estonian Academy of Sciences.

REFERENCES

1. Abel, M. Generalisation of Segal algebras for arbitrary topological algebras. *Period. Math. Hung.*, 2018, **77**(1), 58–68.
2. Abel, M. About some categories of Segal topological algebras. *Poincare J. Anal. Appl.*, (submitted).
3. Abel, M. Initial, terminal and zero objects in the category $S(B)$ of Segal topological algebras. In *Proceedings of the ICTAA 2018; Math. Stud. (Tartu)*, 2018, **7**, 7–24.
4. Abel, M. About products in the category $S(B)$ of Segal topological algebras. In *Proceedings of the ICTAA 2018; Math. Stud. (Tartu)*, 2018, **7**, 25–32.
5. Beidar, K. I., Martindale III, W. S., and Mikhalev, A. V. *Rings with Generalized Identities.* Monographs and Textbooks in Pure and Applied Mathematics, **196**. Marcel Dekker, Inc., New York, 1996.
6. Rotman, J. J. *An Introduction to Homological Algebra.* Second Edition. Springer, New York, 2009.

Korrutised ja kokorrutised Segali topoloogiliste algebrate kategoorias $S(B)$

Mart Abel

Olgu B topoloogiline algebra. Käesolevas töös on näidatud, et Segali topoloogiliste algebrate kategoorias $S(B)$ leiduvad kõik kokorrutised. Lisaks on topoloogilise algebra B jaoks leitud tarvilikud ja piisavad tingimused selleks, et ka kõik korrutised kategoorias $S(B)$ eksisteeriksid.