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Topological spectrum of elements in topological algebras

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Abstract. Properties of the sets of left, right, and two-sided topologically quasi-invertible elements, topological spectra, and topological spectral radii of elements in (not necessarily unital or commutative) topological algebras are studied. We prove the spectral mapping theorem for the topological spectrum of elements in commutative complex (not necessarily unital) topological algebras and show that the topological spectral radius (as a map) is a submultiplicative seminorm in a topological algebra with a functional topological spectrum.

Key words: topological algebra, topological spectrum of an element, F-algebra, continuity of quasi-invertion, functional topological spectrum.

1. INTRODUCTION

First of all we introduce all the notions that will be used later on.

1.1. A *topological algebra* A is a topological linear space over the field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) with an associative *separately continuous* multiplication that turns A into an algebra over $\mathbb{K}$.

An element $x \in A$ is *left* (*right*) *quasi-invertible* in A if there exists an element $y \in A$ (respectively, $z \in A$) such that $y \circ x = y + x - yx = \theta_A$ (respectively, $x \circ z = x + z - xz = \theta_A$). Here and later on, we denote the zero element in A by θ_A. An element $x \in A$ is *quasi-invertible* if it is left and right quasi-invertible. The set of all left (right) quasi-invertible elements in A is denoted by $\mathrm{Qinv}_\ell(A)$ (respectively, by $\mathrm{Qinv}_r(A)$) and the set of all quasi-invertible elements in A by $\mathrm{Qinv}(A)$.

An element $x \in A$ is topologically *left* (*right*) *quasi-invertible* in A if there exists a net $(y_\lambda)_{\lambda\in\Lambda}$ (respectively, $(z_\mu)_{\mu\in\Delta}$) of elements of A such that $(y_\lambda \circ x)_{\lambda\in\Lambda}$ (respectively, $(x \circ z_\mu)_{\mu\in\Delta}$) converges to zero in A. An element $x \in A$ is *topologically quasi-invertible* if it is topologically left and right quasi-invertible. The set of all topologically left (right) quasi-invertible elements in A is denoted by $\mathrm{Tqinv}_\ell(A)$ (respectively, by $\mathrm{Tqinv}_r(A)$) and the set of all topologically quasi-invertible elements in A by $\mathrm{Tqinv}(A)$.

Let A be a topological algebra with unit e. The set of all invertible elements in A is denoted by $\mathrm{Inv}(A)$. Then $\{e\} - \mathrm{Qinv}(A) = \mathrm{Inv}(A)$. Similar equalities hold for left and right invertible elements, quasi-invertible elements, topologically invertible elements, and so on.

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A topological algebra A is a *Q-algebra* if the set $\mathrm{Qinv}(A)$ is open in A, and A is a *TQ-algebra* if the set $\mathrm{Tqinv}(A)$ is open in A. The *left* (*right*) TQ-algebra is defined in a similar way. In particular, when

$$\mathrm{Tqinv}_\ell(A) = \mathrm{Qinv}_\ell(A) \quad (\mathrm{Tqinv}_r(A) = \mathrm{Qinv}_r(A))$$

A is a *left* (respectively, *right*) *advertive algebra* and an *advertive algebra*, when

$$\mathrm{Tqinv}(A) = \mathrm{Qinv}(A).$$

Remark. We know that every Q-algebra is a TQ-algebra (see [1], Proposition 2). But the converse is not true in general, for example: Let $A = \mathbb{C}[t]$ be the unital algebra of all polynomials in $[0,1]$ with complex coefficients endowed with the algebra norm

$$P \to \|P\| = \sum_{i=0}^{n} |a_i| \text{ with } P(t) = \sum_{i=0}^{n} a_i t^i$$

for each $t \in [0,1]$. Then A is a TQ-algebra that is not a Q-algebra (see [5] or [6], p. 73).

1.2. Let A be a topological algebra and $x \in A$. The *spectrum* of x is defined by

$$\mathrm{sp}(x) = \{\lambda \in \mathbb{C}\backslash\{0\} : \lambda^{-1}x \notin \mathrm{Qinv}(A)\} \cup \{0\};$$

the *left* (*right*) *spectrum* of x by

$$\mathrm{sp}_\ell(x) = \{\lambda \in \mathbb{C}\backslash\{0\} : \lambda^{-1}x \notin \mathrm{Qinv}_\ell(A)\} \cup \{0\}$$

$$(\text{respectively, by } \mathrm{sp}_r(x) = \{\lambda \in \mathbb{C}\backslash\{0\} : \lambda^{-1}x \notin \mathrm{Qinv}_r(A)\} \cup \{0\});$$

the *topological spectrum* of x by

$$\mathrm{sp}^t(x) = \{\lambda \in \mathbb{C}\backslash\{0\} : \lambda^{-1}x \notin \mathrm{Tqinv}(A)\} \cup \{0\},$$

and the *left* (*right*) *topological spectrum* of x by

$$\mathrm{sp}^t_\ell(x) = \{\lambda \in \mathbb{C}\backslash\{0\} : \lambda^{-1}x \notin \mathrm{Tqinv}_\ell(A)\} \cup \{0\}$$

$$(\text{respectively, } \mathrm{sp}^t_r(x) = \{\lambda \in \mathbb{C}\backslash\{0\} : \lambda^{-1}x \notin \mathrm{Tqinv}_r(A)\} \cup \{0\}).$$

It is clear that $\mathrm{sp}^t(x) \subseteq \mathrm{sp}(x)$.

Moreover, the *spectral radius* of x is defined by

$$\mathrm{r}(x) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(x)\};$$

the *left* (*right*) *spectral radius* of x by

$$\mathrm{r}_\ell(x) = \sup\{|\lambda| : \lambda \in \mathrm{sp}_\ell(x)\} \ \ (\text{respectively}, \mathrm{r}_r(x) = \sup\{|\lambda| : \lambda \in \mathrm{sp}_r(x)\}),$$

the *topological spectral radius* of x by

$$\mathrm{r}^t(x) = \sup\{|\lambda| : \lambda \in \mathrm{sp}^t(x)\}$$

and the *left* (*right*) *topological spectral radius* of x by

$$\mathrm{r}^t_\ell(x) = \sup\{|\lambda| : \lambda \in \mathrm{sp}^t_\ell(x)\} \ \ (\text{respectively, } \mathrm{r}^t_r(x) = \sup\{|\lambda| : \lambda \in \mathrm{sp}^t_r(x)\}).$$

Then $\mathrm{r}^t(x) \leq \mathrm{r}(x)$ for each $x \in A$.

1.3. For every topological algebra A the set of all non-zero multiplicative linear functionals on A is denoted by $\mathfrak{m}^\sharp(A)$ and the set of all continuous elements in $\mathfrak{m}^\sharp(A)$ by $\mathfrak{m}(A)$.

Let A be a topological algebra such that the set $\mathfrak{m}(A)$ is not empty and $\widehat{x}$ denote the Gelfand transform of x, that is $\widehat{x}(f) = f(x)$ for each $f \in \mathfrak{m}(A)$. Then $\widehat{x}(\mathfrak{m}(A)) \subset \mathrm{sp}^t_\ell(x)$, $\widehat{x}(\mathfrak{m}(A)) \subset \mathrm{sp}^t_r(x)$, and $\widehat{x}(\mathfrak{m}(A)) \subset \mathrm{sp}^t(x)$ for each $x \in A$. We say that A has a *functional left (right or two-sided) topological spectrum* if $\widehat{x}(\mathfrak{m}(A)) = \mathrm{sp}^t_\ell(x)$ (respectively, $\widehat{x}(\mathfrak{m}(A)) = \mathrm{sp}^t_r(x)$ or $\widehat{x}(\mathfrak{m}(A)) = \mathrm{sp}^t(x)$) for each $x \in A$.

1.4. When the underlying topological linear space of a topological algebra A is locally pseudoconvex, then A is called a *locally pseudoconvex algebra*. In this case A has a base $\{U_\lambda : \lambda \in \Lambda\}$ of neighbourhoods of zero consisting of *balanced* (that is, $\mu U_\lambda \subset U_\lambda$ whenever $|\mu| \leq 1$) and *pseudoconvex* (that is, $U_\lambda + U_\lambda \subset \mu U_\lambda$ for some $\mu \geq 2$) sets.

It is well known (see [8], p. 6) that the topology of a locally pseudoconvex algebra A can be given by means of a family $P = \{p_\lambda : \lambda \in \Lambda\}$ of κ_λ-homogeneous seminorms (that is, $p_\lambda(\alpha x) = |\alpha|^{\kappa_\lambda} p_\lambda(x)$ for each $\alpha \in \mathbb{K}$ and $x \in A$), where $\kappa_\lambda \in (0,1]$ for each $\lambda \in \Lambda$. In case every $p_\lambda \in P$ is submultiplicative (that is, $p_\lambda(xy) \leq p_\lambda(x)p_\lambda(y)$ for each $x, y \in A$), A is called a *locally m-pseudoconvex algebra.*

1.5. A *regular* left (right) ideal of A is a left (respectively, right) ideal I of A for which there exists an element $a \in A$ such that $xa - x \in I$ (respectively, $ax - x \in I$) for all $x \in A$ and a two-sided ideal I of A is regular if $ax - x \in I$ and $xa - x \in I$ for all $x \in A$. Note that a two-sided ideal I is regular if the quotient algebra A/I has $[a] = a + I$ as a unit element. In these cases a is a *unit modulo I*.

1.6. In this paper, we will study the properties of left, right, and two-sided topologically quasi-invertible elements;
left, right, and two-sided topological spectra of an element; and left, right, and two-sided topological spectral radii of an element in (not necessarily unital or commutative) topological algebras. We show that (a) the sets $\mathrm{Tqinv}_\ell(A)$, $\mathrm{Tqinv}_r(A)$, and $\mathrm{Tqinv}(A)$ are G_δ-sets in F-algebras; (b) if A is a commutative complex (not necessarily unital) topological algebra, then the topological spectrum of an element has the spectral mapping property; (c) if every element in topological algebra A has a functional topological spectrum, then the topological spectral radius (as a map) is a submultiplicative seminorm on A; and (d) A is a Q-algebra if and only if A is a TQ-algebra and $\mathrm{sp}(a) = \mathrm{sp}^t(a)$ for each $a \in A \setminus \mathrm{Qinv}(A)$.

2. PROPERTIES OF $\mathrm{Tqinv}_\ell(A)$, $\mathrm{Tqinv}_r(A)$, AND $\mathrm{Tqinv}(A)$ FOR AN F-ALGEBRA A

Let A be an F-algebra (a complete and metrizable algebra) with F-norm $\|\cdot\|$, i.e. with a function $x \mapsto \|x\|$ on A such that
1: $\|x\| \geq 0$ for each $x \in A$, and $\|x\| = 0$ if and only if $x = \theta_A$;
2: $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in A$;
3: $(\lambda, x) \mapsto \|\lambda x\|$ is a jointly continuous map from $\mathbb{K} \times A$ to $\mathbb{R}^+$.

We define
$$g^t_\ell(x) = \inf_{u \in A} \|u \circ x\| = \inf_{u \in A} \|u + x - ux\|$$
and
$$g^t_r(x) = \inf_{u \in A} \|x \circ u\| = \inf_{u \in A} \|x + u - xu\|$$
for each $x \in A$. Then
$$\mathrm{Tqinv}_\ell(A) = \{x \in A : g^t_\ell(x) = 0\} \text{ and } \mathrm{Tqinv}_r(A) = \{x \in A : g^t_r(x) = 0\}.$$

Indeed, if $x \in \mathrm{Tqinv}_\ell(A)$, then there is a sequence $(u_n)_{n \in \mathbb{N}}$ in A such that $(u_n \circ x)_{n \in \mathbb{N}}$ converges to θ_A. Since the F-norm is a continuous map, $(\|u_n \circ x\|)_{n \in \mathbb{N}}$ converges to 0. Hence $g^t_\ell(x) = 0$.

Let now $x \in A$ be such that $g_\ell^t(x) = 0$. Then for each $n \in \mathbb{N}$ there is an element u_n in A such that $0 < \|u_n \circ x\| < \frac{1}{n}$. Therefore, $\|u_n \circ x\|$ converges to 0. This means that $(u_n \circ x)_{n\in\mathbb{N}}$ converges to θ_A. Consequently, $x \in \mathrm{Tqinv}_\ell(A)$.

Similarly, we can show that $\mathrm{Tqinv}_r(A) = \{x \in A : g_r^t(x) = 0\}$.

Lemma 1. *Let $(A, \|\cdot\|)$ be an F-algebra. Then the sets*

$$S_{\ell,\lambda}^t = \{x \in A : g_\ell^t(x) < \lambda\} \text{ and } S_{r,\lambda}^t = \{x \in A : g_r^t(x) < \lambda\}$$

are open for any $\lambda > 0$.

Proof. Let $x \in A$, $\lambda > 0$ and $g_\ell^t(x) < \lambda$. By definition of $g_\ell^t(x)$, there is $z \in A$ such that $\|z \circ x\| < \lambda$ and so there is $\varepsilon > 0$ such that

$$\|z \circ x\| < \varepsilon < \lambda.$$

By the continuity of quasi-multiplication, there is a positive δ such that $\|(z\circ x) - (z \circ y)\| < \lambda - \varepsilon$ whenever $\|x - y\| < \delta$. This implies that

$$\|z \circ y\| \leq \|(z\circ y) - (z\circ x)\| + \|(z\circ x)\| < \lambda - \varepsilon + \varepsilon = \lambda$$

for every x and y in A such that $\|x - y\| < \delta$, which implies that the set $\{y : \|x - y\| < \delta\} \subset S_{\ell,\delta}^t$. Therefore, $S_{\ell,\delta}^t$ is open.

Similarly, we can show that $S_{r,\lambda}^t$ is open. □

Corollary 1. *Let A be an F-algebra. Then the set*

$$S_\lambda^t = S_{\ell,\lambda}^t \cap S_{r,\lambda}^t$$

is open for any $\lambda > 0$.

Corollary 2. *The function g_ℓ^t is continuous at all points of* $\mathrm{Tqinv}_\ell(A)$ *and g_r^t is continuous at all points of* $\mathrm{Tqinv}_r(A)$.

Proof. Let x_0 be an element in $\mathrm{Tqinv}(A)$ and $\lambda > 0$, then $x_0 \in S_{\ell,\lambda}^t$. Since $S_{\ell,\lambda}^t$ is open by Lemma 1, there exists a neighbourhood $\mathscr{O}(x_0)$ of x_0 such that $\mathscr{O}(x_0) \subset S_{\ell,\lambda}^t$. Hence $g_\ell^t(x) < \lambda$ for each $x \in \mathscr{O}(x_0)$. This means that g_ℓ^t is continuous at x_0. Since x_0 is an arbitrary element of $\mathrm{Tqinv}(A)$, g_ℓ^t is continuous on $\mathrm{Tqinv}(A)$.

In the same way, it is easy to show that g_r^t is a continuous map at all points of $\mathrm{Tqinv}_r(A)$. □

Proposition 1. *Let A be an F-algebra. Then* $\mathrm{Tqinv}_\ell(A)$, $\mathrm{Tqinv}_r(A)$, *and* $\mathrm{Tqinv}(A)$ *are G_δ-sets.*

Proof. Since

$$\mathrm{Tqinv}_\ell(A) = \bigcap_{n\in\mathbb{N}} \left\{g_\ell^t(x) < \frac{1}{n}\right\} \text{ and } \mathrm{Tqinv}_r(A) = \bigcap_{n\in\mathbb{N}} \left\{g_r^t(x) < \frac{1}{n}\right\},$$

$\mathrm{Tqinv}_\ell(A)$ and $\mathrm{Tqinv}_r(A)$ are G_δ-sets by Lemma 1. Therefore, so is $\mathrm{Tqinv}(A)$ as the intersection of two G_δ-sets. □

Corollary 3. *Let A be an F-algebra and x° denote the quasi-inverse of $x \in A$. If the quasi-inversion $x \mapsto x^\circ$ is discontinuous in A, then* $\mathrm{Tqinv}(A) \setminus \mathrm{Qinv}(A) \neq \emptyset$.

Proof. Suppose that $\mathrm{Tqinv}(A) = \mathrm{Qinv}(A)$. Then $\mathrm{Qinv}(A)$ is a G_δ-set by Proposition 1. Since A is a complete metric space, $\mathrm{Qinv}(A)$ is topologically complete and consequently, according to Theorem 14.9, p. 110 in [9], there is on $\mathrm{Qinv}(A)$ an equivalent metric under which it is complete. By applying Proposition 2.1.8, p. 76 in [4] to $\mathrm{Qinv}(A)$ (with its complete metric) we conclude that the quasi-inversion is continuous on $\mathrm{Qinv}(A)$. □

Corollary 4. *In a very advertive F-algebra the quasi-inversion $x \mapsto x^\circ$ is continuous.*

Corollary 5. *If the quasi-inversion $x \mapsto x^\circ$ is discontinuous in an F-algebra A, then A has a dense proper left and a dense proper right ideal.*

Proof. By Corollary 4, there is an element

$$x \in \mathrm{Tqinv}_\ell(A) \cap \mathrm{Tqinv}_r(A) \setminus \mathrm{Qinv}(A).$$

Suppose that $x \in \mathrm{Qinv}_\ell(A)$. Then there exists an element $a \in A$ such that $a \circ x = \theta_A$. Since $x \in \mathrm{Tqinv}_r(A)$ as well, there is a net $(x_\lambda)_{\lambda\in\Lambda}$ in A such that $(x \circ x_\lambda)_{\lambda\in\Lambda}$ converges to θ_A. Hence, $(x_\lambda)_{\lambda\in\Lambda}$ converges to a in A because

$$x_\lambda = \theta_A \circ x_\lambda = a \circ x \circ x_\lambda \to a$$

and thus $x \in \mathrm{Qinv}(A)$, which is not the case. Similarly, from $x \in \mathrm{Qinv}_r(A)$ it follows that $x \in \mathrm{Qinv}(A)$, which is again impossible. Consequently,

$$x \in (\mathrm{Tqinv}_\ell(A) \setminus \mathrm{Qinv}_\ell(A)) \cap (\mathrm{Tqinv}_r(A) \setminus \mathrm{Qinv}_r(A)).$$

This means that the left ideal $Ax - A$ and the right ideal $xA - A$ are dense in A. Indeed, from $x \in \mathrm{Tqinv}_\ell(A)$ it follows that there exists a net $(x_\lambda)_{\lambda\in\Lambda}$ in A such that $x_\lambda \circ x \to \theta_A$ or $(x_\lambda x - x_\lambda)_{\lambda\in\Lambda} \to x$. As $x_\lambda \in A$ for each $\lambda \in \Lambda$, then $x \in \overline{Ax - A}$. Here $\overline{U}$ denotes the closure of any U in the topology of A. Hence, $A = \overline{Ax - A}$, otherwise $\overline{Ax - A}$ is a regular left ideal in A that contains the regular unit.

Similarly, we can show that $\overline{xA - A} = A$. □

3. TOPOLOGICAL SPECTRUM OF ELEMENTS

Properties of the topological spectrum of elements for unital topological algebras are described in [2]. Now we consider the general (not necessarily unital) case.

3.1. Properties of the (left or right) topological spectrum of elements

Let A be a topological algebra. The left (right) topological spectrum of elements of A has several properties, similar to the left (algebraic) spectrum of elements in case of topological algebras with unity (see [2]). In the present section we show that every topological (not necessarily unital) algebra has similar properties.

Proposition 2. *Let A be a topological (not necessarily unital) algebra. The left, right, and two-sided topological spectra $\mathrm{sp}^t_\ell(x)$, $\mathrm{sp}^t_r(x)$, and $\mathrm{sp}^t(x)$ for each $x \in A$ have the following properties:*

(1) $\mathrm{sp}^t_\ell(\mu x) = \mu\,\mathrm{sp}^t_\ell(x)$, $\mathrm{sp}^t_r(\mu x) = \mu\,\mathrm{sp}^t_r(x)$, *and* $\mathrm{sp}^t(\mu x) = \mu\,\mathrm{sp}^t(x)$ *for any* $x \in A$ *and* $\mu \in \mathbb{C}$;

(2) $\mathrm{sp}^t_\ell(xy) = \mathrm{sp}^t_\ell(yx)$, $\mathrm{sp}^t_r(xy) = \mathrm{sp}^t_r(yx)$ *and* $\mathrm{sp}^t(xy) = \mathrm{sp}^t(yx)$ *for any* $x, y \in A$.

Proof. (1) If $\mu = 0$, then these equalities hold. Let $\mu \neq 0$ and $\lambda \in \mathbb{C}$.

Since

$$\lambda \in \mathrm{sp}^t_\ell(\mu x) \Leftrightarrow \mu^{-1}\lambda \in \mathrm{sp}^t_\ell(x) \Rightarrow \mu(\mu^{-1}\lambda) \in \mu\,\mathrm{sp}^t_\ell(x) \Leftrightarrow \lambda \in \mu\,\mathrm{sp}^t_\ell(x)$$

and

$$\mu\,\mathrm{sp}^t_\ell(x) = \mu\,\mathrm{sp}^t_\ell(\mu^{-1}(\mu x)) \subseteq \mu\mu^{-1}\mathrm{sp}^t_\ell(\mu x) = \mathrm{sp}^t_\ell(\mu x),$$

$\mathrm{sp}^t_\ell(\mu x) = \mu\,\mathrm{sp}^t_\ell(x)$ for each $x \in A$ and $\mu \in \mathbb{C}$.

The proof of $\mathrm{sp}_r^t(\mu x) = \mu \mathrm{sp}_r^t(x)$ for each $x \in A$ and $\mu \in \mathbb{C}$ is similar. Consequently, $\mathrm{sp}^t(\mu x) = \mu \mathrm{sp}^t(x)$ for each $x \in A$ and $\mu \in \mathbb{C}$.

(2) We first show that $xy \in \mathrm{Tqinv}_\ell(A)$ if and only if $yx \in \mathrm{Tqinv}_\ell(A)$. For that, take $xy \in \mathrm{Tqinv}_\ell(A)$. Then there exists a net $(u_\alpha)_{\alpha\in\Lambda}$ in A such that $(u_\alpha \circ xy)_{\alpha\in\Lambda} \to \theta_A$ or $(u_\alpha - u_\alpha xy)_{\alpha\in\Lambda} \to -xy$.
Since

$$((yu_\alpha x - yx) \circ yx)_{\alpha\in\Lambda} = (y[(u_\alpha - u_\alpha xy)x] + yxyx)_{\alpha\in\Lambda} \to y(-xy)x + yxyx = \theta_A,$$

$yx \in \mathrm{Tqinv}_\ell(A)$. By symmetry, if $yx \in \mathrm{Tqinv}_\ell(A)$, then $xy \in \mathrm{Tqinv}_\ell(A)$. Hence, $xy \in \mathrm{Tqinv}_\ell(A)$.

Let now $\lambda \in sp_\ell^t(xy) \setminus \{0\}$, that is, $\lambda^{-1}xy \notin \mathrm{Tqinv}_\ell(A)$. Then $\lambda^{-1}yx \notin \mathrm{Tqinv}_\ell(A)$. Thus, $\mathrm{sp}_\ell^t(xy) \subseteq \mathrm{sp}_\ell^t(yx)$.

Interchanging x and y, we get the reverse inclusion. Therefore, $\mathrm{sp}_\ell^t(xy) = \mathrm{sp}_\ell^t(yx)$.

Similarly, we can show that $\mathrm{sp}_r^t(xy) = \mathrm{sp}_r^t(yx)$ and $\mathrm{sp}^t(xy) = \mathrm{sp}^t(yx)$. □

Lemma 2. *Let A and B be topological algebras and π a continuous homomorphism from A into B. Then*

$$\mathrm{sp}_{\ell,B}^t(\pi(x)) \subseteq \mathrm{sp}_{\ell,A}^t(x), \ \ \mathrm{sp}_{r,B}^t(\pi(x)) \subseteq \mathrm{sp}_{\ell,A}^t(x), \ \ \textit{and} \ \ \mathrm{sp}_B^t(\pi(x)) \subseteq \mathrm{sp}_A^t(x).$$

Proof. If $\lambda = 0$, then the equalities hold. Let $\lambda \neq 0$ and $\lambda \in \mathrm{sp}_{\ell,B}^t(\pi(x))$. Then $\lambda^{-1}\pi(x) \notin \mathrm{Tqinv}_\ell(B)$. Therefore, $\lambda^{-1}x \notin \mathrm{Tqinv}_\ell(A)$. Hence, $\lambda \in \mathrm{sp}_{\ell,A}^t(x)$.

Similarly, $\mathrm{sp}_{r,B}^t(\pi(x)) \subseteq \mathrm{sp}_{r,A}^t(x)$ and $\mathrm{sp}_B^t(\pi(x)) \subseteq \mathrm{sp}_A^t(x)$.

Proposition 3. *Let A and B be topological algebras and π a continuous open homomorphism from A onto B. If there exists a neighbourhood $\mathscr{O}$ of zero in A such that*

$$\mathscr{O} + \ker\pi \subseteq \mathrm{Tqinv}_\ell(A)$$

$$(\mathscr{O} + \ker\pi \subseteq \mathrm{Tqinv}_r(A) \ \ \textit{or} \ \ \mathscr{O} + \ker\pi \subseteq \mathrm{Tqinv}(A)),$$

then

$$\mathrm{sp}_{\ell,B}^t(\pi(x)) = \mathrm{sp}_{\ell,A}^t(x)$$

$$(\textit{respectively,} \ \ \mathrm{sp}_{r,B}^t(\pi(x)) = \mathrm{sp}_{r,A}^t(x) \ \ \textit{or} \ \ \mathrm{sp}_B^t(\pi(x)) = \mathrm{sp}_A^t(x))$$

for each $x \in A$.

Proof. The inclusion $\mathrm{sp}_{\ell,B}^t(\pi(x)) \subseteq \mathrm{sp}_{\ell,A}^t(x)$ holds by Lemma 2. To prove the opposite inclusion, assume that $\lambda \notin \mathrm{sp}_{\ell,B}^t(\pi(x))$ and $\mathscr{O}$ is the neighbourhood of zero as in the statement of Proposition 3. Therefore, $\lambda^{-1}\pi(x) \in \mathrm{Tqinv}_\ell(B)$, which implies that there exists a net $(x_\mu)_{\mu\in\Delta}$ in A such that the net

$$(\pi(x_\mu) \circ \lambda^{-1}\pi(x))_{\mu\in\Delta} \to \theta_B.$$

Since π is open, $\pi(\mathscr{O})$ is a neighbourhood of zero in B. Hence, there exists $\mu_0 \in \Delta$ such that

$$x_\mu \circ \lambda^{-1}x \in \pi^{-1}(\pi(\mathscr{O})) = \mathscr{O} + \ker\pi \subseteq \mathrm{Tqinv}_\ell(A)$$

whenever $\mu > \mu_0$, by the assumption. Now we fix $\mu_1 > \mu_0$. Then there exists a net $(u_\alpha)_{\alpha\in\Lambda}$ in A such that

$$(u_\alpha \circ (x_{\mu_1} \circ \lambda^{-1}x))_{\alpha\in\Lambda} \to \theta_A.$$

Consequently, $\lambda^{-1}x \in \mathrm{Tqinv}_\ell(A)$ or $\lambda \notin \mathrm{sp}_{\ell,A}^t(x)$. Thus, $\mathrm{sp}_{\ell,A}^t(x) \subseteq \mathrm{sp}_{\ell,B}^t(\pi(x))$.

Similarly, $\mathrm{sp}_{r,B}^t(\pi(x)) = \mathrm{sp}_{r,A}^t(x)$ and as $\mathrm{sp}_A^t(x) = \mathrm{sp}_{\ell,A}^t(x) \cup \mathrm{sp}_{r,A}^t(x)$, then

$$\mathrm{sp}_B^t(\pi(x)) = \mathrm{sp}_{\ell,B}^t(\pi(x)) \cup \mathrm{sp}_{r,B}^t(\pi(x)) = \mathrm{sp}_{\ell,A}^t(x) \cup \mathrm{sp}_{r,A}^t(x) = \mathrm{sp}_A^t(x)$$

for each $x \in A$. □

Corollary 6. *Let A be a topological algebra, I a two-sided ideal of A, and π the canonical homomorphism from A onto A/I. If there exists a neighbourhood $\mathscr{O}$ of zero in A such that*

$$\mathscr{O}+I \subseteq \operatorname{Tqinv}_\ell(A)(\mathscr{O}+I \subseteq \operatorname{Tqinv}_r(A) \ \ \textit{or} \ \ \mathscr{O}+I \subseteq \operatorname{Tqinv}(A)),$$

then

$$\operatorname{sp}^t_{\ell,A/I}(\pi(x)) = \operatorname{sp}^t_{\ell,A}(x)(\operatorname{sp}^t_{r,A/I}(\pi(x)) = \operatorname{sp}^t_{r,A}(x) \ \ \textit{or} \ \ \operatorname{sp}^t_{A/I}(\pi(x)) = \operatorname{sp}^t_{A}(x)) \ \ \textit{for each} \ \ x \in A.$$

Proposition 4. *Let A and B be topological algebras and π a topological isomorphism from A into B. If $\pi(A)$ is dense in B, then*

$$\operatorname{Tqinv}_\ell(\pi(A)) = \operatorname{Tqinv}_\ell(B)\cap\pi(A),\ \operatorname{Tqinv}_r(\pi(A)) = \operatorname{Tqinv}_r(B)\cap\pi(A),$$

$$\textit{and} \ \operatorname{Tqinv}(\pi(A)) = \operatorname{Tqinv}(B)\cap\pi(A).$$

Proof. It is easy to see that $\operatorname{Tqinv}_\ell(\pi(A)) \subseteq \operatorname{Tqinv}_\ell(B)\cap\pi(A)$. To prove the opposite inclusion, take $y \in \operatorname{Tqinv}_\ell(B)\cap\pi(A)$ and let $\mathscr{O}$ be a neighbourhood of zero in $\pi(A)$ (then there is a neighbourhood $\mathscr{O}'$ of zero in B with $\mathscr{O} = \mathscr{O}'\cap\pi(A)$). Now there are $x \in A$ such that $y = \pi(x)$ and a balanced neighbourhood $\mathscr{O}_1$ of zero in B such that $\mathscr{O}_1+\mathscr{O}_1-\mathscr{O}_1 y \subset \mathscr{O}$. Since $y \in \operatorname{Tqinv}_\ell(B)$, there exists a net $(y_\mu)_{\mu\in\Delta}$ in B such that $(y_\mu \circ y)_{\mu\in\Delta}$ converge to zero in B. Therefore, there is an index $\mu_0 \in \Delta$ such that $y_\mu \circ y \in \mathscr{O}_1$ whenever $\mu > \mu_0$. Let $\mu_1 \in \Delta$ be such that $\mu_1 > \mu_0$. Then $y_{\mu_1}\circ y \in \mathscr{O}_1$. Since $\pi(A)$ is dense in B there is a net $(u_\alpha)_{\alpha\in\Lambda}$ such that $(\pi(u_\alpha))_{\alpha\in\lambda}$ converge to y_{μ_1}. Hence, there is an index $\alpha_0 \in \Lambda$ such that $\pi(u_\alpha)-y_{\mu_1} \in \mathscr{O}_1$ whenever $\alpha > \alpha_0$.

Since

$$\begin{aligned}
\pi(u_\alpha)\circ\pi(x) &= \pi(u_\alpha)-y_{\mu_1}+\pi(x)+y_{\mu_1}-y_{\mu_1}\pi(x)\\
&\quad -\pi(u_\alpha)\pi(x)+y_{\mu_1}\pi(x)\\
&= \pi(u_\alpha)-y_{\mu_1}+(y_{\mu_1}\circ\pi(x))-(\pi(u_\alpha)-y_{\mu_1})\pi(x)\\
&\quad \subset \mathscr{O}_1+\mathscr{O}_1-\mathscr{O}_1 y \subset \mathscr{O}'
\end{aligned}$$

whenever $\alpha > \alpha_0$, then $(\pi(u_\alpha)\circ\pi(x))_{\alpha\in\Lambda}$ converges to $\theta_{\pi(A)}$ in $\pi(A)$. Thus, $y \in \operatorname{Tqinv}_\ell(\pi(A))$.

Similarly, we can show that $\operatorname{Tqinv}_r(\pi(A)) = \operatorname{Tqinv}_r(B)\cap\pi(A)$ and $\operatorname{Tqinv}(\pi(A)) = \operatorname{Tqinv}(B)\cap\pi(A)$. □

Corollary 7. *Let A and B be topological algebras and π a topological isomorphism from A into B. If*
(a) $\operatorname{Tqinv}_\ell(A)$, $\operatorname{Tqinv}_r(A)$, *and* $\operatorname{Tqinv}(A)$ *are open in A and π is an onto map*
or

(b) $\pi(A)$ *is dense in B,*

then $\operatorname{sp}^t_{\ell,B}(\pi(x)) = \operatorname{sp}^t_{\ell,A}(x)$, $\operatorname{sp}^t_{r,B}(\pi(x)) = \operatorname{sp}^t_{r,A}(x)$, *and* $\operatorname{sp}^t_{B}(\pi(x)) = \operatorname{sp}^t_{A}(x))$ *for each $x \in A$.*

Proof. The inclusion $\operatorname{sp}^t_{\ell,B}(\pi(x)) \subseteq \operatorname{sp}^t_{\ell,A}(x)$ holds for each $x \in A$ (see also the proof of Lemma 2).

If A satisfies the condition (a), then by Proposition 3, the opposite inclusion also holds for each $x \in A$, since π is one to one and onto.

Let now A and π satisfy the condition (b), $x \in A$ and $\lambda \notin \operatorname{sp}^t_{\ell,B}(\pi(x))\setminus\{0\}$. Then $\lambda^{-1}\pi(x) \in \operatorname{Tqinv}_\ell(\pi(A))$ by Proposition 4. Therefore, there exists a net $(x_\mu)_{\mu\in\Delta}$ in A such that $(\pi[x_\mu\circ\lambda^{-1}x])_{\mu\in\Delta} \to \pi(\theta_A)$ in $\pi(A)$. Since π^{-1} is continuous, $(x_\mu\circ\lambda^{-1}x)_{\mu\in\Delta}$ converges to θ_A in A. Hence, $\lambda^{-1}x \in \operatorname{Tqinv}_\ell(A)$, i.e. $\lambda \notin \operatorname{sp}^t_{\ell,A}(x)$. Consequently, $\operatorname{sp}^t_{\ell,A}(x) \subset \operatorname{sp}^t_{\ell,B}(\pi(x))$.

Similarly, $\operatorname{sp}^t_{r,B}(\pi(x)) = \operatorname{sp}^t_{r,A}(x)$ and $\operatorname{sp}^t_{B}(\pi(x)) = \operatorname{sp}^t_{A}(x))$ for each $x \in A$. □

Corollary 8. *Let A and B be topological algebras and π a topological isomorphism from A into B. If $\pi(A)$ is dense in B, then*

$$\mathrm{sp}^t_{\ell,\pi(A)}(\pi(a)) = \mathrm{sp}^t_{\ell,B}(\pi(a)),\ \mathrm{sp}^t_{r,\pi(A)}(\pi(a)) = \mathrm{sp}^t_{r,B}(\pi(a)),$$
$$\text{and}\ \ \mathrm{sp}^t_{\pi(A)}(\pi(a)) = \mathrm{sp}^t_{B}(\pi(a)).$$

Proof. Let $a \in A$. If $\lambda \in \mathrm{sp}^t_{\ell,\pi(A)}(\pi(a)) \setminus \{0\}$, then $\lambda^{-1}\pi(a) \notin \mathrm{Tqinv}_\ell(\pi(A))$. Then $\lambda^{-1}\pi(a) \notin \mathrm{Tqinv}_\ell(B)$ by Proposition 4. Therefore, $\lambda \in \mathrm{sp}^t_{\ell,B}(\pi(a))$.

Now, let $\lambda \in \mathrm{sp}^t_{\ell,B}(\pi(a)) \setminus \{0\}$, i.e. $\lambda^{-1}\pi(a) \notin \mathrm{Tqinv}_\ell(B)$. Therefore, $\lambda^{-1}\pi(a) \notin \mathrm{Tqinv}_\ell(\pi(A))$ by Proposition 4, which implies that $\lambda \in \mathrm{sp}^t_{\ell,\pi(A)}(\pi(a))$.

Similarly, $\mathrm{sp}^t_{r,\pi(A)}(\pi(a)) = \mathrm{sp}^t_{r,B}(\pi(a))$ and $\mathrm{sp}^t_{\pi(A)}(\pi(a)) = \mathrm{sp}^t_{B}(\pi(a))$ for each $a \in A$. □

Corollary 9. *Let A be a Hausdorff topological algebra, $\widetilde{A}$ the completion of A, and τ topological isomorphism from A into $\widetilde{A}$, defined by the completion of A. If $\widetilde{A}$ is an algebra, then*

$$\mathrm{sp}^t_{\ell,\widetilde{A}}(\tau(a)) = \mathrm{sp}^t_{\ell,A}(a),\ \mathrm{sp}^t_{r,\widetilde{A}}(\tau(a)) = \mathrm{sp}^t_{r,A}(a),\ \text{and}\ \mathrm{sp}^t_{\widetilde{A}}(\tau(a)) = \mathrm{sp}^t_{A}(a)$$

for each $a \in A$.

Proof. Since A is a Hausdorff topological algebra, $\tau(A)$ is dense in $\widetilde{A}$. Hence, $\mathrm{sp}^t_{\ell,\widetilde{A}}(\tau(a)) = \mathrm{sp}^t_{\ell,A}(a)$ for each $a \in A$ by Corollary 7b.

Similarly, we can show that $\mathrm{sp}^t_{r,\widetilde{A}}(\tau(a)) = \mathrm{sp}^t_{r,A}(a)$ and $\mathrm{sp}^t_{\widetilde{A}}(\tau(a)) = \mathrm{sp}^t_{A}(a)$ for each $a \in A$. □

3.2. Spectral mapping property for the topological spectrum of an element

We will prove the spectral mapping theorem for the topological spectrum of an element. For that we need

Proposition 5. *Let A be a commutative topological algebra, $n \in \mathbb{N}$, and $x_1, x_2, ..., x_n \in A$. Then $x_1 \circ x_2 \cdots \circ x_n \in \mathrm{Tqinv}(A)$ if and only if $x_i \in \mathrm{Tqinv}(A)$ for each $1 \le i \le n$.*

Proof. Let $x = x_1 \circ x_2 \circ \cdots \circ x_n \in \mathrm{Tqinv}(A)$. Then there exists a net $(z_\alpha)_{\alpha\in\lambda}$ in A such that $(x \circ z_\alpha)_{\alpha\in\Lambda}$ converges to θ_A. Therefore,

$$(x_i \circ (x_1 \circ x_2 \circ \cdots \circ x_{i-1} \circ x_{i+1} \circ \cdots \circ x_n \circ z_\alpha))_{\alpha\in\Lambda}$$

converges to zero for each $i \in \mathbb{N}_n = \{1, 2, ..., n\}$. Now, for each fixed $\alpha \in \Lambda$ we put $\omega^1_\alpha = x_2 \circ x_3 \circ \cdots \circ x_n \circ z_\alpha$, $\omega^i_\alpha = x_1 \circ x_2 \circ \cdots \circ x_{i-1} \circ x_{i+1} \circ \cdots \circ x_n \circ z_\alpha$ for each i with $2 \le i \le n-2$ and $\omega^n_\alpha = x_1 \circ x_2 \circ \cdots \circ x_{n-1} \circ z_\alpha$. Then for any $i \in \mathbb{N}$ the net $(x_i \circ \omega^i_\alpha)_{\alpha\in\Lambda}$ converges to zero. Hence, $x_i \in \mathrm{Tqinv}(A)$ for $1 \le i \le n$.

Conversely, if $x_i \in \mathrm{Tqinv}(A)$ for any $1 \le i \le n$, then using induction, it is sufficient to consider the case $n = 2$. Since $x_1 \in \mathrm{Tqinv}(A)$, there exists a net $(z_\alpha)_{\alpha\in\Lambda}$ in A such that $(x_1 \circ z_\alpha)_{\alpha\in\Lambda}$ converges to zero, so for any (open) neighbourhood U of zero in A there is an index $\alpha_0 \in \Lambda$ such that $x_1 \circ z_\alpha \in U$ whenever $\alpha \ge \alpha_0$. Let $\alpha_1 \in \Lambda$ be a fixed index such that $\alpha_1 > \alpha_0$. Since $x_2 \in \mathrm{Tqinv}(A)$, there exists a net $(\nu_\beta)_{\beta\in\Delta}$ in A such that $(\nu_\beta \circ x_2)_{\beta\in\Delta}$ converges to zero. Now $(\nu_\beta \circ x_2 \circ (x_1 \circ z_{\alpha_1}))_{\beta\in\Delta}$ converges to $x_1 \circ z_{\alpha_1}$. Since U is a neighbourhood of $x_1 \circ z_{\alpha_1}$, there exists $\beta_0 \in \Delta$ such that $(\nu_\beta \circ z_{\alpha_1}) \circ x_1 \circ x_2 \in U$ for each $\beta \ge \beta_0$. Hence, there is a net $(\nu_\beta \circ z_{\alpha_1})_{\beta\in\Delta}$ in A such that $((\nu_\beta \circ z_{\alpha_1}) \circ (x_1 \circ x_2))_{\beta\in\Delta}$ converges to zero. Consequently, $x_1 \circ x_2 \in \mathrm{Tqinv}(A)$. □

Proposition 6. *Let A be a nonunital topological algebra and A_1 the unitization of A. Then*

$$\mathrm{sp}^t_{A_1}((a, 0)) = \mathrm{sp}^t_{A}(a)$$

for each $a \in A$.

Proof. If $\lambda \notin \mathrm{sp}^t_{A_1}((a,0))$, then $(\lambda^{-1}a,0) \in \mathrm{Tqinv}(A_1)$. Therefore, there exist nets $((a_\alpha,\lambda_\alpha))_{\alpha\in\Lambda}$ and $((b_\beta,\lambda_\beta))_{\beta\in\Delta}$ elements of A_1 such that

$$((a_\alpha,\lambda_\alpha)\circ(\lambda^{-1}a,0))_{\alpha\in\Lambda} \text{ converges to } (\theta_A,0)$$

and

$$((\lambda^{-1}a,0)\circ(b_\beta,\lambda_\beta))_{\beta\in\Delta} \text{ converges to } (\theta_A,0).$$

Since

$$\begin{aligned}(a_\alpha,\lambda_\alpha)\circ(\lambda^{-1}a,0) &= (a_\alpha,\lambda_\alpha)+(\lambda^{-1}a,0)-(a_\alpha,\lambda_\alpha)(\lambda^{-1}a,0)\\ &= (a_\alpha+\lambda^{-1}a-a_\alpha\lambda^{-1}a-\lambda_\alpha(\lambda^{-1}a),\lambda_\alpha)\\ &= (a_\alpha\circ(\lambda^{-1}a)-\lambda_\alpha(\lambda^{-1}a),\lambda_\alpha)\end{aligned}$$

for each $\alpha\in\Lambda$ (similarly $(\lambda^{-1}a,0)\circ(b_\beta,\lambda_\beta) = ((\lambda^{-1}a)\circ b_\beta-(\lambda^{-1}a)\lambda_\beta,\lambda_\beta)$ for each $\alpha\in\Lambda$), then

$$\lim_\alpha(a_\alpha\circ(\lambda^{-1}a)-\lambda_\alpha(\lambda^{-1}a)) = \theta_A \text{ and } \lim_\alpha\lambda_\alpha = 0 \text{ at the same time}$$

and

$$\lim_\beta((\lambda^{-1}a)\circ b_\beta-(\lambda^{-1}a)\lambda_\beta) = \theta_A \text{ and } \lim_\beta\lambda_\beta = 0 \text{ at the same time.}$$

Hence, $(a_\alpha\circ(\lambda^{-1}a))_{\alpha\in\Lambda}$ and $((\lambda^{-1}a)\circ b_\beta)_{\beta\in\Lambda}$ converge to θ_A. This means that $\lambda^{-1}a\in\mathrm{Tqinv}(A)$ or $\lambda\notin\mathrm{sp}^t_A(a)$. Hence, $\mathrm{sp}^t_A(a)\subseteq\mathrm{sp}^t_{A_1}((a,0))$.

Let now $\lambda\notin\mathrm{sp}^t_A(a)$. Then $\lambda^{-1}a\in\mathrm{Tqinv}(A)$. Hence there are nets $(a_\alpha)_{\alpha\in\Lambda}$ and $(b_\beta)_{\beta\in\Delta}$ in A such that $(a_\alpha\circ(\lambda^{-1}a))_{\alpha\in\Lambda}$ and $((\lambda^{-1}a)\circ b_\beta)_{\beta\in\Delta}$ converge to θ_A. Therefore,

$$((a_\alpha,0)\circ(\lambda^{-1}a,0))_{\alpha\in\Lambda} = (a_\alpha\circ(\lambda^{-1}a),0)_{\alpha\in\Lambda} \text{ converges to } (\theta_A,0)$$

and

$$((\lambda^{-1}a,0)\circ(b_\beta,0))_{\beta\in\Delta} = ((\lambda^{-1}a)\circ b_\beta,0)_{\beta\in\Delta} \text{ converges to } (\theta_A,0).$$

Thus, $(\lambda^{-1}a,0)\in\mathrm{Tqinv}(A_1)$ or $\lambda\notin\mathrm{sp}^t_{A_1}((a,0))$. Consequently, $\mathrm{sp}^t_{A_1}((a,0))\subseteq\mathrm{sp}^t_A(a)$ for each $a\in A$. Therefore, $\mathrm{sp}^t_{A_1}((a,0)) = \mathrm{sp}^t_A(a)$ for each $a\in A$. □

Next we will give a new proof to the result of Najmi (see [7], Lemma 2.1, p. 33).

Proposition 7. *Let A be a commutative complex topological algebra. Then*

$$\mathrm{sp}^t(p(x)) = p(\mathrm{sp}^t(x))$$

for any complex non-constant polynomial p and every $x\in A$.

Proof. When A is a unital algebra, the proof has been given in [2] (see the proof of Proposition 7).

If A is not a unital algebra, we considered the unitarization A_1 of A. Then $\mathrm{sp}^t_A(x) = \mathrm{sp}^t_{A_1}((x,0))$ by Proposition 6, which implies that

$$p(\mathrm{sp}^t_A(x)) = p(\mathrm{sp}^t_{A_1}(x,0)) = \mathrm{sp}^t_{A_1}(p(x,0)) = \mathrm{sp}^t_A(p(x))$$

for each $x\in A$ (see Proposition 7 in [2]). □

3.3. Properties of the topological spectral radius

Several properties of the topological spectral radius of an element in unital topological algebras are presented in [2]. Here we give properties of the left, right, and two-sided topological spectral radii of an element in topological (not necessarily unital) algebras.

Proposition 8. *Let A be a topological (not necessarily unital) algebra. The left, right, and two-sided topological spectral radii r^t_ℓ, r^t_r, and r^t of an element have the following properties:*

(1) $\mathrm{r}^t_\ell(\mu x) = |\mu|\mathrm{r}^t_\ell(x)$, $\mathrm{r}^t_r(\mu x) = |\mu|\mathrm{r}^t_r(x)$, *and* $\mathrm{r}^t(\mu x) = |\mu|\mathrm{r}^t(x)$ *for all* $x \in A$ *and* $\mu \in \mathbb{C}$*;*

(2) $\mathrm{r}^t_\ell(xy) = \mathrm{r}^t_\ell(yx)$, $\mathrm{r}^t_r(xy) = \mathrm{r}^t_r(yx)$, *and* $\mathrm{r}^t(xy) = \mathrm{r}^t(yx)$ *for all* $x, y \in A$*;*

(3) *if A is commutative, then* $\mathrm{r}^t(x^n) = \mathrm{r}^t(x)^n$ *for each* $x \in A$ *and all* $n \in \mathbb{N}$.

Proof. (1) Take $x \in A$ and $\mu \in \mathbb{C}\setminus\{0\}$ (the case when $\mu = 0$ is trivial). Then by Proposition 2, $\mathrm{sp}^t_\ell(\mu x) = \mu\mathrm{sp}^t_\ell(x)$ for each $\mu \in \mathbb{C}$. Since

$$\mathrm{r}^t_\ell(\mu x) = \sup\{|\lambda| : \lambda \in \mu\mathrm{sp}^t_\ell(x)\},$$

then $|\lambda| \leq |\mu|\mathrm{r}^t_\ell(x)$ for each $\lambda \in \mathrm{sp}^t_\ell(\mu x)$ and $\mu \in \mathbb{C}$. Hence

$$\mathrm{r}^t_\ell(\mu x) \leq |\mu|\mathrm{r}^t_\ell(x) = |\mu|\mathrm{r}^t_\ell(\mu^{-1}(\mu x)) \leq \frac{|\mu|}{|\mu|}\mathrm{r}^t_\ell(\mu x),$$

from which it follows that $\mathrm{r}^t_\ell(\mu x) = |\mu|\mathrm{r}^t_\ell(x)$ for all $x \in A$ and $\mu \in \mathbb{C}$.

Similarly, we can show the rest of the relations in (1).

(2) If $x, y \in A$, then $\mathrm{sp}^t(xy) = \mathrm{sp}^t(yx)$ by Proposition 2, which implies that $\mathrm{r}^t(xy) = \mathrm{r}^t(yx)$ for each $x, y \in A$.

Similarly, we can show the rest of the relations in (2).

(3) If $x \in A$ and $n \in \mathbb{N}$, then $\mathrm{sp}^t(x^n) = \mathrm{sp}^t(x)^n$ by Proposition 7, which implies that $\mathrm{r}^t(x^n) = \mathrm{r}^t(x)^n$ for each $x \in A$ and all $n \in \mathbb{N}$. □

Proposition 9. *If every element of a topological algebra A has a functional left (right or two-sided) topological spectrum, then*

(1) $\mathrm{r}^t_\ell(x+y) \leq \mathrm{r}^t_\ell(x) + \mathrm{r}^t_\ell(y)$ *(respectively,* $\mathrm{r}^t_r(x+y) \leq \mathrm{r}^t_r(x) + \mathrm{r}^t_r(y)$ *and* $\mathrm{r}^t(x+y) \leq \mathrm{r}^t(x) + \mathrm{r}^t(y)$*) for all* $x, y \in A$*;*

(2) $\mathrm{r}^t_\ell(xy) \leq \mathrm{r}^t_\ell(x)\mathrm{r}^t_\ell(y)$ *(respectively,* $\mathrm{r}^t_r(xy) \leq \mathrm{r}^t_r(x)\mathrm{r}^t_r(y)$ *and* $\mathrm{r}^t(xy) \leq \mathrm{r}^t(x)\mathrm{r}^t(y)$*) for all* $x, y \in A$.

Proof. We only prove the first relations in (1) and (2). The rest of the relations can be proved similarly.

(1) Since every element of A has a functional left topological spectrum, we can present every $\lambda \in \mathrm{sp}^t_\ell(x+y)$ in the form $\lambda = \varphi_0(x) + \varphi_0(y)$ for some $\varphi_0 \in m(A)$. Since $|\lambda| \leq |\varphi_0(x)| + |\varphi_0(y)| \leq \mathrm{r}^t_\ell(x) + \mathrm{r}^t_\ell(y)$, condition (1) is fulfilled.

(2) Since every element of A has a functional left topological spectrum, we can present every $\lambda \in \mathrm{sp}^t_\ell(xy)$ in the form $\lambda = \varphi_0(x)\varphi_0(y)$ for some $\varphi_0 \in m(A)$. Since $|\lambda| = |\varphi_0(x)| \cdot |\varphi_0(y)| \leq \mathrm{r}^t_\ell(x) \cdot \mathrm{r}^t_\ell(y)$, condition (2) is fulfilled. □

Corollary 10. *Let A be a topological algebra. If every element of A has a functional topological spectrum, then the topological spectral radius is a submultiplicative seminorm on A.*

Proof. By Propositions 8 and 9 the topological spectral radius is a submultiplicative seminorm on A. □

Corollary 11. *Let A be a commutative Hausdorff locally m-pseudoconvex algebra. Then the topological spectral radius is a submultiplicative seminorm on A.*

Proof. By Corollary 5.4 in [3], A is a simplicial algebra and, by Proposition 5.2 in [3], A has a functional topological spectrum. Then the topological spectral radius is a submultiplicative seminorm on A by Corollary 10. $\square$

Proposition 10. *A topological algebra A is a Q-algebra if and only if A is a TQ-algebra and* $\mathrm{sp}(a) = \mathrm{sp}^t(a)$ *for each* $a \in A \setminus \mathrm{Qinv}(A)$.

Proof. If A is a Q-algebra, then $\mathrm{Tqinv}(A) = \mathrm{Qinv}(A)$ (see Proposition 2 in [1], p. 16). Hence, A is a TQ-algebra and $\mathrm{sp}(a) = \mathrm{sp}^t(a)$ for each $a \in A$.

Conversely, let A be a TQ-algebra and $\mathrm{sp}(a) = \mathrm{sp}^t(a)$ for each $a \in A \setminus \mathrm{Qinv}(A)$. Then $1 \in \mathrm{sp}(a)$. Since $a \in A \setminus \mathrm{Tqinv}(A)$, we get $\mathrm{Qinv}(A) = \mathrm{Tqinv}(A)$. Consequently, A is a Q-algebra. $\square$

4. CONCLUSION

The spectral mapping theorem holds for the topological spectrum of elements in commutative (not necessarily unital) topological algebras and the topological spectral radius (as a map) is a submultiplicative seminorm in case of topological algebras with a functional topological spectrum.

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Elementide topoloogilised spektrid topoloogilistes algebrates

Mati Abel ja Yuliana de Jesús Zárate-Rodríguez

On kirjeldatud topoloogilise algebra elementide ühe- ja kahepoolsete topoloogiliste spektrite omadusi ning näidatud, et funktsionaalse topoloogilise spektriga topoloogilistes algebrates on elementide topoloogiline spektraalraadius submultiplikatiivne poolnorm.