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MATHEMATICS

Periodic polynomial spline histopolation

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Abstract. Periodic polynomial spline histopolation with arbitrary placement of histogram knots is studied. Spline knots are considered coinciding with histogram knots. The main problem is the existence and uniqueness of the histopolant for any degree of spline and for any number of partition points. The results for arbitrary grid give as particular cases known assertions for the uniform grid but different techniques are used.

Key words: histopolation, interpolation, periodic spline, existence and uniqueness of histopolant.

1. INTRODUCTION

The given histopolation problem may in general be reduced to an equivalent interpolation problem and the derivative of the interpolant is the histopolant. On the contrary, a certain integral of the histopolant is the solution of a corresponding interpolation problem. This correspondence keeps the periodicity only in one direction, namely, the derivative of a periodic interpolant is periodic but not vice versa. This means that, at periodic histopolation, some problems like, e.g., convergence or error estimates cannot be reduced to similar problems at periodic interpolation. Fortunately, when asking about the existence and uniqueness of the solution in spline spaces we are successful because the uniqueness problem could be solved for the corresponding homogeneous problems in finite dimensional spaces and the periodicity would be preserved in both directions. The existence and uniqueness of the solution at periodic polynomial spline histopolation is the main problem in this paper. Several cases are treated and the reader can see that different tools are needed in the proofs of assertions.

2. THE HISTOPOLATION PROBLEM FOR PERIODICITY

For a given grid Δ_n of points $a = x_0 < x_1 < \ldots < x_n = b$ define the spline space

$$X_m(\Delta_n) = \{S | S : [x_{i-1}, x_i] \to \mathbb{R} \; is \; in \; \mathcal{P}_m \; (the \; set \; of \; all \; polynomials \\ of \; degree \; at \; most \; m) \; for \; i = 1, \ldots, n, \; S \in C^{m-1}[a,b]\}.$$

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It is known that $\dim X_m(\Delta_n) = n+m$. The space $X_{p,m}(\Delta_n)$ of periodic splines is

$$X_{p,m}(\Delta_n) = \left\{ S \in X_m(\Delta_n) | S^{(j)}(a) = S^{(j)}(b), \quad j = 0,1,\dots,m-1 \right\}.$$

Then $\dim X_{p,m}(\Delta_n) = n$ and this could be shown, e.g., in following way.

Lemma 1. *Let X be a vector space with $\dim X = n$ and $\phi_i, i = 1,\dots,k$, be linear functionals defined on X that are linearly independent. Then $\dim(\cap_{i=1}^{k} \ker \phi_i) = n-k$.*

To use this result we take functionals $\phi_i(S) = S^{(i)}(b) - S^{(i)}(a), i = 0,\dots,m-1$, defined on $X_m(\Delta_n)$. Their linear independence could be verified on polynomials $p_i(x) = x^{i+1}, i = 0,\dots,m-1$.

Denote the sizes of the intervals $h_i = x_i - x_{i-1},\ i = 1,\dots,n$. In the periodic histopolation problem we have to find $S \in X_{p,m}(\Delta_n)$ such that

$$\int_{x_{i-1}}^{x_i} S(x)dx = z_i h_i, \quad i = 1,\dots,n, \tag{1}$$

for given numbers z_i. Conditions (1) are called histopolation conditions.

Our main task in this paper is to answer the question: When has the formulated periodic histopolation problem a unique solution for any given values $z_i,\ i = 1,\dots,n$?

As our problem is linear, this question could be reformulated equivalently as follows: When the corresponding homogeneous problem has only trivial solution, i.e., when

$$S \in X_{p,m}(\Delta_n), \quad \int_{x_{i-1}}^{x_i} S(x)dx = 0, \quad i = 1,\dots,n, \quad \text{does it imply } S = 0 \text{ ?}$$

3. EXISTENCE AND UNIQUENESS

In this section we first indicate the cases where the solution exists and is unique.

Proposition 2. *For m even the periodic histopolation problem has a unique solution.*

Proof. Let $m = 2k$. Consider in $X_m(\Delta_n)$ the seminorm $\|S\| = \left(\int_a^b \left(S^{(k)}(x)\right)^2 dx\right)^{1/2}$. Suppose that $S \in X_{p,m}(\Delta_n)$. Then we get by using integration by parts and periodicity properties of the spline S

$$\begin{aligned}
\|S\|^2 &= \int_a^b S^{(k)}(x)S^{(k)}(x)dx \\
&= S^{(k)}(x)S^{(k-1)}(x)|_a^b - \int_a^b S^{(k+1)}(x)S^{(k-1)}(x)dx \\
&= \dots = (-1)^{k-1}S^{(2k-1)}(x)S(x)|_a^b + (-1)^k \int_a^b S^{(2k)}(x)S(x)dx \\
&= (-1)^k \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} S^{(2k)}(x)S(x)dx \\
&= (-1)^k \sum_{i=1}^{n} S^{(2k)}\left(\frac{x_{i-1}+x_i}{2}\right) \int_{x_{i-1}}^{x_i} S(x)dx.
\end{aligned}$$

Let now, in addition, $\int_{x_{i-1}}^{x_i} S(x)dx = 0, i = 1,\dots,n$. Then $\|S\| = 0$ or $S \in \mathcal{P}_{k-1}$. It is immediate to check that a periodic polynomial S is constant. The homogeneous histopolation conditions then yield $S = 0$, which completes the proof. $\square$

Recall that the sign change zero of a function f is a number z such that $f(z) = 0$ and there exists $\varepsilon_0 > 0$ such that $f(z-\varepsilon)f(z+\varepsilon) < 0$ for all $\varepsilon \in (0,\varepsilon_0)$. If $S \in X_m(\Delta_n)$, then let $Z(S)$ be the number of sign change zeros of S in the interval $[x_0,x_n)$. In the case $m = 0$ we talk here about sign change point z requiring only $f(z-\varepsilon)f(z+\varepsilon) < 0$ for all $\varepsilon \in (0,\varepsilon_0)$.

Lemma 3 (see, e.g., [13]). *For $S \in X_{p,m}(\Delta_n)$ it holds*

$$Z(S) \le \begin{cases} n-1, & \text{if } n \text{ is odd,} \\ n, & \text{if } n \text{ is even.} \end{cases}$$

This holds for all $m \in \mathbb{N} \cup \{0\}$.

Lemma 4. *If $S \in X_m(\Delta_n)$, $\int_{x_{j-1}}^{x_j} S(x)dx = 0, \quad j = 1,\ldots,n$, and $S(x) = 0$, where $x \in [x_{i-1},x_i]$ for some i, then $S(x) = 0, x \in [a,b]$.*

Proof. If $S(x) = 0, x \in [x_{i-1},x_i]$, then for $x \in [x_i,x_{i+1}]$ use the Taylor expansion

$$\begin{aligned} S(x) &= S(x_i) + S'(x_i)(x-x_i) + \ldots + \frac{S^{(m-1)}(x_i)}{(m-1)!}(x-x_i)^{m-1} \\ &+ \frac{S^{(m)}(x_i+0)}{m!}(x-x_i)^m = \frac{S^{(m)}(x_i+0)}{m!}(x-x_i)^m. \end{aligned}$$

As $\int_{x_i}^{x_{i+1}} S(x)dx = 0$, it holds $S^{(m)}(x_i+0) = 0$ and $S(x) = 0,\ x \in [x_i,x_{i+1}]$.

We may continue going from x_{i+1} to the right or similarly from x_{i-1} to the left and establish $S(x) = 0,\ x \in [a,b]$. □

Proposition 5. *For m odd and n odd the periodic histopolation problem has a unique solution.*

Proof. Let $S \in X_{p,m}(\Delta_n)$ and $\int_{x_{i-1}}^{x_i} S(x)dx = 0, i = 1,\ldots,n$. Let $S \ne 0$. If $S(x) = 0,\ x \in [x_{i-1},x_i]$, then by Lemma 4 it holds $S = 0$, which is already a contradiction. The condition $S(x) \ge 0$ for all $x \in [x_{i-1},x_i]$ and $S(\xi) > 0$ for some $\xi \in [x_{i-1},x_i]$ gives $\int_{x_{i-1}}^{x_i} S(x)dx > 0$, which is not the case. Similarly, $S(x) \le 0$ for all $x \in [x_{i-1},x_i]$ and $S(\xi) < 0$ for some $\xi \in [x_{i-1},x_i]$ do not take place. Thus, there are sign change zeros $\eta_i \in (x_{i-1},x_i),\ i = 1,\ldots,n$, of S and $Z(S) \ge n$. But by Lemma 3 it holds $Z(S) \le n-1$, which is a contradiction. This means that the homogeneous problem has only a trivial solution. □

Let us remark that the proof of Proposition 5 is valid for arbitrary $m \ge 1$ and n odd.

Proposition 6. *For $m = 1$ and n even the homogeneous periodic histopolation problem has a non-trivial solution.*

Proof. Take $\eta_i = (x_{i-1}+x_i)/2,\ i = 1,\ldots,n$. Let $c \ne 0$. Consider the function $S(x) = c_i(x-\eta_i),\ x \in [x_{i-1},x_i],\ i = 1,\ldots,n$. It holds $\int_{x_{i-1}}^{x_i} S(x)dx = 0,\ i = 1,\ldots,n$, for any choice of numbers c_i. The choice of $c_i = (-2c)/h_i$ for $i = 1,3,\ldots$ and $c_i = (2c)/h_i$ for $i = 2,4,\ldots$ ensures that $S \in X_{p,1}(\Delta_n),\ S \ne 0$, with

$$S(x_0) = S(x_2) = \ldots = S(x_n) = c$$

and

$$S(x_1) = S(x_3) = \ldots = S(x_{n-1}) = -c.$$

□

Proposition 7. *For m odd and $n = 2$ the homogeneous periodic histopolation problem has a non-trivial solution.*

Proof. For $m = 1$ the assertion is already proved by Proposition 6. We prove the general case by induction.

Denote $\eta_i = (x_{i-1} + x_i)/2,\ i = 1,2$. Let $m = 2k-1$ and $S \in X_{p,m}(\Delta_2)$ be such that $S \neq 0$ and

$$S(x) = c_{1,i}(x-\eta_i) + c_{3,i}(x-\eta_i)^3 + \ldots + c_{2k-1,i}(x-\eta_i)^{2k-1}, x \in [x_{i-1}, x_i],\ i = 1,2. \tag{2}$$

Clearly, this holds for the spline S from the proof of Proposition 6 in the case $m = 1$. Define

$$S_1(x) = c_{0,i} + \int_{\eta_i}^{x} S(s)ds$$

or

$$S_1(x) = c_{0,i} + \frac{c_{1,i}}{2}(x-\eta_i)^2 + \frac{c_{3,i}}{4}(x-\eta_i)^4 + \ldots + \frac{c_{2k-1,i}}{2k}(x-\eta_i)^{2k}, x \in [x_{i-1}, x_i]. \tag{3}$$

Then $S_1' = S$ and (3) implies that, for any numbers $c_{0,i}$,

$$S_1(x_{i-1}+0) = S_1(x_i - 0), \quad i = 1,2. \tag{4}$$

If $c_{0,1}$ and $c_{0,2}$ are such that

$$S_1(x_1 - 0) = S_1(x_1 + 0), \tag{5}$$

then $S_1 \in X_{p,m+1}(\Delta_2)$. Next, define $\bar{S}$ by

$$\bar{S}(x) = \int_{\eta_i}^{x} S_1(s)ds$$

or

$$\bar{S}(x) = c_{0,i}(x-\eta_i) + \frac{c_{1,i}}{2\cdot 3}(x-\eta_i)^3 + \ldots + \frac{c_{2k-1,i}}{2k(2k+1)}(x-\eta_i)^{2k+1}, x \in [x_{i-1}, x_i].$$

We see that $\bar{S}$ has the form (2), $\bar{S}' = S_1$, and

$$\bar{S}(x_{i-1}+0) = -\bar{S}(x_i - 0), \quad i = 1,2. \tag{6}$$

If, in addition to (5), we have

$$\bar{S}(x_1 - 0) = \bar{S}(x_1 + 0), \tag{7}$$

then $\bar{S} \in X_{p,m+2}(\Delta_2)$ due to (4)–(7).

It remains to show that by (5) and (7) we can determine suitable numbers $c_{0,1}$ and $c_{0,2}$. Equation (5) is, in fact,

$$\begin{aligned} c_{0,1} - c_{0,2} &= \frac{c_{1,2}}{2}\left(\frac{-h_2}{2}\right)^2 + \ldots + \frac{c_{2k-1,2}}{2k}\left(\frac{-h_2}{2}\right)^{2k} \\ &\quad - \left(\frac{c_{1,1}}{2}\left(\frac{h_1}{2}\right)^2 + \ldots + \frac{c_{2k-1,1}}{2k}\left(\frac{h_1}{2}\right)^{2k}\right) \end{aligned}$$

and (7) is

$$\begin{aligned} c_{0,1}\frac{h_1}{2} + c_{0,2}\frac{h_2}{2} &= \frac{c_{1,2}}{2\cdot 3}\left(\frac{-h_2}{2}\right)^3 + \ldots + \frac{c_{2k-1,2}}{2k(2k+1)}\left(\frac{-h_2}{2}\right)^{2k+1} \\ &\quad - \left(\frac{c_{1,1}}{2\cdot 3}\left(\frac{h_1}{2}\right)^3 + \ldots + \frac{c_{2k-1,1}}{2k(2k+1)}\left(\frac{h_1}{2}\right)^{2k+1}\right). \end{aligned}$$

But this system has the non-zero determinant $(h_1 + h_2)/2$. However, as $S \neq 0$ then $\bar{S} \neq 0$. □

We say that the grid $x_0 < x_1 < \ldots < x_n$ is pairwise uniform if n is even and for any i even it holds that $x_{i+1} - x_i = h_1$, $x_{i+2} - x_{i+1} = h_2$ or $x_{i+1} - x_i = h_2$, $x_{i+2} - x_{i+1} = h_1$.

Corollary 8. *The homogeneous periodic histopolation problem has a non-trivial solution for m odd and the pairwise uniform grid.*

In particular, the case of the uniform grid for m odd and n even is included in Corollary 8. This result could be found in [10,13].

In general, we state as an open problem the following.

Conjecture. *For m odd and n even the homogeneous periodic histopolation problem has a non-trivial solution.*

Define the subspace of $X_{p,m}(\Delta_n)$ as

$$X_{0,p,m}(\Delta_n) = \left\{ S \in X_{p,m}(\Delta_n) \mid \int_{x_{i-1}}^{x_i} S(x)dx = 0, i = 1, \ldots, n \right\}.$$

For m odd and n even it may be that $X_{0,p,m}(\Delta_n) \neq \{0\}$ (if the Conjecture is true, then always). It is natural to ask what $\dim X_{0,p,m}(\Delta_n)$ is in this case.

Remove from the grid $\Delta_n : a = x_0 < x_1 < \ldots < x_n = b$ a knot x_i. We get the grid $\Delta'_{n-1} : a = x_0 < x_1 < \ldots < x_{i-1} < x_{i+1} < \ldots < x_n = b$ with the number of subintervals $n-1$, which is odd. By Proposition 5 it holds $X_{0,p,m}(\Delta'_{n-1}) = \{0\}$. Clearly, $X_{p,m}(\Delta'_{n-1}) + X_{0,p,m}(\Delta_n) \subset X_{p,m}(\Delta_n)$. The sum $X_{p,m}(\Delta'_{n-1}) + X_{0,p,m}(\Delta_n)$ is a direct sum that follows from the relation

$$X_{p,m}(\Delta'_{n-1}) \cap X_{0,p,m}(\Delta_n) = \{0\}.$$

Thus, the equality

$$\dim X_{p,m}(\Delta_n) = \dim X_{p,m}(\Delta'_{n-1}) + \dim X_{0,p,m}(\Delta_n)$$

implies that $\dim X_{0,p,m}(\Delta_n) = 1$.

The obtained results about the existence of non-trivial solutions for the homogeneous problem yield the following.

Theorem 9. *For m even or m and n odd the periodic histopolation problem has for each z_i, $i = 1, \ldots, n$, the unique solution. For m odd and n even there may exist (if the Conjecture is true, then always exist) z_i, $i = 1, \ldots, n$, such that the periodic histopolation problem does not have a solution.*

4. BIBLIOGRAPHICAL NOTES

In this section we acquaint the reader with a subjective list of works on periodic spline interpolation and histopolation. The results about the existence and uniqueness of a solution for periodic polynomial spline interpolation could be found in [1]. A short overview of existence results by several authors are presented in [10], which contains also convergence estimates for problems on a uniform grid with interpolation knots not necessarily in grid points. The paper [9] contains results about properties of periodic interpolating polynomial splines on subintervals. The existence and uniqueness results of periodic solutions for the uniform grid case in several papers are based on the theory of circulant matrices, see, e.g. [4,5]. The general non-uniform grid is considered in [6] for low degree periodic splines with convergence estimates. The work [12] gives error estimates for the periodic quadratic spline interpolation problem arising from the histopolation problem with these splines. In [8] the existence and uniqueness problem of solution in periodic quartic polynomial spline histopolation ($m = 4$) is stated generally but solved only for the uniform grid. Unlike in the other studies, the spline representation via moments is used. Our Proposition 2 gives here the answer for the general grid case. The periodic interpolation problem on a uniform grid with certain non-polynomial functions is studied in [3], and histopolation in [2]. Interpolation with periodic polynomial splines of the defect greater than minimal is studied in [11,14,15]. Cubic spline histopolation on a general grid is treated in [7] from several aspects, including methods of the practical construction of the histopolant.

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Perioodiliste polünomiaalsete splainidega histopoleerimine

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Antud võrgul $a = x_0 < x_1 < \ldots < x_n = b$ vaadeldakse polünomiaalseid splaine S, mis on igas osalõigus $[x_{i-1}, x_i]$, $i = 1, \ldots, n$, ülimalt m astme polünoomid ja kuuluvad ruumi $C^{m-1}[a,b]$. Splain S on perioodiline, kui $S^{(j)}(a) = S^{(j)}(b)$, $j = 0, \ldots, m-1$. Histopoleerimisülesandes nõutakse, et $\int_{x_{i-1}}^{x_i} S(x)dx = (x_i - x_{i-1})z_i$, $i = 1, \ldots, n$, kus z_i on antud arvud. Artiklis on põhiprobleemiks vastuse otsimine küsimusele, millal selline histopoleerimisülesanne on igasuguste arvude z_i korral üheselt lahenduv. Probleemile oli lahendus varem teada ühtlase võrgu korral, kus sõlmed on võetud $x_i = a + ih, i = 0, \ldots, n, h = (b-a)/n$. Selles artiklis näitame suvalise võrgu korral, et lahend on ühene, kui m on paaris, samuti on lahend ühene, kui m ja n on mõlemad paaritud. Tõestame, et lahend ei ole ühene ehk vastaval homogeensel ülesandel on mittetriviaalne lahend, kui $m = 1$ ja n on paaris, samuti kui m on paaritu ning $n = 2$. Üldine juht, kus m on paaritu ja n paaris, näib olevat raske probleem ning selle oletatav lahendus on meil sõnastatud hüpoteesina. On tähelepanuväärne, et kui ühtlase võrgu korral on võimalik anda kõikide juhtude jaoks ühtse meetodiga vastus, siis üldise võrgu korral kasutame erinevate juhtude puhul lahenduse saamiseks erinevaid tõestusvõtteid.