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MATHEMATICS

# Pointwise approximation of modified conjugate functions by matrix operators of their Fourier series

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**Abstract.** We extend the results presented by Xh. Z. Krasniqi (Slight extensions of some theorems on the rate of pointwise approximation of functions from some subclasses of $L^p$. *Acta Comment. Univ. Tartu. Math.*, 2013, **17**, 89–101) and W. Lenski and B. Szal (Approximation of functions belonging to the class $L^p(\omega)$ by linear operators. *Acta Comment. Univ. Tartu. Math.*, 2009, **13**, 11–24) to the case when a conjugate function depends on $r$ and where in the measures of estimations $r$-differences of the entries are used.

**Key words:** rate of approximation, summability, Fourier series.

## 1. INTRODUCTION

Let $L^p$ $(1 \leq p < \infty)$ be the class of all $2\pi$-periodic real-valued functions, integrable in the Lebesgue sense with the $p$th power over $Q = [-\pi, \pi]$ with the norm

$$\|f\| = \|f(\cdot)\|_{L^p} = \left( \int_Q |\, f(t)\,|^p \, dt \right)^{1/p}.$$

Given a function of class $L^p$ let us consider its conjugate trigonometric Fourier series

$$\widetilde{S}f(x) := \sum_{\nu=1}^{\infty} (a_\nu(f) \sin \nu x - b_\nu(f) \cos \nu x)$$

with the partial sums $\widetilde{S}_k f$. We know that if $f \in L^1$, then

$$\widetilde{f}(x) := -\frac{1}{\pi} \int_0^\pi \psi_x(t) \frac{1}{2} \cot \frac{t}{2} dt = \lim_{\varepsilon \to 0^+} \widetilde{f}(x, \varepsilon) = \lim_{\varepsilon \to 0^+} \widetilde{f}_r(x, \varepsilon),$$

where, for $r \in \mathbb{N}$,

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$$\widetilde{f}_r(x,\varepsilon) := \begin{cases} -\frac{1}{\pi}\left(\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\varepsilon}^{\frac{2(m+1)\pi}{r}-\varepsilon} + \int_{\frac{2[r/2]\pi}{r}+\varepsilon}^{\frac{(2[r/2]+1)\pi}{r}}\right)\psi_x(t)\,\frac{1}{2}\cot\frac{t}{2}dt \text{ for an odd } r, \\ -\frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\varepsilon}^{\frac{2(m+1)\pi}{r}-\varepsilon}\psi_x(t)\,\frac{1}{2}\cot\frac{t}{2}dt \text{ for an even } r \end{cases}$$

and $\widetilde{f}(x,\varepsilon) = \widetilde{f}_1(x,\varepsilon) := -\frac{1}{\pi}\int_{\varepsilon}^{\pi}\psi_x(t)\,\frac{1}{2}\cot\frac{t}{2}dt$, with $\psi_x(t) := f(x+t) - f(x-t)$, exist for almost all $x$ (cf. [5, Theorem (3.1)IV]).

Let $A := (a_{n,k})$ be an infinite matrix of real numbers such that

$$a_{n,k} \geq 0 \text{ when } k,n = 0,1,2,..., \lim_{n\to\infty} a_{n,k} = 0 \text{ and } \sum_{k=0}^{\infty} a_{n,k} = 1.$$

We will use the notations $A_{n,r} = \sum_{k=0}^{\infty}|a_{n,k} - a_{n,k+r}|$ for $r \in \mathbb{N}$ and

$$\widetilde{T}_{n,A}f(x) := \sum_{k=0}^{\infty} a_{n,k}\widetilde{S}_k f(x) \quad (n = 0,1,2,...)$$

for the $A$-transformation of $\widetilde{S}f$.

In this paper, we will estimate the deviation $\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}_r(x,\varepsilon)\right|$ by the function of modulus of continuity type, i.e. nondecreasing continuous function $\widetilde{\omega}$ having the following properties: $\widetilde{\omega}(0) = 0$, $\widetilde{\omega}(\delta_1+\delta_2) \leq \widetilde{\omega}(\delta_1) + \widetilde{\omega}(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1+\delta_2 \leq 2\pi$. We will also consider functions from the following subclass $L^p(\widetilde{\omega})_\beta$ of $L^p$ :

$$L^p(\widetilde{\omega})_\beta = \left\{f \in L^p : \widetilde{\omega}_\beta(f,\delta)_{L^p} = O(\widetilde{\omega}(\delta)) \text{ when } \delta \in [0,2\pi] \text{ and } \beta \geq 0\right\},$$

where

$$\widetilde{\omega}_\beta f(\delta)_{L^p} = \sup_{0\leq|t|\leq\delta}\left\{\left|\sin\frac{rt}{2}\right|^\beta \|\psi_\cdot(t)\|_{L^p}\right\}.$$

It is clear that for $\beta > \alpha \geq 0, \widetilde{\omega}_\beta f(\delta)_{L^p} \leq \widetilde{\omega}_\alpha f(\delta)_{L^p}$ and it is easy to see that $\widetilde{\omega}_0 f(\cdot)_{L^p} = \widetilde{\omega} f(\cdot)_{L^p}$ is the classical integral modulus of continuity of $f$.

The above deviation was estimated with $r = 1$ in [2] and generalized in [1] as follows:

**Theorem** [1, Theorem 8, p. 95]**.** *If $f \in L^p(\widetilde{\omega})_\beta$ with $\beta < 1 - \frac{1}{p}$, where $\widetilde{\omega}$ satisfies the conditions*

$$\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\gamma}|\psi_x(t)|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p} = O_x\left((n+1)^\gamma\right) \tag{1}$$

*and*

$$\left\{\int_0^{\pi/(n+1)}\left(\frac{t|\psi_x(t)|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{1/p} = O_x\left((n+1)^{-1}\right) \tag{2}$$

*with $0 < \gamma < \beta + \frac{1}{p}$, then*

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right)\right| = O_x\left((n+1)^{\beta+\frac{1}{p}+1} A_{n,1}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

In our theorems we generalize the above results using $\widetilde{f}_r(x,\varepsilon)$ with $r \in \mathbb{N}$ instead of $\widetilde{f}_1(x,\varepsilon) = \widetilde{f}(x,\varepsilon)$. In the paper $\sum_{k=a}^{b} = 0$ when $a > b$.

## 2. STATEMENT OF THE RESULTS

First we will present the estimates of the quantity $\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r(x,\varepsilon)\right|$. Finally, we will formulate some remarks and corollaries.

**Theorem 1.** *Let $f\in L^p$, $0\leq\beta<1-\frac{1}{p}$ and let a function of modulus of continuity type $\widetilde{\omega}$ satisfy the conditions: for $r\in\mathbb{N}$*

$$\left\{\int_0^{\frac{\pi}{r(n+1)}}\left(\frac{t\left|\psi_x(t)\right|\left|\sin\frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}(t)}\right)^p dt\right\}^{1/p}=O_x\left((n+1)^{-1}\right), \tag{3}$$

*for a natural $r\geq 3$*

$$\left\{\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\left(\frac{\left|\psi_x(t)\right|\left|\sin\frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}\left(t-\frac{2m\pi}{r}\right)}\right)^p dt\right\}^{1/p}=O_x(1), \tag{4}$$

*where $m\in\left\{1,\dots\left[\frac{r}{2}\right]\right\}$ when $r$ is an odd natural number or $m\in\left\{1,\dots\left[\frac{r}{2}\right]-1\right\}$ when $r$ is an even natural number, and for $r\in\mathbb{N}$*

$$\left\{\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\left(\frac{\left|\psi_x(t)\right|\left|\sin\frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}(t)\left(t-\frac{2m\pi}{r}\right)^{\gamma}}\right)^p dt\right\}^{1/p}=O_x\left((n+1)^{\gamma}\right), \tag{5}$$

*with $0<\gamma<\beta+\frac{1}{p}$, where $m\in\left\{0,\dots\left[\frac{r}{2}\right]\right\}$ when $r$ is an odd natural number or $m\in\left\{0,\dots\left[\frac{r}{2}\right]-1\right\}$ when $r$ is an even natural number. Moreover, let $\widetilde{\omega}$ satisfy, for a natural $r\geq 2$, the conditions:*

$$\left\{\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{\left|\psi_x(t)\right|\left|\sin\frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}\left(\frac{2(m+1)\pi}{r}-t\right)}\right)^p dt\right\}^{1/p}=O_x(1), \tag{6}$$

$$\left\{\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\left(\frac{\left|\psi_x(t)\right|\left|\sin\frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}(t)\left(\frac{2(m+1)\pi}{r}-t\right)^{\gamma}}\right)^p dt\right\}^{1/p}=O_x\left((n+1)^{\gamma}\right), \tag{7}$$

*with $0<\gamma<\beta+\frac{1}{p}$, where $m\in\left\{0,\dots\left[\frac{r}{2}\right]-1\right\}$. If a matrix $A$ is such that*

$$\left[\sum_{l=0}^{n}\sum_{k=l}^{r+l-1}a_{n,k}\right]^{-1}=O(1) \tag{8}$$

*and*

$$\sum_{k=0}^{\infty}(k+1)^2a_{n,k}=O\left((n+1)^2\right) \tag{9}$$

*are true for $r\in\mathbb{N}$, then*

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{r(n+1)}\right)\right|=O_x\left((n+1)^{\beta+\frac{1}{p}+1}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

**Theorem 2.** *Let $f \in L^p$, $0 \le \beta < 1 - \frac{1}{p}$ and let a function of modulus of continuity type $\widetilde{\omega}$ satisfy, for $r \in \mathbb{N}$, the conditions:*

$$\left\{ \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} \left( \frac{|\psi_x(t)| \left|\sin \frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}\left(t - \frac{2m\pi}{r}\right)} \right)^p dt \right\}^{1/p} = O_x(1), \tag{10}$$

*and (5) with $0 < \gamma < \beta + \frac{1}{p}$, where $m \in \left\{0, \ldots \left[\frac{r}{2}\right]\right\}$ when $r$ is an odd natural number or $m \in \left\{0, \ldots \left[\frac{r}{2}\right] - 1\right\}$ when $r$ is an even natural number. Moreover, let $\widetilde{\omega}$ satisfy for natural $r \ge 2$, the conditions (6) and (7) with $0 < \gamma < \beta + \frac{1}{p}$, where $m \in \left\{0, \ldots \left[\frac{r}{2}\right] - 1\right\}$. If a matrix $A$ is such that* (8) *and*

$$\sum_{k=0}^{\infty} (k+1) a_{n,k} = O(n+1) \tag{11}$$

*are true for $r \in \mathbb{N}$ then*

$$\left| \widetilde{T}_{n,A} f(x) - \widetilde{f}_r\left(x, \frac{\pi}{r(n+1)}\right) \right| = O_x\left( (n+1)^{\beta+\frac{1}{p}+1} A_{n,r} \widetilde{\omega}\left(\frac{\pi}{n+1}\right) \right).$$

We can observe that if $f \in L^p(\widetilde{\omega})_\beta$ and

$$\int_0^{\delta} \left( \frac{u^{\beta}}{\widetilde{\omega}(u)} \right)^p du = O(1) \text{ when } \delta \to 0^+ \tag{12}$$

is fulfilled, then the conditions (3)–(7) and (10) always hold with $\|\psi_\cdot(t)\|_{L^p}$ instead of $|\psi_x(t)|$. Hence from Theorems 1 and 2 we can obtain the following corollary:

**Corollary 1.** *Let $f \in L^p(\widetilde{\omega})_\beta$ with $0 \le \beta < 1 - \frac{1}{p}$, where $\widetilde{\omega}$ satisfy condition* (12). *If a matrix $A$ is such that* (8) *and* (9) *or* (8) *and* (11) *are true for $r \in \mathbb{N}$, then*

$$\left\| \widetilde{T}_{n,A} f(\cdot) - \widetilde{f}_r\left(\cdot, \frac{\pi}{r(n+1)}\right) \right\|_{L^p} = O\left( (n+1)^{\beta+\frac{1}{p}+1} A_{n,r} \widetilde{\omega}\left(\frac{\pi}{n+1}\right) \right).$$

**Corollary 2.** *We can observe that in the case $r = 1$, the conditions* (3)–(7) *in Theorem 1 reduce to* (1) *and* (2). *Thus we obtain the results from* [2] *and* [1].

**Remark 1.** If we consider the following more natural conditions

$$\left\{ \int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} \left( \frac{\left(t - \frac{2m\pi}{r}\right)^{-\gamma} |\psi_x(t)| \left|\sin \frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}(t)} \right)^p dt \right\}^{1/p} = O_x\left( (n+1)^{\gamma-\frac{1}{p}} \right),$$

$$\left\{ \int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}} \left( \frac{\left(\frac{2(m+1)\pi}{r} - t\right)^{-\gamma} |\psi_x(t)| \left|\sin \frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}(t)} \right)^p dt \right\}^{1/p} = O_x\left( (n+1)^{\gamma-\frac{1}{p}} \right),$$

for $\gamma \in \left(\frac{1}{p}, \frac{1}{p} + \beta\right)$ where $\beta > 0$, instead of (5) and (7), and

$$\left\{ \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} \left( \frac{|\psi_x(t)| \left|\sin \frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}\left(t - \frac{2m\pi}{r}\right)} \right)^p dt \right\}^{1/p} = O_x\left( (n+1)^{-\frac{1}{p}} \right),$$

$$\left\{\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{|\psi_x(t)|\left|\sin\frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}\left(\frac{2(m+1)\pi}{r}-t\right)}\right)^{p}dt\right\}^{1/p}=O_x\left((n+1)^{-\frac{1}{p}}\right)$$

instead of conditions $(6)$ and $(10)$, respectively, then our estimate takes the form

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}\left(x,\frac{\pi}{r(n+1)}\right)\right|=O_x\left((n+1)^{\beta+1}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

**Remark 2.** We note that our extra conditions $(8)$, $(9)$, and $(11)$ for a lower triangular infinite matrix $A$ always hold.

**Corollary 3.** *Under the above remarks and the obvious inequality*

$$A_{n,r}\le rA_{n,1}\ \textit{for}\ r\in\mathbb{N}, \tag{13}$$

*our results also improve and generalize the mentioned result of Krasniqi* [1].

**Remark 3.** We note that instead of $L^p(\widetilde{\omega})_\beta$ one can consider other subclasses of $L^p$ generated by any function of modulus continuity type, e.g. $\widetilde{\omega}_x$ such that

$$\widetilde{\omega}_x(f,\delta)=\sup_{|t|\le\delta}|\psi_x(t)|\le\widetilde{\omega}_x(\delta)\ \textit{or}\ \widetilde{\omega}_x(f,\delta)=\frac{1}{\delta}\int_0^{\delta}|\psi_x(t)|\,dt\le\widetilde{\omega}_x(\delta).$$

**Remark 4.** We note that our condition $(12)$ holds if we take $\widetilde{\omega}(\delta)=\delta^{\alpha}$ with $0<\alpha<\beta+\frac{1}{p}$.

## 3. AUXILIARY RESULTS

We begin this section with some notations from [4] and [5, Section 5 of Chapter II]. Let for $r=1,2,\ldots$

$$D^{\circ}_{k,r}(t)=\frac{\sin\frac{(2k+r)t}{2}}{2\sin\frac{rt}{2}},\ \widetilde{D^{\circ}}_{k,r}(t)=\frac{\cos\frac{(2k+r)t}{2}}{2\sin\frac{rt}{2}}$$

and

$$\widetilde{D}_{k,r}(t)=\frac{\cos\frac{rt}{2}-\cos\frac{(2k+r)t}{2}}{2\sin\frac{rt}{2}}=\frac{\cos\frac{rt}{2}}{2\sin\frac{rt}{2}}-\widetilde{D^{\circ}}_{k,r}(t).$$

It is clear by [5] that

$$\widetilde{S}_kf(x)=-\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)\widetilde{D}_{k,1}(t)\,dt$$

and

$$\widetilde{T}_{n,A}f(x)=-\frac{1}{\pi}\int_{-\pi}^{\pi}f(x+t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D}_{k,1}(t)\,dt.$$

Now we present a very useful property of the modulus of continuity.

**Lemma 1** ([5])**.** *A function $\widetilde{\omega}$ of a modulus of continuity type on the interval $[0,2\pi]$ satisfies the condition* $\delta_2^{-1}\widetilde{\omega}(\delta_2)\le 2\delta_1^{-1}\widetilde{\omega}(\delta_1)$ for $\delta_2\ge\delta_1>0$.

Next, we present the known estimates.

**Lemma 2** ([5]). *If* $0 < |t| \le \pi$, *then*

$$\left|\widetilde{D^{\circ}}_{k,1}(t)\right| \le \frac{\pi}{2|t|}, \quad \left|\widetilde{D}_{k,1}(t)\right| \le \frac{\pi}{|t|}$$

*and for any real t we have*

$$\left|D^{\circ}_{k,1}(t)\right| \le k+\frac{1}{2}, \quad \left|\widetilde{D}_{k,1}(t)\right| \le \frac{1}{2}k(k+1)|t|, \quad \left|\widetilde{D}_{k,1}(t)\right| \le k+1.$$

**Lemma 3** ([3,4]). *Let* $r \in N$, $l \in Z$ *and* $(a_n) \subset \mathbb{C}$ *If* $t \neq \frac{2l\pi}{r}$, *then for every* $m \ge n$

$$\sum_{k=n}^{m} a_k \sin kt = -\sum_{k=n}^{m}(a_k - a_{k+r})\widetilde{D^{\circ}}_{k,r}(t) + \sum_{k=m+1}^{m+r} a_k \widetilde{D^{\circ}}_{k,-r}(t) - \sum_{k=n}^{n+r-1} a_k \widetilde{D^{\circ}}_{k,-r}(t),$$

$$\sum_{k=n}^{m} a_k \cos kt = \sum_{k=n}^{m}(a_k - a_{k+r}) D^{\circ}_{k,r}(t) - \sum_{k=m+1}^{m+r} a_k D^{\circ}_{k,-r}(t) + \sum_{k=n}^{n+r-1} a_k D^{\circ}_{k,-r}(t).$$

## 4. PROOFS OF THEOREMS

### 4.1. Proof of Theorem 1

It is clear that for an odd $r$

$$\begin{aligned}
\widetilde{T}_{n,A}f(x) - \widetilde{f}_r\left(x, \frac{\pi}{r(n+1)}\right) &= \frac{1}{\pi}\left(\sum_{m=0}^{[r/2]} \int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} + \sum_{m=0}^{[r/2]-1} \int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\right) \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \widetilde{D}^{\circ}_{k,1}(t)\,dt \\
&\quad - \frac{1}{\pi}\int_{0}^{\frac{\pi}{r(n+1)}} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k}\widetilde{D}_{k,1}(t)\,dt \\
&\quad - \frac{1}{\pi}\left(\sum_{m=1}^{[r/2]} \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} + \sum_{m=0}^{[r/2]-1} \int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\right) \psi_x(t) \sum_{k=0}^{\infty} a_{n,k}\widetilde{D}_{k,1}(t)\,dt \\
&= J_1(x) + J_2(x) + J_0(x) + J_3(x) + J_4(x)
\end{aligned}$$

and for an even $r$

$$\begin{aligned}
\widetilde{T}_{n,A}f(x) - \widetilde{f}_r\left(x, \frac{\pi}{r(n+1)}\right) &= \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\left(\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} + \int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\right) \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \widetilde{D}^{\circ}_{k,1}(t)\,dt + J_0(x) \\
&\quad - \frac{1}{\pi}\left(\sum_{m=1}^{[r/2]-1} \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} + \sum_{m=0}^{[r/2]-1} \int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\right) \psi_x(t) \sum_{k=0}^{\infty} a_{n,k}\widetilde{D}_{k,1}(t)\,dt \\
&= J_1^{'}(x) + J_2(x) + J_0(x) + J_3^{'}(x) + J_4(x).
\end{aligned}$$

Then

$$\begin{aligned} & \left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{r(n+1)}\right)\right| \\ \leq\ & |J_0(x)|+|J_1(x)|+\left|J_1^{'}(x)\right|+|J_2(x)|+|J_3(x)|+\left|J_3^{'}(x)\right|+|J_4(x)|. \end{aligned}$$

By Lemma 3,

$$\frac{1}{2}\sum_{k=0}^{\infty}a_{n,k}\cos\frac{(2k+1)t}{2}=\frac{1}{2}\left(\sum_{k=0}^{\infty}a_{n,k}\cos kt\cos\frac{t}{2}-\sum_{k=0}^{\infty}a_{n,k}\sin kt\sin\frac{t}{2}\right)$$

$$\begin{aligned} =\ & \frac{\cos\frac{t}{2}}{2}\left(\sum_{k=0}^{\infty}(a_{n,k}-a_{n,k+r})D_{k,r}^{\circ}(t)+\sum_{k=0}^{r-1}a_{n,k}D_{k,-r}^{\circ}(t)\right) \\ & -\frac{\sin\frac{t}{2}}{2}\left(-\sum_{k=0}^{\infty}(a_{n,k}-a_{n,k+r})\widetilde{D^{\circ}}_{k,r}(t)-\sum_{k=0}^{r-1}a_{n,k}\widetilde{D^{\circ}}_{k,-r}(t)\right). \end{aligned}$$

Therefore,

$$|J_1(x)|+\left|J_1^{'}(x)\right|\leq\frac{2}{\pi}\sum_{m=0}^{[r/2]}\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\frac{|\psi_x(t)|}{\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|}A_{n,r}dt.$$

Using the estimates $\left|\sin\frac{t}{2}\right|\geq\frac{|t|}{\pi}$ for $t\in[0,\pi]$, $\left|\sin\frac{rt}{2}\right|\geq\frac{rt}{\pi}-2m$ for $t\in\left[\frac{2m\pi}{r},\frac{2m\pi}{r}+\frac{\pi}{r}\right]$, where $m\in\{0,...,[r/2]\}$, we obtain

$$|J_1(x)|+\left|J_1^{'}(x)\right|\leq\frac{2\pi}{r}A_{n,r}\sum_{m=0}^{[r/2]}\left[\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\left(\frac{|\psi_x(t)|}{\widetilde{\omega}(t)\left(t-\frac{2m\pi}{r}\right)^{\gamma}}\left|\sin\frac{rt}{2}\right|^{\beta}\right)^{p}dt\right]^{\frac{1}{p}}$$

$$\cdot\left[\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\left(\frac{\widetilde{\omega}(t)\left(t-\frac{2m\pi}{r}\right)^{\gamma}}{t\left(t-\frac{2m\pi}{r}\right)\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}}\text{ with } q=p(p-1)^{-1}.$$

Hence by (5)

$$|J_1(x)|+\left|J_1^{'}(x)\right|=O_x(1)A_{n,r}\sum_{m=0}^{[r/2]}(n+1)^{\gamma}\left[\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\left(\frac{\widetilde{\omega}(t)\left(t-\frac{2m\pi}{r}\right)^{\gamma}}{t\left(t-\frac{2m\pi}{r}\right)\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}}.$$

Using Lemma 1 we get

$$\left[\int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}}\left(\frac{\widetilde{\omega}(t)\left(t-\frac{2m\pi}{r}\right)^{\gamma}}{t\left(t-\frac{2m\pi}{r}\right)\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}}$$

$$= \left[\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\left(\frac{\widetilde{\omega}\left(t+\frac{2m\pi}{r}\right)t^{\gamma}}{t\left(t+\frac{2m\pi}{r}\right)\left|\sin\frac{rt+2m\pi}{2}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}} \leq \left[2^{q}\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}\left(\frac{\widetilde{\omega}(t)}{t^{2-\gamma}\left|\frac{rt}{\pi}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}}$$

$$\leq \frac{\widetilde{\omega}\left(\frac{\pi}{r(n+1)}\right)}{\frac{\pi}{r(n+1)}}\left[4^{q}\left(\frac{\pi}{r}\right)^{\beta q}\int_{\frac{\pi}{r(n+1)}}^{\frac{\pi}{r}}t^{(\gamma-1-\beta)q}dt\right]^{\frac{1}{q}} = O\left((n+1)^{1-\gamma+\beta+\frac{1}{p}}\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right),$$

for $0<\gamma<\beta+\frac{1}{p}$. Therefore

$$|J_1(x)|+\left|J_1^{'}(x)\right| = O_x\left((n+1)^{1+\beta+\frac{1}{p}}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

Next,

$$|J_2(x)| \leq \frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\frac{|\psi_x(t)|}{\left|\sin\frac{t}{2}\sin\frac{rt}{2}\right|}A_{n,r}dt.$$

Using the estimates $\left|\sin\frac{t}{2}\right| \geq \frac{|t|}{\pi}$ for $t\in[0,\pi]$, $\left|\sin\frac{rt}{2}\right| \geq 2(m+1)-\frac{rt}{\pi}$ for $t\in\left[\frac{2(m+1)\pi}{r}-\frac{\pi}{r},\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}\right]$, where $m\in\{0,...,[r/2]-1\}$, we get

$$|J_2(x)| \leq \frac{\pi}{r}A_{n,r}\sum_{m=0}^{[r/2]-1}\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\left(\frac{|\psi_x(t)|\left|\sin\frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}(t)\left(\frac{2(m+1)\pi}{r}-t\right)^{\gamma}}\right)^{p}dt\right]^{\frac{1}{p}}$$

$$\cdot\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\left(\frac{\widetilde{\omega}(t)\left(\frac{2(m+1)\pi}{r}-t\right)^{\gamma}}{t\left(\frac{2(m+1)\pi}{r}-t\right)\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}}, \text{ where } q=p(p-1)^{-1}.$$

Analogously as before, by (7)

$$|J_2(x)| \leq O_x(1)A_{n,r}\sum_{m=0}^{[r/2]-1}(n+1)^{\gamma}\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\left(\frac{\widetilde{\omega}(t)\left(\frac{2(m+1)\pi}{r}-t\right)^{\gamma}}{t\left(\frac{2(m+1)\pi}{r}-t\right)\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{\frac{1}{q}}$$

$$= O_x\left((n+1)^{1+\beta+\frac{1}{p}}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right),$$

for $0<\gamma<\beta+\frac{1}{p}$.

Further, by Lemma 2

$$|J_3(x)| + \left|J_3'(x)\right| \le 2\sum_{m=1}^{[r/2]} \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} \frac{|\psi_x(t)|}{t} dt$$

$$\le \quad 2\sum_{m=1}^{[r/2]} \left[ \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} \left( \frac{|\psi_x(t)|}{\widetilde{\omega}\left(t-\frac{2m\pi}{r}\right)} \left|\sin\frac{rt}{2}\right|^{\beta} \right)^p dt \right]^{1/p}$$

$$\cdot \left[ \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} \left( \frac{\widetilde{\omega}\left(t-\frac{2m\pi}{r}\right)}{\left|\sin\frac{rt}{2}\right|^{\beta}} \right)^q dt \right]^{1/q}, \text{ where } q = p(p-1)^{-1}.$$

Hence, by condition $(4)$ for $0 \le \beta < 1 - \frac{1}{p}$

$$|J_3(x)| + \left|J_3'(x)\right| = O_x(1) \left[ \int_0^{\frac{\pi}{r(n+1)}} \left( \frac{\widetilde{\omega}(t)}{\left|\sin\frac{rt}{2}\right|^{\beta}} \right)^q dt \right]^{1/q}$$

$$\le O_x(1)\, \widetilde{\omega}\left(\frac{\pi}{r(n+1)}\right) \left[ \int_0^{\frac{\pi}{r(n+1)}} \frac{dt}{\left|\sin\frac{rt}{2}\right|^{\beta q}} \right]^{1/q} = O_x\left( (n+1)^{\beta-1/q}\, \widetilde{\omega}\left(\frac{\pi}{n+1}\right) \right).$$

Finally, we note that applying condition $(8)$ we have

$$[(n+1)A_{n,r}]^{-1} = \left[\sum_{l=0}^{n} A_{n,r}\right]^{-1} \le \left[\sum_{l=0}^{n}\sum_{k=l}^{\infty} |a_{n,k} - a_{n,k+r}|\right]^{-1}$$

$$\le \quad \left[\sum_{l=0}^{n} \left|\sum_{k=l}^{\infty} (a_{n,k} - a_{n,k+r})\right|\right]^{-1} = \left[\sum_{l=0}^{n}\sum_{k=l}^{r+l-1} a_{n,k}\right]^{-1} = O(1),$$

whence

$$|J_3(x)| + \left|J_3'(x)\right| = O_x\left( (n+1)^{\beta+1/p} A_{n,r} \widetilde{\omega}\left(\frac{\pi}{n+1}\right) \right).$$

By Lemma 2, (9), (3), and (8) for $0 \le \beta < 1 - \frac{1}{p}$

$$|J_0(x)| \le \frac{1}{2\pi} \int_0^{\frac{\pi}{r(n+1)}} |\psi_x(t)| \sum_{k=0}^{\infty} a_{n,k} k(k+1)\, t\, dt \le \frac{(n+1)^2}{2\pi} \int_0^{\frac{\pi}{r(n+1)}} |\psi_x(t)|\, t\, dt$$

$$\le \frac{(n+1)^2}{2\pi} \left\{ \int_0^{\frac{\pi}{r(n+1)}} \left( \frac{t\,|\psi_x(t)|}{\widetilde{\omega}(t)} \right)^p \left|\sin\frac{rt}{2}\right|^{\beta p} dt \right\}^{1/p} \left\{ \int_0^{\frac{\pi}{r(n+1)}} \left( \frac{\widetilde{\omega}(t)}{\sin^{\beta}\frac{rt}{2}} \right)^q dt \right\}^{1/q}$$

$$= O_x(n+1)\, \widetilde{\omega}\left(\frac{\pi}{r(n+1)}\right) (n+1)^{\beta-\frac{1}{q}} = O_x\left( (n+1)^{1+\beta+\frac{1}{p}} A_{n,r} \widetilde{\omega}\left(\frac{\pi}{n+1}\right) \right).$$

Analogously as in the estimate of $|J_3(x)|+\left|J_3'(x)\right|$ we get

$$\begin{aligned} |J_4(x)| &\leq \sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\frac{|\psi_x(t)|}{t}dt \\ &\leq \sum_{m=0}^{[r/2]-1}\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{|\psi_x(t)|}{\widetilde{\omega}\left(\frac{2(m+1)\pi}{r}-t\right)}\left|\sin\frac{rt}{2}\right|^{\beta}\right)^{p}dt\right]^{1/p} \\ &\cdot \left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{\widetilde{\omega}\left(\frac{2(m+1)\pi}{r}-t\right)}{\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{1/q}, \text{ where } q=p(p-1)^{-1}. \end{aligned}$$

Hence, by (6)

$$|J_4(x)|=O_x\left((n+1)^{\beta+1/p}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

Thus our proof is complete. □

### 4.2. Proof of Theorem 2

As in the above proof

$$\begin{aligned} &\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{r(n+1)}\right)\right| \\ \leq\ & |J_1(x)|+\left|J_1'(x)\right|+|J_2(x)|+|J_0(x)+J_3(x)|+\left|J_0(x)+J_3'(x)\right|+|J_4(x)| \end{aligned}$$

and

$$|J_1(x)|+\left|J_1'(x)\right|+|J_2(x)|=O_x\left((n+1)^{1+\beta+\frac{1}{p}}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

Next, by Lemma 2, (11), (10) for $0\leq\beta<1-\frac{1}{p}$, and (8), we get

$$\begin{aligned} &|J_0(x)+J_3(x)|+\left|J_0(x)+J_3'(x)\right| \\ \leq\ & \frac{2(n+1)}{\pi}\sum_{m=0}^{[r/2]}\left[\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\left(\frac{|\psi_x(t)|\left|\sin\frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}\left(t-\frac{2m\pi}{r}\right)}\right)^{p}dt\right]^{1/p} \\ &\cdot\left[\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\left(\frac{\widetilde{\omega}\left(t-\frac{2m\pi}{r}\right)}{\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{1/q} \\ =\ & O_x(n+1)\,\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\left[\int_0^{\frac{\pi}{r(n+1)}}\left(\frac{1}{\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{1/q} \\ =\ & O_x\left((n+1)^{1+\beta+\frac{1}{p}}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right). \end{aligned}$$

Finally, applying Lemma 2 and conditions (11), (6), and (8) we obtain

$$
\begin{aligned}
|J_4(x)| &\leq \frac{n+1}{\pi}\sum_{m=0}^{[r/2]-1}\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{|\psi_x(t)|\left|\sin\frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}\left(\frac{2(m+1)\pi}{r}-t\right)}\right)^{p}dt\right]^{1/p}\\
&\cdot\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{\widetilde{\omega}\left(\frac{2(m+1)\pi}{r}-t\right)}{\left|\sin\frac{rt}{2}\right|^{\beta}}\right)^{q}dt\right]^{1/q}\\
&= O_x\left((n+1)^{1+\beta+\frac{1}{p}}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).
\end{aligned}
$$

Collecting the partial estimates we get our statement. □

## 5. CONCLUSIONS

We investigated pointwise approximation of modified conjugate functions by matrix operators of their Fourier series. In particular, we estimated the deviation $\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r(x,\varepsilon)\right|$ by the function of modulus of continuity type in the case when conjugate function $\widetilde{f}_r$ depends on $r$. In the obtained results the measures of approximation depend on $r$-differences of the entries.

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## Modifitseeritud kaasfunktsioonide punktiviisi lähendamine nende Fourier' ridade maatriksoperaatoritega

Włodzimierz Łenski ja Bogdan Szal

Olgu $L^p$ $(1 \leq p < \infty)$ reaalväärtustega $2\pi$-perioodiliste lõigul $[-\pi,\pi]$ Lebesgue'i mõttes integreeruvate funktsioonide klass ja $\tilde{S}f(x)$ funktsiooni $f \in L^p$ trigonomeetriline Fourier' kaasrida. On teada, et funktsiooni $f \in L^1$ kaasfunktsioon $\tilde{f}$ on esitatav teatud viisil defineeritud funktsioonide $\tilde{f}_r(x,\varepsilon)$ ($r$ on naturaalarv) kaudu piirväärtusena

$$\tilde{f}(x) = \lim_{\varepsilon\to 0+}\tilde{f}_r(x,\varepsilon).$$

Olgu $A$ teatud omadustega mittenegatiivne regulaarne maatriks ja $\tilde{T}_{n,A}f(x)$ rea $\tilde{S}f$ $A$-teisendus. Artiklis on antud hinnang vahele $|\tilde{T}_{n,A}f(x)-\tilde{f}_r(x,\varepsilon)|$ pidevuse mooduli tüüpi funktsiooni abil.